Therefore, there are 9 numbers that are divisible by 5 and not by 2. If I add this together with the number of two-digit numbers that are divisible by two (45), I see that there are 54 two-digit numbers divisible by 2 or 5. Whatever is leftover from the two digit numbers are the ones that are not divisible by either 2 or 5. This amount is:
90 – 54 = 36. Thus, there are 36 two-digit numbers that are not divisible by either 2 or 5.

20. The number of different outcomes for a student's test, *N*, is related to the number of possible answers for each question on the test, *T*:
$N = T_1 \cdot T_2 \cdot T_3 \cdot T_4 \cdot T_5 \cdot T_6 \cdot T_7 \cdot T_8 \cdot T_9 \cdot T_{10}$
$N = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
$N = 1024$
A perfect score is only 1 out of these 1024 outcomes; therefore, there is a 1 in 1024 chance that the student will get a perfect score.

21. This question is solved by constant application of the Fundamental Counting Principle.
If an item from each category is selected:
$O = 3 \cdot 5 \cdot 4 \cdot 2$
$O = 120$
If no soup is selected:
$O = 5 \cdot 4 \cdot 2$
$O = 40$
If no sandwich is selected:
$O = 3 \cdot 4 \cdot 2$
$O = 24$
If no drink is selected:
$O = 3 \cdot 5 \cdot 2$
$O = 30$
If no dessert is selected:
$O = 3 \cdot 5 \cdot 4$
$O = 60$
If no soup or sandwich is selected:
$O = 4 \cdot 2$
$O = 8$
If no soup or drink selected:
$O = 5 \cdot 2$
$O = 10$
If no soup or dessert is selected:
$O = 5 \cdot 4$
$O = 20$
If no sandwich or drink is selected:
$O = 3 \cdot 2$
$O = 6$
If no sandwich or dessert is selected:
$O = 3 \cdot 4$
$O = 12$
If no drink or dessert is selected:
$O = 3 \cdot 5$
$O = 15$
If only a soup, sandwich, drink or dessert is selected:
O = 3, 5, 4, 2
$T_{\text{otal}} = 120 + 40 + 24 + 30 + 60 + 8 + 10 + 20 + 6 + 12 + 15 + 3 + 5 + 4 + 2$
$T_{\text{otal}} = 359$
Therefore, 359 meals are possible if you do not have to choose an item from a category.

Lesson 4.2: Introducing Permutations and Factorial Notation, page 243

1. a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$6! = 720$

b) $9 \cdot 8! = 9 \cdot (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$9 \cdot 8! = 9 \cdot 40320$
$9 \cdot 8! = 362880$

c) $\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$
$\frac{5!}{3!} = 5 \cdot 4 \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$
$\frac{5!}{3!} = 5 \cdot 4 \cdot \frac{3!}{3!}$
$\frac{5!}{3!} = 5 \cdot 4 \cdot 1$
$\frac{5!}{3!} = 20$

d) $\frac{8!}{7!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$
$\frac{8!}{7!} = 8 \cdot \frac{7}{7} \cdot \frac{6}{6} \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$
$\frac{8!}{7!} = 8 \cdot \frac{7!}{7!}$
$\frac{8!}{7!} = 8 \cdot 1$
$\frac{8!}{7!} = 8$

e) $3! \cdot 2! = (3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)$
$3! \cdot 2! = 6 \cdot 2$
$3! \cdot 2! = 12$

f) $\frac{9!}{4!3!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)}$
$\frac{9!}{4!3!} = \frac{9}{3} \cdot \frac{8}{2} \cdot \frac{7}{1} \cdot 6 \cdot 5 \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$
$\frac{9!}{4!3!} = \frac{9}{3} \cdot \frac{8}{2} \cdot 7 \cdot 6 \cdot 5 \cdot \frac{4!}{4!}$
$\frac{9!}{4!3!} = \frac{9}{3} \cdot \frac{8}{2} \cdot 7 \cdot 6 \cdot 5 \cdot 1$
$\frac{9!}{4!3!} = 3 \cdot 4 \cdot 7 \cdot 6 \cdot 5$
$\frac{9!}{4!3!} = 2520$

2. a) e.g., There are six permutations of Ken, Sarah, and Raj. I figured this out by making a table showing each permutation and the three possible positions. Also, $3 \cdot 2 \cdot 1 = 6$

	Position 1	**Position 2**	**Position 3**
Permutation 1	Ken	Sarah	Raj
Permutation 2	Ken	Raj	Sarah
Permutation 3	Sarah	Ken	Raj
Permutation 4	Sarah	Raj	Ken
Permutation 5	Raj	Ken	Sarah
Permutation 6	Raj	Sarah	Ken

b) Let L represent the total number of permutations:
$L = 3 \cdot 2 \cdot 1$
$L = 3!$

3. a) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

b) $9 \cdot 8 \cdot 7 = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$9 \cdot 8 \cdot 7 = \frac{9!}{6!}$

c) $\frac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{12!}{12!}$

$\frac{15 \cdot 14 \cdot 13}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{15!}{12!4!}$

d) $100 \cdot 99 = 100 \cdot 99 \cdot \frac{98!}{98!}$

$100 \cdot 99 = \frac{100!}{98!}$

4. Expressions a), c), and d) are undefined because factorial notation is only defined for natural numbers.

5. a) $8 \cdot 7 \cdot 6! = 8 \cdot 7 \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

$8 \cdot 7 \cdot 6! = 8 \cdot 7 \cdot 720$

$8 \cdot 7 \cdot 6! = 56 \cdot 720$

$8 \cdot 7 \cdot 6! = 40\,320$

b) $\frac{12!}{10!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$\frac{12!}{10!} = 12 \cdot 11 \cdot \frac{10}{10} \cdot \frac{9}{9} \cdot \frac{8}{8} \cdot \frac{7}{7} \cdot \frac{6}{6} \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$

$\frac{12!}{10!} = 12 \cdot 11 \cdot \frac{10!}{10!}$

$\frac{12!}{10!} = 12 \cdot 11 \cdot 1$

$\frac{12!}{10!} = 132$

c) $\frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}$

$\frac{8!}{2! \cdot 6!} = \frac{8}{2} \cdot \frac{7}{1} \cdot \frac{6}{6} \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$

$\frac{8!}{2! \cdot 6!} = \frac{8}{2} \cdot 7 \cdot \frac{6!}{6!}$

$\frac{8!}{2! \cdot 6!} = \frac{8}{2} \cdot 7 \cdot 1$

$\frac{8!}{2! \cdot 6!} = 4 \cdot 7$

$\frac{8!}{2! \cdot 6!} = 28$

d) $\frac{7 \cdot 6!}{5!} = \frac{7 \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$\frac{7 \cdot 6!}{5!} = 7 \cdot 6 \cdot \frac{5}{5} \cdot \frac{4}{4} \cdot \frac{3}{3} \cdot \frac{2}{2} \cdot \frac{1}{1}$

$\frac{7 \cdot 6!}{5!} = 7 \cdot 6 \cdot \frac{5!}{5!}$

$\frac{7 \cdot 6!}{5!} = 7 \cdot 6 \cdot 1$

$\frac{7 \cdot 6!}{5!} = 42$

e) $4\left(\frac{6!}{2! \cdot 2!}\right) = 4\left[\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (2 \cdot 1)}\right]$

$4\left(\frac{6!}{2! \cdot 2!}\right) = 4\left(\frac{6}{2} \cdot \frac{5}{1} \cdot 4 \cdot 3 \cdot \frac{2}{2} \cdot \frac{1}{1}\right)$

$4\left(\frac{6!}{2! \cdot 2!}\right) = 4\left(\frac{6}{2} \cdot 5 \cdot 4 \cdot 3 \cdot \frac{2!}{2!}\right)$

$4\left(\frac{6!}{2! \cdot 2!}\right) = 4\left(\frac{6}{2} \cdot 5 \cdot 4 \cdot 3 \cdot 1\right)$

$4\left(\frac{6!}{2! \cdot 2!}\right) = 4(3 \cdot 5 \cdot 4 \cdot 3)$

$4\left(\frac{6!}{2! \cdot 2!}\right) = 4(180)$

$4\left(\frac{6!}{2! \cdot 2!}\right) = 720$

f) $4! + 3! + 2! + 1! = (4 \cdot 3 \cdot 2 \cdot 1) + (3 \cdot 2 \cdot 1) + (2 \cdot 1) + 1$

$4! + 3! + 2! + 1! = 24 + 6 + 2 + 1$

$4! + 3! + 2! + 1! = 33$

6. a) $\frac{n!}{(n-1)!} = \frac{(n)(n-1)(n-2)(n-3)...(3)(2)(1)}{(n-1)(n-2)(n-3)...(3)(2)(1)}$

$\frac{n!}{(n-1)!} = \frac{(n)(n-1)!}{(n-1)!}$

$\frac{n!}{(n-1)!} = n$

b) $(n+4)(n+3)(n+2)!$

$=(n+4)(n+3)\left[(n+2)(n+1)(n)(n-1)...(3)(2)(1)\right]$

$=(n+4)(n+3)(n+2)(n+1)(n)(n-1)...(3)(2)(1)$

$=(n+4)!$

c) $\frac{(n+1)!}{n!}=\frac{(n+1)(n)(n-1)(n-2)...(3)(2)(1)}{(n)(n-1)(n-2)...(3)(2)(1)}$

$\frac{(n+1)!}{n!}=\frac{(n+1)(n!)}{n!}$

$\frac{(n+1)!}{n!}=n+1$

d) $\frac{n!}{(n-3)!}=\frac{(n)(n-1)(n-2)(n-3)(n-4)(n-5)...(3)(2)(1)}{(n-3)(n-4)(n-5)...(3)(2)(1)}$

$\frac{n!}{(n-3)!}=\frac{(n)(n-1)(n-2)(n-3)!}{(n-3)!}$

$\frac{n!}{(n-3)!}=(n)(n-1)(n-2)$

$\frac{n!}{(n-3)!}=(n^2-n)(n-2)$

$\frac{n!}{(n-3)!}=n^3-3n^2+2n$

e) $\frac{(n+5)!}{(n+3)!}=\frac{(n+5)(n+4)(n+3)(n+2)(n+1)...(3)(2)(1)}{(n+3)(n+2)(n+1)...(3)(2)(1)}$

$\frac{(n+5)!}{(n+3)!}=\frac{(n+5)(n+4)(n+3)!}{(n+3)!}$

$\frac{(n+5)!}{(n+3)!}=(n+5)(n+4)$

$\frac{(n+5)!}{(n+3)!}=n^2+9n+20$

f) $\frac{(n-2)!}{(n-1)!}=\frac{(n-2)(n-3)(n-4)...(3)(2)(1)}{(n-1)(n-2)(n-3)(n-4)...(3)(2)(1)}$

$\frac{(n-2)!}{(n-1)!}=\frac{(n-2)!}{(n-1)(n-2)!}$

$\frac{(n-2)!}{(n-1)!}=\frac{1}{n-1}$

7. There are nine students in the lineup, so there are nine possible positions. Let L represent the total number of permutations:

$L = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 9!$
$L = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 72 \cdot 7 \cdot 30 \cdot 4 \cdot 6$
$L = 72 \cdot 210 \cdot 24$
$L = 362\ 880$

There are 362 880 permutations for the nine students at the Calgary Stampede.

8. There are five students in the club and there are five possible positions. Let L represent the total number of permutations:

$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 5!$
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 20 \cdot 6$
$L = 120$

There are 120 different ways to select members for the five positions.

9. There are six activities to do and there are six days. Let L represent the total number of permutations:

$L = 6!$
$L = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 30 \cdot 4 \cdot 6$
$L = 120 \cdot 6$
$L = 720$

There are 720 different ways they can sequence these activities over the six days.

10. There are 28 movies, so there are 28 possible spots for the movies to go. Let L represent the total number of permutations:

$L = 28!$
$L = 3.048\ldots \times 10^{29}$

There are about 3.05×10^{29} possible permutations of the movie list.

11. a) $\frac{(n+1)!}{n!}=10$

$\frac{(n+1)(n)(n-1)...(3)(2)(1)}{(n)(n-1)...(3)(2)(1)}=10$

$\frac{(n+1)(n!)}{n!}=10$

$n+1=10$

$n=9$

Check $n = 9$

LS	RS
$\frac{(9+1)!}{9!}$	10
$\frac{10!}{9!}$	
$\frac{10 \cdot 9!}{9!}$	
10	

There is one solution, $n = 9$.

b) $$\frac{(n+2)!}{n!}=6$$

$$\frac{(n+2)(n+1)(n)(n-1)...(3)(2)(1)}{(n)(n-1)...(3)(2)(1)}=6$$

$$\frac{(n+2)(n+1)(n!)}{n!}=6$$

$$(n+2)(n+1)=6$$

$$n^2+n+2n+2=6$$

$$n^2+3n+2=6$$

$$n^2+3n-4=0$$

$$(n+4)(n-1)=0$$

$n+4=0$ or $n-1=0$

$n=-4$ $n=1$

Check n = –4

LS	RS
$\frac{(-4+2)!}{(-4)!}$	6
$\frac{(-2)!}{(-4)!}$ is undefined	

Check n = 1

LS	RS
$\frac{(1+2)!}{1!}$	6
$\frac{3!}{1!}$	
$\frac{3\cdot 2\cdot 1!}{1!}$	
$3\cdot 2$	
6	

There is one solution, n = 1.

c) $$\frac{(n-1)!}{(n-2)!}=8$$

$$\frac{(n-1)(n-2)(n-3)...(3)(2)(1)}{(n-2)(n-3)...(3)(2)(1)}=8$$

$$\frac{(n-1)(n-2)!}{(n-2)!}=8$$

$$n-1=8$$

$$n=9$$

Check n = 9

LS	RS
$\frac{(9-1)!}{(9-2)!}$	8
$\frac{8!}{7!}$	
$\frac{8\cdot 7!}{7!}$	
8	

There is one solution, n = 9.

d) $$\frac{3(n+1)!}{(n-1)!}=126$$

$$\frac{3(n+1)(n)(n-1)(n-2)...(3)(2)(1)}{(n-1)(n-2)...(3)(2)(1)}=126$$

$$\frac{3(n+1)(n)(n-1)!}{(n-1)!}=126$$

$$3(n+1)(n)=126$$

$$3(n^2+n)=126$$

$$3(n^2+n)-126=0$$

$$3\left[(n^2+n)-42\right]=0$$

$$3(n^2+n-42)=0$$

$$3(n+7)(n-6)=0$$

$n+7=0$ or $n-6=0$

$n=-7$ $n=6$

Check n = –7

LS	RS
$\frac{3(-7+1)!}{(-7-1)!}$	126
$\frac{3(-6)!}{(-8)!}$ is undefined	

Check n = 6

LS	RS
$\frac{3(6+1)!}{(6-1)!}$	126
$\frac{3(7!)}{5!}$	
$\frac{3\cdot(7\cdot 6\cdot 5!)}{5!}$	
$3\cdot 7\cdot 6$	
126	

There is one solution, n = 6.

12. There are eight more players left to organize so there are eight more spots left in the batting order. Let L represent the number of permutations:
$L = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 8!$
$L = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 8 \cdot 42 \cdot 20 \cdot 6$
$L = 336 \cdot 120$
$L = 40\ 320$
There are 40 320 possible batting orders.

13. There are 7 possible digits to use and there are 7 digits in each serial number. Let L represent the number of permutations:
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 7!$
There are 7! possible serial numbers. This makes sense because, e.g., the integer in the factorial (7 in this case) for the number of permutations is normally equal to the number of spots in which there are things to place. There are seven spots in the serial number so this means that the number of permutations should be 7! which matches the answer that was found.

14. There are 5 cars to be arranged between the engine and the caboose so there are 5 spots in which the cars can be lined up. Let L represent the number of permutations:
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 5!$
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 20 \cdot 6$
$L = 120$
There are 120 ways for the cars to be arranged between the engine and the caboose.

15. There would be 7 chuckwagons behind Brant's so there are 7 spots where the other drivers could finish. Let L represent the number of permutations:
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 7!$
$L = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 42 \cdot 20 \cdot 6$
$L = 42 \cdot 120$
$L = 5040$
If Brant's wagon wins, there are 5040 different orders in which the eight chuckwagons can finish.

16. a) e.g., YKONU, YUKNO, YKNOU
b) There are five letters so there are five spots to put the letters. Let L represent the number of permutations:
$L = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$L = 5!$
There are 5! possible permutations. This makes sense because e.g., the integer in the factorial (5 in this case) for the number of permutations is normally equal to the number of spots in which there are things to place. There are five spots to place the letters so this means that the number of permutations should be 5! which matches the answer that was found.

17. a) e.g., Using trial and error, I have the following calculations:
$1! = 1$, $2^1 = 2$; $2! = 2$, $2^2 = 4$;
$3! = 6$, $2^3 = 8$; $4! = 24$, $2^4 = 16$
I notice that for $n = 4$, $n!$ is greater than 2^n. This continues for $n \geq 4$ because 2^4 will keep getting multiplied by 2, while 4! will keep getting multiplied by numbers greater than 2 to obtain the higher factorials.
b) e.g., Using what I have in a), I know that for $n < 4$, $n!$ is not greater than 2^n. The calculations for these values of n are shown in a). Thus for $n = \{1, 2, 3\}$, $n!$ is less than 2^n.

18. e.g., First, figure out how many ways Darlene and Arnold can be placed next to each other in the line. This can be found using a table.

Darlene	Arnold
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9
9	10
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8
10	9

From the table I can see that there are 18 different ways for Darlene and Arnold to be placed next to each other in the line. For every one of those 18 ways, there are 8 other dancers to be placed in 8 different spots in the line. Let L represent the number of permutations:
$L = 18(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$L = 18(8!)$
$L = 18(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$L = 18(8 \cdot 42 \cdot 20 \cdot 6)$
$L = 18(336 \cdot 120)$
$L = 18(40\ 320)$
$L = 725\ 760$
There are 725 760 possible arrangements of the dancers for the Red River Jig.