Lesson 4.3: Permutations When All Objects Are Distinguishable, page 255

1. a) $_5P_2 = \frac{5!}{(5-2)!}$

$_5P_2 = \frac{5!}{3!}$

$_5P_2 = \frac{5 \cdot 4 \cdot 3!}{3!}$

$_5P_2 = 5 \cdot 4$

$_5P_2 = 20$

b) $_8P_6 = \frac{8!}{(8-6)!}$

$_8P_6 = \frac{8!}{2!}$

$_8P_6 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!}$

$_8P_6 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

$_8P_6 = 20160$

c) $_{10}P_5 = \frac{10!}{(10-5)!}$

$_{10}P_5 = \frac{10!}{5!}$

$_{10}P_5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5!}$

$_{10}P_5 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$

$_{10}P_5 = 30240$

d) $_9P_0 = \frac{9!}{(9-0)!}$

$_9P_0 = \frac{9!}{9!}$

$_9P_0 = 1$

e) $_7P_7 = \frac{7!}{(7-7)!}$

$_7P_7 = \frac{7!}{0!}$

$_7P_7 = \frac{7!}{1}$

$_7P_7 = 7!$

$_7P_7 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$_7P_7 = 5040$

f) $_{15}P_5 = \frac{15!}{(15-5)!}$

$_{15}P_5 = \frac{15!}{10!}$

$_{15}P_5 = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10!}{10!}$

$_{15}P_5 = 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11$

$_{15}P_5 = 360360$

2. a) e.g.,

Permutation	President	Vice-President
1	Katrina	Jess
2	Katrina	Nazir
3	Katrina	Mohamad
4	Jess	Katrina
5	Jess	Nazir
6	Jess	Mohamad
7	Nazir	Jess
8	Nazir	Katrina
9	Nazir	Mohamad
10	Mohamad	Nazir
11	Mohamad	Jess
12	Mohamad	Katrina

There are 12 different ways that a president and vice-president can be elected.

b) $_nP_r = \frac{n!}{(n-r)!}$

$_4P_2 = \frac{4!}{(4-2)!}$

$_4P_2 = \frac{4!}{2!}$

$_4P_2 = \frac{4 \cdot 3 \cdot 2!}{2!}$

$_4P_2 = 4 \cdot 3$

$_4P_2 = 12$

The formula for $_nP_r$ gives an answer of 12. This matches my results from part a).

3. a) $_nP_r = \frac{n!}{(n-r)!}$

$_6P_4 = \frac{6!}{(6-4)!}$

$_6P_4 = \frac{6!}{2!}$

$_6P_4 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!}$

$_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3$

$_6P_4 = 360$

There are 360 different ways the chocolate bars can be distributed.

b) $_nP_r = \frac{n!}{(n-r)!}$

$_6P_1 = \frac{6!}{(6-1)!}$

$_6P_1 = \frac{6!}{5!}$

$_6P_1 = \frac{6 \cdot 5!}{5!}$

$_6P_1 = 6$

The chocolate bars can be distributed in 6 different ways.

4. ${}_{10}P_8$ is larger. e.g., I know this by looking at the formula for ${}_nP_r$. The numerator is the same for both values since n is the same. The denominator will be smaller for the first value since it has a greater r. When you divide a numerator by two different denominators, the final value is greater for the one with the smaller denominator. Based on this, I know that ${}_{10}P_8$ has the larger value since its expansion has the smaller denominator.

5. $${}_9P_3 = \frac{9!}{(9-3)!}$$
$${}_9P_3 = \frac{9!}{6!}$$
$${}_9P_3 = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!}$$
$${}_9P_3 = 9 \cdot 8 \cdot 7$$
$${}_9P_3 = 504$$
There are 504 different ways the positions can be filled.

6. $${}_{15}P_4 = \frac{15!}{(15-4)!}$$
$${}_{15}P_4 = \frac{15!}{11!}$$
$${}_{15}P_4 = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11!}{11!}$$
$${}_{15}P_4 = 15 \cdot 14 \cdot 13 \cdot 12$$
$${}_{15}P_4 = 32760$$
There are 32 760 possible executive committees.

7. $${}_8P_8 = \frac{8!}{(8-8)!}$$
$${}_8P_8 = \frac{8!}{0!}$$
$${}_8P_8 = \frac{8!}{1}$$
$${}_8P_8 = 8!$$
$${}_8P_8 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$${}_8P_8 = 40320$$
Therefore, 40 320 different signals could be created.

8. $${}_{5000}P_3 = \frac{5000!}{(5000-3)!}$$
$${}_{5000}P_3 = \frac{5000!}{4997!}$$
$${}_{5000}P_3 = \frac{5000 \cdot 4999 \cdot 4998 \cdot 4997!}{4997!}$$
$${}_{5000}P_3 = 5000 \cdot 4999 \cdot 4998$$
$${}_{5000}P_3 = 124\ 925\ 010\ 000$$
There are about 124 925 010 000 different ways the tickets can be drawn.

9. e.g., Assuming that any of the 10 digits can be put in any of the 8 remaining spots for the SINs, let S represent the number of social insurance numbers:
$S = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$
$S = 10^8$
$S = 100\ 000\ 000$
There are 100 000 000 different SINs that can be registered in each of these groups of provinces and territories.

10. a) $${}_{12}P_5 = \frac{12!}{(12-5)!}$$
$${}_{12}P_5 = \frac{12!}{7!}$$
$${}_{12}P_5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7!}$$
$${}_{12}P_5 = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$$
$${}_{12}P_5 = 95040$$
There are 95 040 ways the coach can select the starting five players.

b) $${}_{11}P_4 = \frac{11!}{(11-4)!}$$
$${}_{11}P_4 = \frac{11!}{7!}$$
$${}_{11}P_4 = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7!}$$
$${}_{11}P_4 = 11 \cdot 10 \cdot 9 \cdot 8$$
$${}_{11}P_4 = 7920$$
There are 7920 ways the coach can select the starting five players, if the tallest student must start at the centre position.

c) $${}_{10}P_3 = \frac{10!}{(10-3)!}$$
$${}_{10}P_3 = \frac{10!}{7!}$$
$${}_{10}P_3 = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$$
$${}_{10}P_3 = 10 \cdot 9 \cdot 8$$
$${}_{10}P_3 = 720$$
Multiply by 2, since Sandy and Natasha can play the guard positions in either order. (720)(2) = 1440
There are 1440 ways in which the coach can select the starting five players, if Sandy and Natasha must play the two guard positions.

11. a) $n \geq 0$ and
$n - 1 \geq 0$
$n \geq 1$
Therefore, the expression is defined for $n \geq 1$, where $n \in I$.
b) $n + 2 \geq 0$
$n \geq -2$
Therefore, the expression is defined for $n \geq -2$, where $n \in I$.

c) $n+1 \geq 0$ AND $n \geq 0$

$n \geq -1$

Therefore, the expression is defined for $n \geq 0$, where $n \in$ I.

d) $n+5 \geq 0$ AND $n+3 \geq 0$

$n \geq -5$ $\quad$ $n \geq -3$

Therefore, the expression is defined for $n \geq -3$, where $n \in$ I.

12. a) $_6P_4 = \dfrac{6!}{(6-4)!}$

$_6P_4 = \dfrac{6!}{2!}$

$_6P_4 = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!}$

$_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3$

$_6P_4 = 360$

There are 360 ways to draw the four marbles if you do not replace the marble each time.

b) Let L represent the number of ways:

$L = 6 \cdot 6 \cdot 6 \cdot 6$

$L = 6^4$

$L = 1296$

There are 1296 ways to draw the four marbles if you replace the marble each time.

c) e.g., Yes; if you replace the marble, there are more possibilities for the next draw.

13. a) $_{20}P_5 = \dfrac{20!}{(20-5)!}$

$_{20}P_5 = \dfrac{20!}{15!}$

$_{20}P_5 = \dfrac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{15!}$

$_{20}P_5 = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

$_{20}P_5 = 1860480$

There are 1 860 480 different ways to award the scholarships.

b) Let L represent the number of ways:

$L = 20 \cdot 20 \cdot 20 \cdot 20 \cdot 20$

$L = 20^5$

$L = 3\,200\,000$

There are 3 200 000 different ways to award the scholarships.

14. a) $_{10}P_4 = \dfrac{10!}{(10-4)!}$

$_{10}P_4 = \dfrac{10!}{6!}$

$_{10}P_4 = \dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!}$

$_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7$

$_{10}P_4 = 5040$

There are 5040 different phone numbers possible.

b) Subtract the total possible numbers by the answer to part a).

104 = 10 000

10 000 – 5040 = 4960

There are 4960 different phone numbers.

15. a) I need to solve $\dfrac{n!}{(n-2)!} = 20$.

$n \geq 0$ AND $n-2 \geq 0$

$n \geq 2$

Therefore, $\dfrac{n!}{(n-2)!} = 20$ is defined for $n \geq 2$, where $n \in$ I.

$$\frac{n!}{(n-2)!} = 20$$

$$\frac{(n)(n-1)(n-2)(n-3)...(3)(2)(1)}{(n-2)(n-3)...(3)(2)(1)} = 20$$

$$\frac{(n)(n-1)(n-2)!}{(n-2)!} = 20$$

$$(n)(n-1) = 20$$

$$n^2 - n = 20$$

$$n^2 - n - 20 = 0$$

$$(n+4)(n-5) = 0$$

$n+4=0$ or $n-5=0$

$n=-4$ $\quad$ $n=5$

The root $n = -4$ is not a solution to $n \geq 2$.

Check $n = 5$

LS	RS
$_5P_2$	20
$\dfrac{5!}{(5-2)!}$	
$\dfrac{5!}{3!}$	
$\dfrac{5 \cdot 4 \cdot 3!}{3!}$	
$5 \cdot 4$	
20	

There is one solution, $n = 5$.

b) I need to solve $\dfrac{(n+1)!}{(n+1-2)!} = 72$.

$n+1 \geq 0$ AND $n+1-2 \geq 0$

$n \geq -1$ $\quad$ $n-1 \geq 0$

$n \geq 1$

Therefore, $\dfrac{(n+1)!}{(n+1-2)!} = 72$ is defined for $n \geq 1$, where $n \in$ I.

$$\frac{(n+1)!}{(n+1-2)!}=72$$
$$\frac{(n+1)!}{(n-1)!}=72$$
$$\frac{(n+1)(n)(n-1)(n-2)...(3)(2)(1)}{(n-1)(n-2)...(3)(2)(1)}=72$$
$$\frac{(n+1)(n)(n-1)!}{(n-1)!}=72$$
$$(n+1)(n)=72$$
$$n^2+n=72$$
$$n^2+n-72=0$$
$$(n+9)(n-8)=0$$

$n+9=0$ or $n-8=0$

$n=-9$ $\quad$ $n=8$

The root n = –9 is not a solution to $n \geq 1$.

Check n = 8

LS	RS
${}_{8+1}P_2$	72
${}_9P_2$	
$\frac{9!}{(9-2)!}$	
$\frac{9!}{7!}$	
$\frac{9\cdot 8\cdot 7!}{7!}$	
$9\cdot 8$	
72	

There is one solution, n = 8.

16. a) The equation I need to solve is $\frac{6!}{(6-r)!}=30$.

$6-r\geq 0$

$r\leq 6$

Therefore, $\frac{6!}{(6-r)!}=30$ is defined for $0 \leq r \leq 6$, where $r\in$ I.

$$\frac{6!}{(6-r)!}=30$$
$$\frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(6-r)!}=30$$
$$\frac{720}{(6-r)!}=30$$
$$(6-r)!=\frac{720}{30}$$
$$(6-r)!=24$$
$$6-r=4$$
$$-r=-2$$
$$r=2$$

Check r = 2

LS	RS
${}_6P_2$	30
$\frac{6!}{(6-2)!}$	
$\frac{6!}{4!}$	
$\frac{6\cdot 5\cdot 4!}{4!}$	
$6\cdot 5$	
30	

There is one solution, r = 2.

b) The equation I need to solve is $2\left[\frac{7!}{(7-r)!}\right]=420$.

$7-r\geq 0$

$r\leq 7$

Therefore, $2\left[\frac{7!}{(7-r)!}\right]=420$ is defined for $0 \leq r \leq 7$, where $r\in$ I.

$$2\left[\frac{7!}{(7-r)!}\right]=420$$
$$\frac{7!}{(7-r)!}=210$$
$$\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(7-r)!}=210$$
$$\frac{5040}{(7-r)!}=210$$
$$(7-r)!=\frac{5040}{210}$$
$$(7-r)!=24$$
$$7-r=4$$
$$-r=-3$$
$$r=3$$

Check r = 3

LS	RS
$2({}_7P_3)$	420
$2\left[\frac{7!}{(7-3)!}\right]$	
$2\left(\frac{7!}{4!}\right)$	
$2\left(\frac{7\cdot 6\cdot 5\cdot 4!}{4!}\right)$	
$2(7\cdot 6\cdot 5)$	
$2(210)$	
420	

There is one solution, r = 3.

17. e.g.,

LS	RS
${}_nP_n$	${}_nP_{n-1}$
$\frac{n!}{(n-n)!}$	$\frac{n!}{[n-(n-1)]!}$
$\frac{n!}{0!}$	$\frac{n!}{[n-n+1]!}$
$\frac{n!}{1}$	$\frac{n!}{1!}$
$n!$	$\frac{n!}{1}$
	$n!$

LS = RS

18. a) e.g., The formulas for both ${}_nP_n$ and ${}_nP_r$ have a numerator of $n!$. However, the formula for ${}_nP_n$ has a denominator of 1 and the formula for ${}_nP_r$ has a denominator of $(n - r)!$.
b) e.g., A group of friends each order a different flavour of ice cream from a shop with 12 flavours. How many possibilities are there if the group is 12 people? If the group is 7 people?

19. a) $n = 52$ and $r = 5$

$${}_{52}P_5 = \frac{52!}{(52-5)!}$$
$${}_{52}P_5 = \frac{52!}{47!}$$
$${}_{52}P_5 = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47!}{47!}$$
$${}_{52}P_5 = 52\cdot 51\cdot 50\cdot 49\cdot 48$$
$${}_{52}P_5 = 311875200$$

There are 311 875 200 possible arrangements.
b) $n = 26$ and $r = 5$

$${}_{26}P_5 = \frac{26!}{(26-5)!}$$
$${}_{26}P_5 = \frac{26!}{21!}$$
$${}_{26}P_5 = \frac{26\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21!}{21!}$$
$${}_{26}P_5 = 26\cdot 25\cdot 24\cdot 23\cdot 22$$
$${}_{26}P_5 = 7893600$$

Likelihood = $\frac{7\ 893\ 600}{311\ 875\ 200}\cdot 100\%$
Likelihood = $0.025...\cdot 100\%$
Likelihood = 2.531...%
Therefore, there is about a 2.53% chance that an arrangement contains black cards only.

c) $n = 13$ and $r = 5$

$${}_{13}P_5 = \frac{13!}{(13-5)!}$$
$${}_{13}P_5 = \frac{13!}{8!}$$
$${}_{13}P_5 = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8!}{8!}$$
$${}_{13}P_5 = 13\cdot 12\cdot 11\cdot 10\cdot 9$$
$${}_{13}P_5 = 154440$$

Likelihood = $\frac{154440}{311875200}\cdot 100\%$
Likelihood = $0.000\ldots \cdot 100\%$
Likelihood = 0.049…%
Therefore, there is about a 0.05% chance that an arrangement contains hearts only.

20. e.g.,
$${}_{n+1}P_2 - {}_nP_1 = \frac{(n+1)!}{(n+1-2)!} - \frac{n!}{(n-1)!}$$
$$= \frac{(n+1)!}{(n-1)!} - \frac{n!}{(n-1)!}$$
$$= \frac{(n+1)! - n!}{(n-1)!}$$
$$= \frac{n+1\cdot n\cdot n-1\cdot n-2... - n\cdot n-1\cdot n-2...}{n-1\cdot n-2...}$$
$$= \frac{n\cdot n\cdot -1\cdot n-2...(n+1-1)}{n-1\cdot n-2...}$$
$$= n(n)$$
$$= n^2$$

21. e.g.,
$${}_nP_{r+1} = \frac{n!}{[n-(r+1)]!}$$
$$= \frac{n!}{(n-r-1)!}$$
$$= \frac{n!}{(n-r-1)!}\cdot\frac{n-r}{n-r}$$
$$= \frac{(n-r)n!}{(n-r)!}$$
$$= (n-r)\frac{n!}{(n-r)!}$$
$$= (n-r)\,{}_nP_r$$

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a) e.g., January 5, April 23, July 24, and October 15 would be 5, 113, 205, and 288.
b) 365 · 365 · 365 · 365 = 17 748 900 630
c) i) $\frac{{}_{365}P_4}{365^4} = 0.983...$ or about 98.4%

ii) $1 - \frac{{}_{365}P_4}{365^4} = 0.016...$ or about 1.6%