8.5 Modelling Data with Sinusoidal Functions

GOAL

Determine the sinusoidal function that best models a set of data, and use your model to solve a problem.

YOU WILL NEED

- graphing technology

EXPLORE...

- Think of naturally occurring phenomena that you can model with a sinusoidal function. How many phenomena can the class come up with?

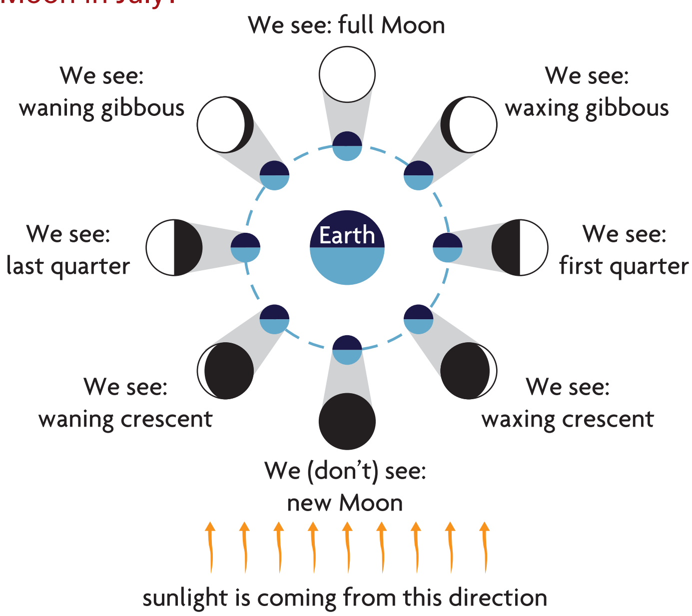
INVESTIGATE the Math

Kelly lives in Winnipeg, Manitoba. She walks her dog at the same time each evening. She noticed that a different percent of the Moon is illuminated each evening, so she decided to look for patterns. She recorded the following data for three months, beginning on April 1.

Percent of the Moon Visible Each Evening

Day	1	3	5	7	9	11	13	15	17	19	21	23
Percent (%)	2	0	6	20	41	64	85	96	100	95	82	63
Day	25	27	29	31	33	35	37	39	41	43	45	47
Percent (%)	43	25	10	1	0	2	12	30	52	75	93	98
Day	49	51	53	55	57	59	61	63	65	67	69	71
Percent (%)	99	89	72	53	33	17	5	0	4	17	38	61
Day	73	75	77	79	81	83	85	87	89	91		
Percent (%)	83	97	100	97	86	69	50	31	14	3		

? Kelly is planning a camping trip in July, and she wants her trip to occur during a full Moon. When can she expect to see a full Moon in July?



A. Create a scatter plot on graph paper. Indicate the day along the horizontal axis and the percent of the Moon illuminated along the vertical axis. Draw a curve that best approximates the trend in the data.

B. How would you describe your graph? Explain.

C. The synodic period is the length of time between full Moons. Estimate the synodic period using your graph.

D. Enter the data into your graphing calculator. Create a scatter plot.

E. Determine the equation of the sinusoidal regression function that models the data.

F. Comment on the accuracy of your regression equation.

G. Determine the synodic period using your regression equation.

H. On what date in July will the full Moon occur? Explain.

Reflecting

I. The synodic period of the Moon is actually 29.53 days. Compare your result with this value.

J. Would you rely on a prediction made from Kelly's data? Explain.

K. To save time, Betty entered only every fourth data point in her graphing calculator. Would she get the same sinusoidal regression equation that you did? Would Betty's prediction be as reliable? Explain.

APPLY the Math

EXAMPLE 1 Solving an interpolation problem using a sinusoidal model

Celeste lives in Red Deer, Alberta. The predicted hours of daylight for two consecutive years are shown in the tables below. In the second year, the spring equinox will occur on March 20 and the fall equinox will occur on September 23. Compare the hours of daylight on these two days.

Hours of Daylight in Red Deer This Year		
Date	**Day Number**	**Length of Day (h)**
Jan. 1	1	7.812
Feb. 1	32	9.113
Mar. 1	60	10.896
Apr. 1	91	12.998
May 1	121	14.944
Jun. 1	152	16.455
Jul. 1	182	16.690
Aug. 1	213	15.494
Sep. 1	244	13.595
Oct. 1	274	11.597
Nov. 1	305	9.580
Dec. 1	335	8.064

Predicted Hours of Daylight in Red Deer Next Year		
Date	**Day Number**	**Length of Day (h)**
Jan. 1	366	7.808
Feb. 1	397	9.100
Mar. 1	425	10.880
Apr. 1	456	12.982
May 1	486	14.929
Jun. 1	517	16.447
Jul. 1	547	16.694
Aug. 1	578	15.507
Sep. 1	609	13.611
Oct. 1	639	11.613
Nov. 1	670	9.595
Dec. 1	700	8.073
Jan. 1	731	7.803

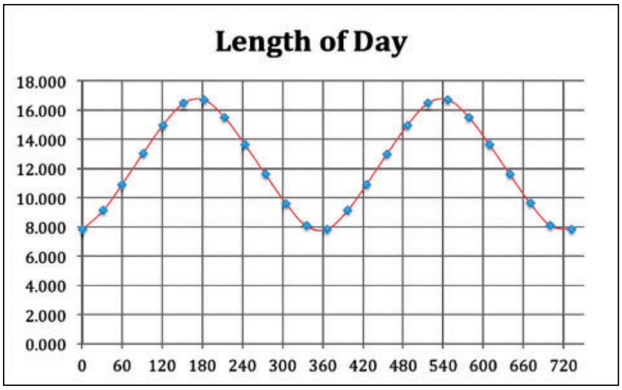
Celeste's Solution

I entered the data into a spreadsheet.

	A	B
1	Day Number	Length of Day (h)
2	1	7.812
3	32	9.113
4	60	10.896
25	700	8.073
26	731	7.803

I entered the day numbers in column A, so the independent variable would precede the dependent variable.

I created a graph I could use to compare day number to hours of daylight.



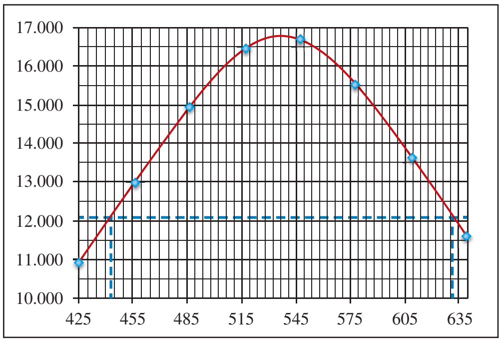
The data is sinusoidal.
One maximum point is at about $x = 180$ and the next maximum point is at about $x = 545$.
Therefore, the period is about $545 - 180$, or 365 days. This makes sense, because a year is 365 days long.

I saw that the graph formed a sine wave. This makes sense because the hours of daylight are determined by the tilt of the Earth as it orbits the Sun in a repeating cycle.

I determined the period.

March 1 of next year is day 425, so March 20 will be 19 days later, or day 444.
September 1 of next year is day 609, so September 23 will be 22 days later, or day 631.

I determined the day numbers for the spring and fall equinoxes.



I zoomed in on the area of the graph that I wanted to study.

I determined the points on the graph for both dates. I looked at the *y*-coordinate of each point.

By interpolation, I can see that days 444 and 631 both have about 12.1 h of daylight.

The spring and fall equinoxes both have about the same amount of daylight, about 12.1 h.

These values are approximate, because I interpolated the data on a graph.

Your Turn

Suppose that the summer solstice will be on June 21 and the winter solstice will be on December 21 this year. Determine the hours of daylight in Red Deer on each day. Round to two decimal places.

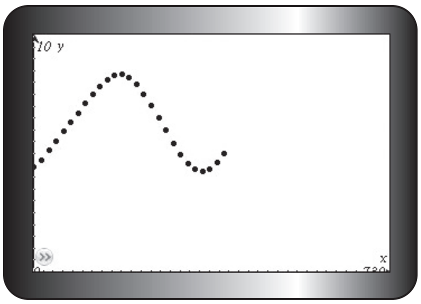
EXAMPLE 2 Solving an extrapolation problem using a sinusoidal model

Tara's high school class is planning a trip to British Columbia to study eco-tourism next year. The students want to finish the tour, on May 21, by watching the Sun set over the Pacific Ocean. Tara found the following data for the current year and the following January. All the times are in standard time, in hours after noon, as decimal values. At what time should the class be at the vantage point on the cliff?

Time of Sunset on West Coast of British Columbia This Year								
Date	**Day**	**Time of Sunset**	**Date**	**Day**	**Time of Sunset**	**Date**	**Day**	**Time of Sunset**
Jan. 1	1	4.42	May 16	136	7.83	Sep. 28	271	5.97
Jan. 16	16	4.72	May 31	151	8.13	Oct. 13	286	5.43
Jan. 31	31	5.12	Jun. 15	166	8.32	Oct. 28	301	4.93
Feb. 15	46	5.53	Jun. 30	181	8.35	Nov. 12	316	4.58
Mar. 2	61	5.95	Jul. 15	196	8.22	Nov. 27	331	4.32
Mar. 17	76	6.33	Jul. 30	211	7.92	Dec. 12	346	4.23
Apr. 1	91	6.72	Aug. 14	226	7.52	Dec. 27	361	4.33
Apr. 16	106	7.10	Aug. 29	241	7.03	Jan. 11	376	4.60
May 1	121	7.48	Sep. 13	256	6.50	Jan. 26	391	4.98

Tara's Solution

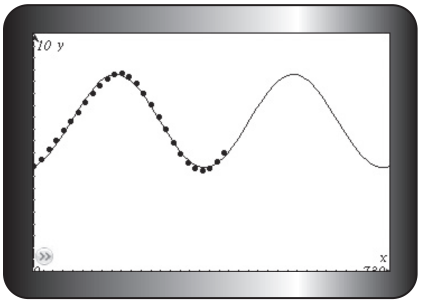
I entered the data into my graphing calculator and created a scatter plot.



I think there might be minimum points when $x = 0$ and $x = 365$.

I examined the points. It made sense that the period is 365 days or 1 year.

The period is about 365 days or about 1 year.

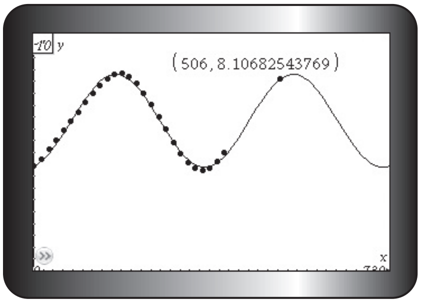


I determined the equation of the sinusoidal regression function and graphed the function.

$y = 1.966... \sin(0.0172...x - 1.334...) + 6.342...$

May 21 is day 141 of this year. Next year, May 21 will be 365 + 141, or day 506.

If next year was going to be a leap year, I would have added 366 instead of 365 to the day number.



I determined the time when the Sun will set on May 21 of next year using my calculator.

$$0.107...9\ \cancel{h} \cdot \frac{60\text{ min}}{1\ \cancel{h}} = 6.4\text{ min}$$

8.107... h is equivalent to 8:06 p.m. standard time, to the nearest minute.

I converted the time from a decimal in hours to hours and minutes.

I added an hour to account for daylight-saving time.

The Sun will set at 8:06 p.m. standard time. Since British Columbia is on daylight-saving time in May, the Sun will set at 9:06 p.m.

We should arrive at the vantage point on the cliff at least 10 min before 9:06 to be sure that we do not miss the sunset.

This value is an estimate, since I extrapolated using a graph.

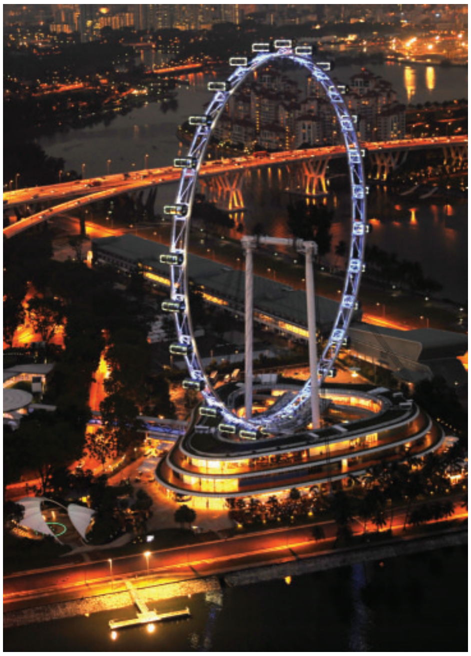
Your Turn

Suppose that Tara's class decides to finish its tour on June 25 instead of May 21. At what time should they arrive at the vantage point on the cliff?

EXAMPLE 3 Solving a problem using a sinusoidal model

In 2011, the Singapore Flyer was the largest Ferris wheel in the world. The table below gives the height of a rider from the ground at different times.

Time (min)	Height (ft)
0	49
9.25	295
18.50	541
27.75	295
37.00	49
46.25	295
55.50	541
64.75	295
74.00	49

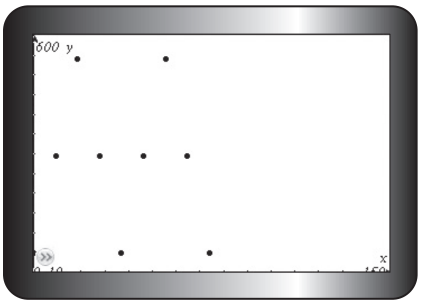


The Singapore Flyer, built on land reclaimed from the sea, first turned in 2008.

Jordy got on the Singapore Flyer at noon and rode it for four consecutive rotations. His friend Yale was in a building directly across from the Singapore Flyer, at a height of 400 ft. When was Jordy level with Yale?

Yale's Solution

I entered the given data into my calculator and created a scatter plot.



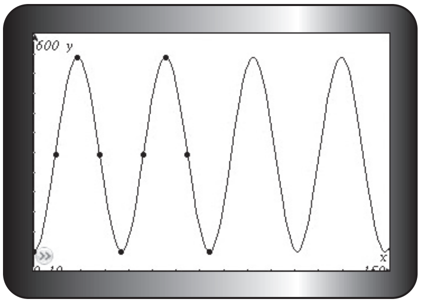
The minimum heights are at 0, 37, and 74 min. The period appears to be about 37 min.

I examined the points.

I determined the equation of the sinusoidal regression function:

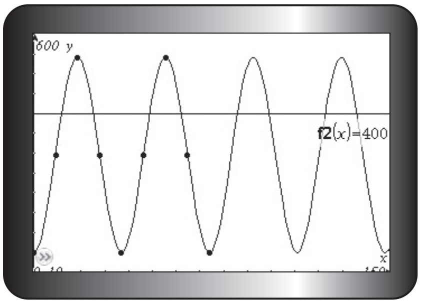
$y = 246 \sin (0.169... x - 1.570...) + 295$

I graphed the function.



Jordy rode the Ferris wheel for four rotations, so I adjusted the window to show just over four periods.

I added a graph of the horizontal line $y = 400$.



This line crosses the graph eight times.

The two graphs intersect at 11.85 min, 25.15 min, 48.85 min, 62.15 min, 85.85 min, 99.15 min, 122.85 min, and 136.15 min.

I determined all the intersection points of the line and the sine wave.

I verified the intersection points.

11.85 min + 37 min or 48.85 min
48.85 min + 37 min or 85.85 min
85.85 min + 37 min or 122.85 min

25.15 min + 37 min or 62.15 min
62.15 min + 37 min or 99.15 min
99.15 min + 37 min or 136.15 min

The intersection points are correct.

I verified my results. If they are correct, then adding one period to any time will result in another time when the Ferris wheel is in the same position.

Jordy was level with me at the following times after the Ferris wheel started:

11.85 min	85.85 min
25.15 min	99.15 min
48.85 min	122.85 min
62.15 min	136.15 min

Your Turn

At what time was Jordy at a height of 500 ft for the fifth time?

In Summary

Key Idea

- If the data points on a scatter plot seem to follow a regular periodic pattern of increasing and decreasing curves, then there may be a sinusoidal relationship between the independent and dependent variables.

Need to Know

- If the points on a scatter plot show a sinusoidal trend, then graphing technology can be used to determine the equation of the sinusoidal regression function that models the data. Use radian mode when performing sinusoidal regression.
- Interpolated or extrapolated values can be predicted by reading values from a graph or by using the equation of the sinusoidal regression function.
- When the data in a set repeats in a regular, periodic pattern, interpolation or extrapolation can be used to make predictions.

CHECK Your Understanding

1. A cuckoo clock is hanging on a wall. The table below gives the height of the pendulum above the ground as the clock ticks.

Time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
Height (in.)	65.0	66.5	68.0	66.5	65.0	66.5	68.0

a) Plot the data.

b) Determine the equation of a sinusoidal regression function that models the movement of the pendulum.

2. A water wheel, with a radius of 5 ft, is 1 ft above the ground. When water falls onto the wheel, the wheel completes one revolution every 2 s.

a) Complete the table for a point on the water wheel as it rotates.

Revolution	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2
Time (s)	0		1		2				4
Height (ft)	11		1		11				

b) Determine the equation of a sinusoidal regression function that models the data.

c) How high is the point at 1.15 s? State your answer in feet and inches, to the nearest inch.

d) How high is the point at 7.2 s? State your answer in feet and inches, to the nearest inch.

PRACTISING

3. While on her nightly walks with her dog, Kelly noticed that the Moon appeared to be at different angles on different days. She researched the maximum altitude of the Moon, in degrees, and obtained this data, starting on April 1.

Day	1	3	5	7	9	11	13	15	17	19
Angle (°)	41.2	50.6	58.4	63.0	63.1	57.9	48.2	36.1	30.1	20.5
Day	21	23	25	27	29	31	33	35	37	39
Angle (°)	16.7	19.1	25.9	34.9	44.5	53.4	60.3	63.5	61.7	54.9
Day	41	43	45	47	49	51	53	55	57	59
Angle (°)	44.5	32.7	24.5	19.0	16.9	20.8	28.6	38.0	47.5	56.0

a) Graph Kelly's data, and estimate the period.
b) Explain why a sinusoidal function can be used to model the data.
c) Determine the equation of a sinusoidal regression function that models the data.
d) What is the maximum angle of the Moon?
e) Estimate the altitude of the Moon, in degrees, on day 100. Round to the nearest tenth of a degree.
f) What is the first day in May that the altitude of the Moon will drop below 30°?

4. The Bay of Fundy, in the Maritimes, has the highest tides in the world. The height of the water, in metres above the seabed, is shown for one point over 36 h.

Hour	Height (m)	Hour	Height (m)	Hour	Height (m)	Hour	Height (m)
1	3.5	10	7.0	19	4.6	28	1.6
2	2.4	11	6.4	20	5.9	29	1.9
3	1.8	12	5.3	21	6.9	30	2.7
4	1.9	13	4.0	22	7.2	31	4.0
5	2.7	14	2.7	23	6.8	32	5.4
6	3.8	15	1.8	24	5.9	33	6.5
7	5.2	16	1.7	25	4.6	34	7.1
8	6.3	17	2.2	26	3.2	35	7.0
9	7.0	18	3.3	27	2.1	36	6.3

In 2011, the Bay of Fundy was under consideration as one of the Seven Natural Wonders of North America.

a) Explain why a sinusoidal function can be used to model the data.
b) Graph the data. Determine the equation of a sinusoidal regression function that models the height of the water. Does the regression equation match the data closely?

c) How high is the water at high tide, to the nearest tenth of a metre? How high is the water at low tide?
d) How long, to the nearest minute, does it take for the tide to cycle from high tide to low tide and back again?
e) Simon plans to go fishing at hour 50. How high, to the nearest tenth of a metre, will the tide be when he begins fishing?

5. A boat needs to be in water that is at least 4 m deep to move safely. For how many consecutive hours is it safe for this boat to sail in the Bay of Fundy? (Use the data in question 4.)

6. The data for monthly temperatures in Winnipeg is given in the table below.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Average High (°C)	−12.2	−5.2	0.3	10.7	17.6	21.8	25.4	25.3	19.1	10.8	−1.3	−7.5
Average Low (°C)	−21.3	−15.7	−10.0	−2.6	3.4	9.9	12.3	11.6	5.6	−0.4	−10.3	−17.2
Average (°C)	−16.9	−10.4	−4.9	4.1	10.5	15.9	18.9	18.5	12.4	5.2	−5.9	−12.4
Record Daily High (°C)	4.9	8.5	16.6	28.1	31.8	31.4	32.3	33.0	31.9	26.3	17.2	8.9

a) Determine the equation of a sinusoidal regression function that models the average high temperatures.
b) Determine the equation of a sinusoidal regression function that models the average low temperatures.
c) Determine the equation of a sinusoidal regression function that models the average temperatures.
d) Compare the three equations you determined in parts a) to c). How are they similar? How are they different?
e) Do you think that the record daily high temperatures form a sinusoidal pattern? Explain.
f) Estimate the average high temperature on October 1, to the nearest tenth of a degree Celsius. Assume that the average monthly temperature is the temperature on the 15th of the month.

7. Sally will be travelling to Yellowknife in June of next year. She collected the following data for average temperatures.

Average Temperatures Last Year			Average Temperatures This Year		
Month	Month Number	Average Temperature (°C)	Month	Month Number	Average Temperature (°C)
Jan.	1	–25.6	Jan.	13	–24.2
Feb.	2	–23.5	Feb.	14	–16.3
Mar.	3	–20.3	Mar.	15	–9.9
Apr.	4	–5.0	Apr.	16	0.8
May	5	1.4	May	17	5.6
Jun.	6	12.2	Jun.	18	14.8
Jul.	7	15.8	Jul.	19	17.9
Aug.	8	14.8	Aug.	20	15.5
Sep.	9	10.4	Sep.	21	7.8
Oct.	10	–1.6	Oct.	22	0.4
Nov.	11	–10.9	Nov.	23	–12.0
Dec.	12	–22.0	Dec.	24	–21.9

a) Determine the equation of a sinusoidal regression function that models the average temperatures.

b) Determine the maximum and minimum values of the sinusoidal regression function. What do these values show about the accuracy of the regression function?

c) Based on your sinusoidal regression function, what will be the average temperature in June of next year?

Yellowknife, situated on Great Slave Lake within the traditional territories of the Yellowknives Dene First Nation, was founded by gold prospectors in the 1930s. The town is also known as Somba k'e, "where the money is" in Dogrib.

8. The following table gives the hours of daylight in Regina, which is at a latitude of 50°. The hours of daylight are calculated as the time between sunrise and sunset each day.

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	8.5	10.1	11.8	13.7	16.4	17.1	15.6	14.6	12.7	10.8	9.1	8.1

a) Determine the equation of a sinusoidal regression function that models the data.

b) Determine the range between the maximum and minimum hours of daylight in Regina. Round to the nearest minute.

c) Which day has the most hours of daylight?

d) Determine the hours of daylight in Regina on day 30. Round to the nearest minute.

e) Determine which days of the year have about 15.0 h of daylight.

The Saskatchewan Legislative Building, completed in 1912, is set in a park on Lake Wascana in Regina.

9. This table gives the average monthly temperature and precipitation in Edmonton.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Average Monthly Temperature (°C)	–13.9	–5.4	–3.6	5.3	10.6	14.7	17.4	17.3	11.6	5.3	–4.0	–8.7
Average Monthly Precipitation (mm)	22	9	26	36	57	88	66	53	45	17	23	13

a) Predict what the average temperature will be next January, to the nearest tenth of a degree Celsius. Explain what you did.

b) Predict what the average precipitation will be next January, to the nearest millimetre. Explain what you did.

c) Which of your two predictions do you think is more reliable? Explain your answer.

10. Catherine swings a pendulum back and forth. Its minimum height from the floor is 40 cm, and its maximum height is 70 cm. The pendulum falls from the highest point to the lowest point in 0.5 s.

a) Create a table of values for height versus time, for $t = 0$ to $t = 3$ s.

b) Determine the equation of the sinusoidal regression function that models the height of the pendulum over time.

c) How high will the pendulum be after 7.25 s?

11. During a physics experiment, Carol held down a weight attached to a spring and then released the weight. The height of the weight above the table is shown.

Time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Height (in.)	5	11	17	11	5	11	17	11	5

a) Determine the equation of a sinusoidal regression function that models the data.

b) How high above the table will the spring be after 0.75 s? How high will the spring be after 5.3 s? Round your answers to the nearest tenth.

Closing

12. This table gives the average monthly temperature for Prince George, British Columbia.

Month	**Jan.**	**Feb.**	**Mar.**	**Apr.**	**May**	**Jun.**	**Jul.**	**Aug.**	**Sep.**	**Oct.**	**Nov.**	**Dec.**
Average Monthly Temperature (°C)	–8.0	–3.9	0.9	6.1	11.0	14.4	16.5	16.0	11.3	5.7	–1.6	–6.5

Jerry says, "To predict the average monthly temperature for Prince George next November, I just have to look at the table. The temperature will be -1.6 °C. I don't need to bother doing a regression." Do you agree with Jerry? Explain.

Extending

13. The average monthly temperatures, in degrees Celsius, for two communities are given.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
Hay River, NWT (°C)	−24.5	−21.2	−15.8	−3.5	5.9	12.3	15.8	14.4	8.2	0.9	−12.4	−20.9
Mar del Plata, Argentina (°C)	20.3	19.9	18.0	14.6	11.3	8.5	8.1	8.9	10.5	13.1	15.9	18.5

a) Determine the equation of a sinusoidal regression function that models the temperatures in each city. Does each function fit the data closely?

b) Compare the graphs of the two functions.

c) How does the location of each community affect its average monthly temperatures?

d) On about what dates would you expect the two communities to have the same temperature? Explain.

e) The average monthly temperatures, in degrees Celsius, for City X are given below. Is this city north or south of the equator? Provide your reasoning.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
City X (°C)	22.0	23.0	22.5	21.0	18.5	17.0	15.5	15.0	15.0	16.5	18.0	20.0

14. The word “month” comes from the word “Moon.” Therefore, you might think that the Moon goes through 12 cycles in a year, since there are 12 months in a year. However, the lunar year is actually made up of about 13 cycles as its sidereal period is 27.32 days, taking into consideration that the Moon is both rotating about its axis and revolving about the Earth simultaneously.

a) Why do you think that the sidereal period is shorter than the synodic period (the time from full moon to full moon)?

b) How does the graph of the sidereal period compare with the graph of the synodic period?