8.3 The Graphs of Sinusoidal Functions

GOAL

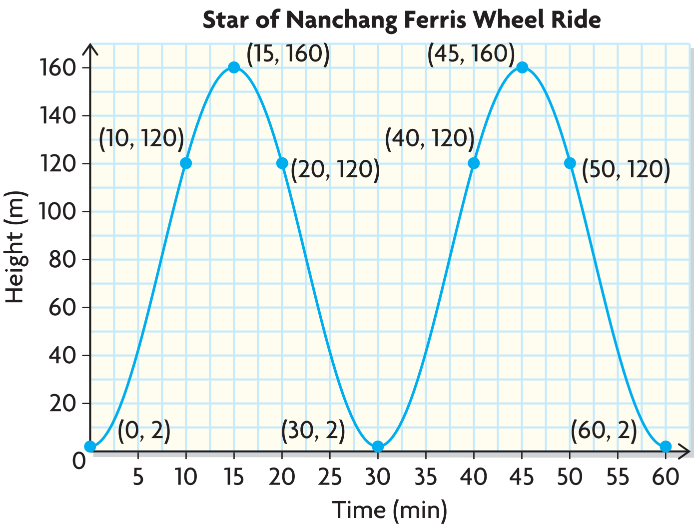
Identify characteristics of the graphs of sinusoidal functions.

YOU WILL NEED

- ruler
- graph paper
- graphing technology

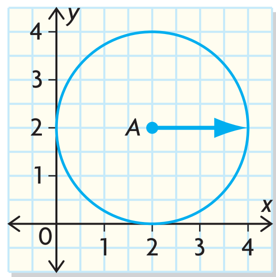
INVESTIGATE the Math

Students in Simone's graduating class went on an exchange trip to China. While they were there, they rode the Star of Nanchang, one of the tallest Ferris wheels in the world. Simone graphed the **sinusoidal function** that represented her ride.



EXPLORE...

- Sketch the graph of the vertical position of the tip of this pointer as a function of degrees of rotation as the pointer spins clockwise for two rotations. How does your graph compare with the graph of the spinner in Lesson 8.2?



sinusoidal function

Any periodic function whose graph has the same shape as that of $y = \sin x$.

? How can you describe Simone's ride using the graph?

A. How can you tell, from Simone's graph, that the lowest part of the Ferris wheel is 2 m off the ground?

B. Determine the maximum value of the graph. What is the height of the Ferris wheel?

C. Determine the range of the graph. What does this value represent?

D. Determine the amplitude of the graph. What does this value represent?

E. Determine the equation of the midline. What does this value represent?

F. Determine the period of the graph. Explain your method.

G. What length of time is needed for the Star of Nanchang to make one full revolution?

H. How long does it take to get to the top of the Ferris wheel from the bottom?

Reflecting

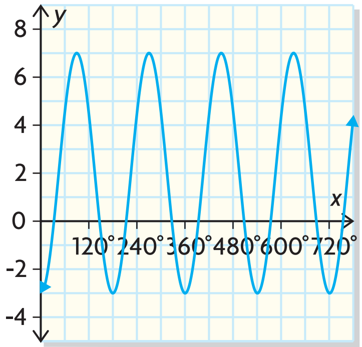
I. Suppose that a circle of lights were placed on the Star of Nanchang as shown by the red ring, 26 m from the circumference. Sketch Simone's graph on a grid. On the same grid, sketch the graph of the movement of one of the lights. How are the graphs the same? How are they different?

J. George Ferris built the first Ferris wheel in 1893. It was 80.4 m high, and it took 9 min to make one complete revolution. Assume that the minimum height of this Ferris wheel was 0 m. Graph the movement of one of its passenger cars. Compare your graph with Simone's graph. How are the graphs the same, and how are they different?

APPLY the Math

EXAMPLE 1 Describing the graph of a sinusoidal function in degree measure

The graph of a sinusoidal function is shown. Describe this graph by determining its range, the equation of its midline, its amplitude, and its period.



Terry's Solution

Range:
Minimum value $= -3$
Maximum value $= 7$

I located the minimum and maximum values of the graph to help me determine the range.

The range of the graph is
$\{y \mid -3 \le y \le 7, y \in R\}$.

I wrote the range using these values.

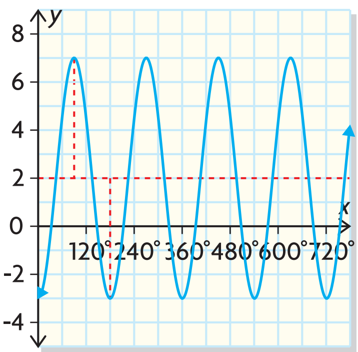
Equation of the midline:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$y = \frac{7 + (-3)}{2}$$

$$y = 2$$

I knew that the midline is the horizontal line halfway between the minimum value and the maximum value.



I verified my solution by looking at the graph. The graph goes 5 units above this horizontal line and 5 units below it.

Amplitude:
Amplitude $= 7 - 2$
Amplitude $= 5$
The amplitude is 5 units.

The amplitude is the vertical distance between the maximum value and the midline.

Period:
There is a minimum value at 180°.
The next minimum value is at 360°.
Period $= 360° - 180°$
Period $= 180°$
The graph goes through one complete cycle every 180°.

To determine the period, I chose two consecutive minimum points. The difference between the *x*-values of these points is the period.

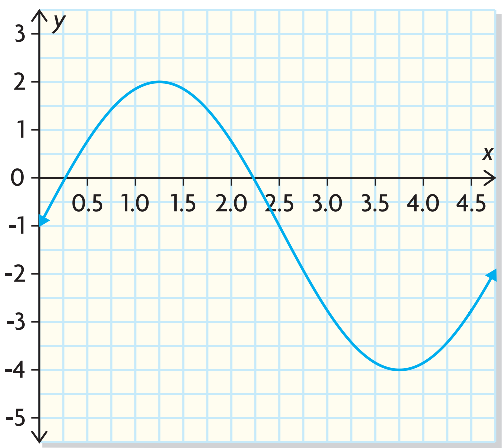
Your Turn

Sketch the graph of a sinusoidal function with the following characteristics:

- The domain is $\{x \mid 0° \le x \le 360°, x \in \mathrm{R}\}$.
- The range is $\{y \mid -5 \le y \le 9, y \in \mathrm{R}\}$.
- The period is 120°.
- The y-intercept is 2.

EXAMPLE 2 Describing the graph of a sinusoidal function in radian measure

The graph of a sinusoidal function is shown. Describe this graph by determining its range, the equation of its midline, its amplitude, and its period.



Bonnie's Solution

Range:
Minimum value $= -4$
Maximum value $= 2$

I located the minimum and maximum values of the graph.

The range of the graph is $\{y \mid -4 \le y \le 2, y \in \mathrm{R}\}$.

I wrote the range using these values.

Equation of the midline:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$y = \frac{2 + (-4)}{2}$$

$$y = -1$$

Amplitude:

Amplitude $= 2 - (-1)$

Amplitude $= 3$

The amplitude is 3 units.

Period:

The maximum value is at $x = 1.25$.

The minimum value is at $x = 3.75$.

Period $= 2(3.75 - 1.25)$

Period $= 2(2.5)$

Period $= 5$

The graph goes through one complete cycle every 5 radians.

The midline is the horizontal line halfway between the minimum and maximum values.

I verified my solution by looking at the graph. The graph goes 3 units above this line and 3 units below it.

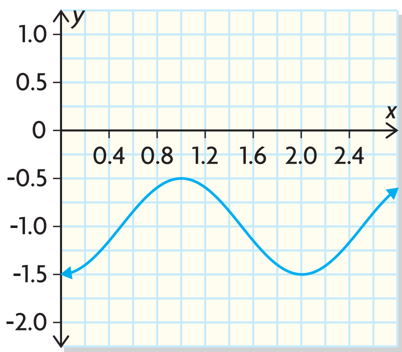
The amplitude is the vertical distance between the maximum value and the equation of the midline.

Since the graph does not show more than one period, I decided to determine the period by using the maximum point and the minimum point.

The horizontal distance between these two points represents half of a period, so the period is twice this distance.

Your Turn

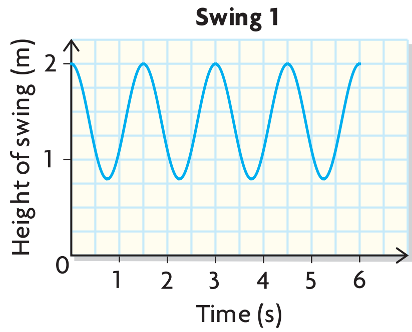
Determine the range, amplitude, period, and equation of the midline of this sinusoidal function.



EXAMPLE 3 Connecting a sinusoidal function to oscillating motion

For a physics project, Morgan and Lily had to graph and analyze an example of simple harmonic motion. Morgan swung on a swing, and Lily used a motion detector to measure Morgan's height above the ground over time, as she swung back and forth. The girls then graphed their data as shown. At the end of each cycle, the swing returned to its initial position, which resulted in a sinusoidal graph.

a) Interpret the graph.
b) Determine Morgan's height above the ground at 4 s.



Lily's Solution

a) Range:
Maximum value = 2.0
Minimum value = 0.8
The range of the height of the swing is
$\{y \mid 0.8 \leq y \leq 2.0, y \in \mathrm{R}\}$.

I knew the maximum and minimum values from the data I collected using the motion detector. These values match the maximum and minimum values on the graph.

Equation of the midline:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

$$y = \frac{2.0 + 0.8}{2}$$

$$y = 1.4$$

I determined the equation of the midline using the range.

I checked my answer by placing my ruler along the line $y = 1.4$ on the graph. Half of the graph is above this line, and half of the graph is below it.

Amplitude:

$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$\text{Amplitude} = \frac{2.0 - 0.8}{2}$$

$$\text{Amplitude} = 0.6$$

The amplitude is half of the vertical distance between the minimum value and the maximum value.

The graph begins at a maximum $(t = 0)$ and reaches the next maximum at $t = 1.5$, so the period is 1.5 s.
This is the length of time for one complete swing, either forward or backward.

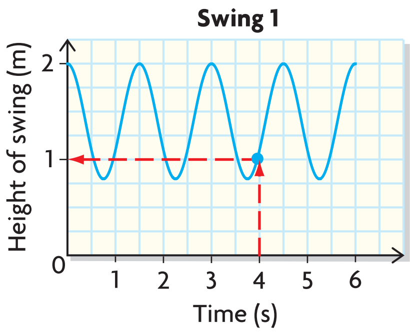
I identified two consecutive maximum points and determined the horizontal distance between them. This matches the time that I determined from the motion detector.

The graph of our simple harmonic model is a sinusoidal function.

- The period of one swing is 1.5 s.
- Morgan swung to a maximum height of 2.0 m.
- The swing is at its minimum height, 0.8 m, each time it passes its position at rest.
- The equation of the midline is $y = 1.4$.
- The amplitude of the swing is 0.6 m.

I summarized our findings for our report.

b)



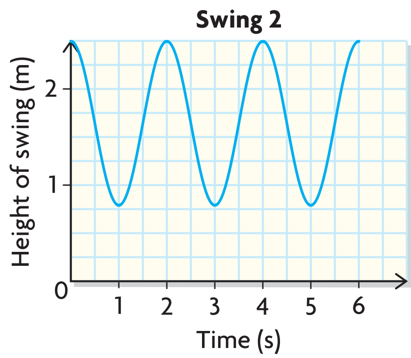
I examined the graph. I drew a vertical line from the horizontal axis, at $t = 4$ s, to a point on the graph. To determine the height, I drew a horizontal line from this point to the vertical axis.

Interpolating from the graph, at $t = 4$ s, Morgan's height is 1 m.

Your Turn

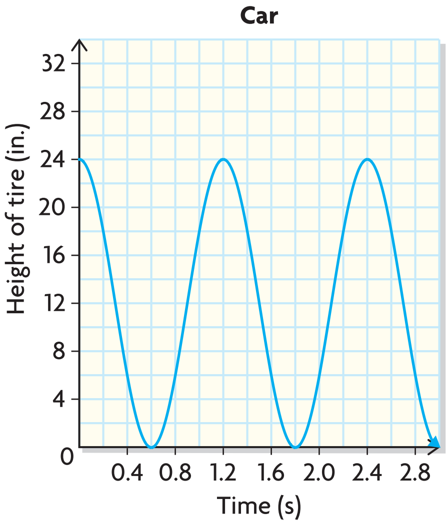
a) To collect a different data set, Lily swung on a different swing, and Morgan created the graph to the right from the data she collected. Interpret this graph.

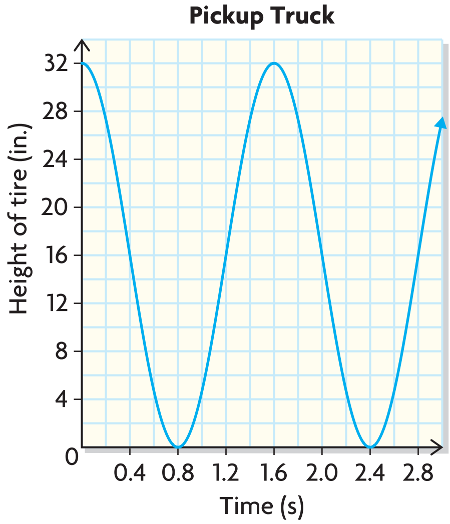
b) Compare the two swings.



EXAMPLE 4 Comparing two sinusoidal functions

Alexis and Colin own a car and a pickup truck. They noticed that the odometers of the two vehicles gave different values for the same distance. As part of their investigation into the cause, they put a chalk mark on the outer edge of a tire on each vehicle. The following graphs show the height of the tires as they rotated while the vehicles were driven at the same slow, constant speed. What can you determine about the characteristics of the tires from these graphs?





Alexis's Solution

The minimum value of both graphs is 0. This makes sense, because the tires move along the ground.

> I examined both graphs. The minimum value is on the horizontal axis.

Maximum values:
Maximum value for car = 24 in.
Maximum value for truck = 32 in.
These values represent the maximum height of the chalk mark, which is the diameter of each tire.
The diameter of the truck tire is 8 in. greater than the diameter of the car tire.

> I compared the maximum values and interpreted what they meant.

Midline of car graph:

$$y = \frac{0 + 24}{2}$$
$$y = 12$$

Midline of truck graph:

$$y = \frac{0 + 32}{2}$$
$$y = 16$$

> I determined the midline of each graph.

The centre of the car tire is 12 in. above the ground.
The centre of the truck tire is 16 in. above the ground.

This makes sense, because the axle would be at the centre of each tire, and the height of the centre of each tire above the ground should equal the radius of the tire.

Periods:

Period of car graph = 1.2 s

Period of truck graph = 1.6 s

Since the truck graph has a greater period, the truck tire takes longer to make one rotation.

The truck travels farther in one rotation of the wheels than the car.

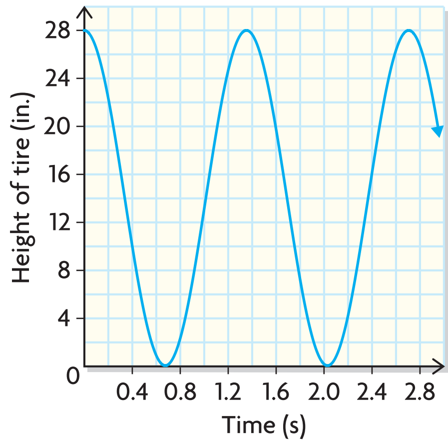
I determined the period of each graph.

Your Turn

Alexis installed tires with a larger diameter on the car. She obtained this graph as she tracked the vertical position of a chalk mark.

a) Compare this graph with the original graph for the car tire.

b) A speedometer and an odometer operate based on the number of revolutions that a wheel makes. Discuss, as a class, how larger tires might affect the speedometer and the odometer.



In Summary

Key Ideas

- Sinusoidal functions can be used as models to solve problems that involve repeating or periodic behaviour.
- Functions whose graphs have the same shape and characteristics as the sine function are called sinusoidal functions.

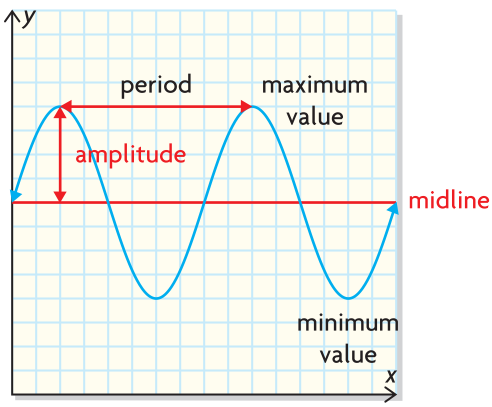
Need to Know

- You can determine the characteristics of a sinusoidal function from its graph:
  - The **period** is the horizontal distance between consecutive maximum values or consecutive minimum values. It is also twice the horizontal distance between a maximum value and the next minimum value.
  - The **equation of the midline** is the average of the maximum and minimum values:

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

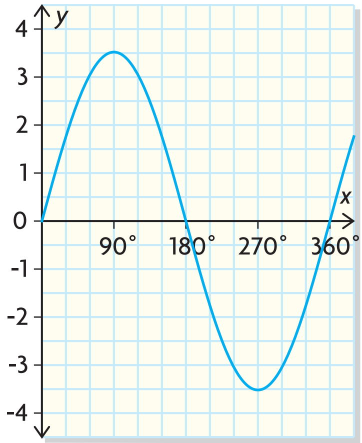
  - The **amplitude** is the positive vertical distance between the midline and either a maximum or minimum value. It is also half of the vertical distance between a maximum value and a minimum value.

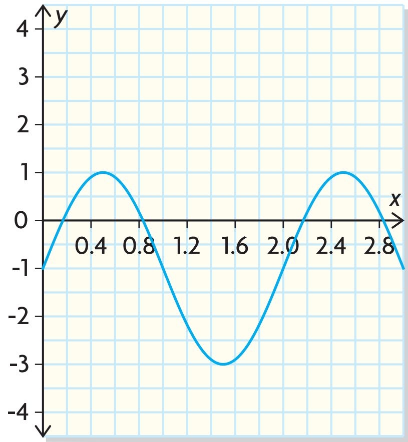
$$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$



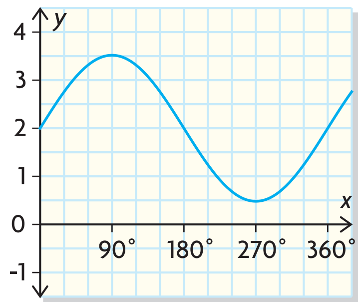
CHECK Your Understanding

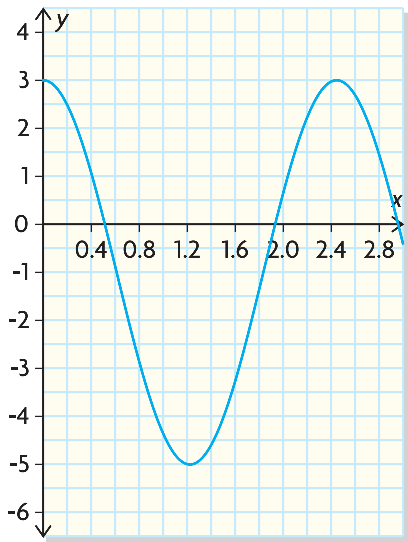
1. Determine the range and amplitude of each graph.

a) 

b) 

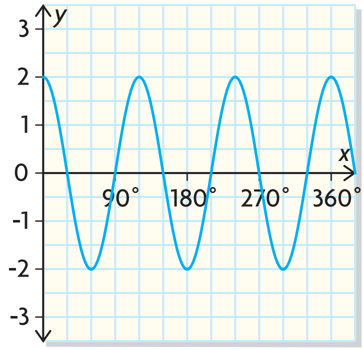
2. Determine the equation of the midline and the amplitude of each graph.

a) 

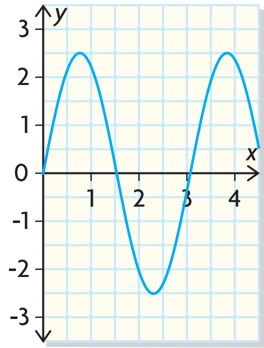
b) 

3. Determine the period of each graph.

a)



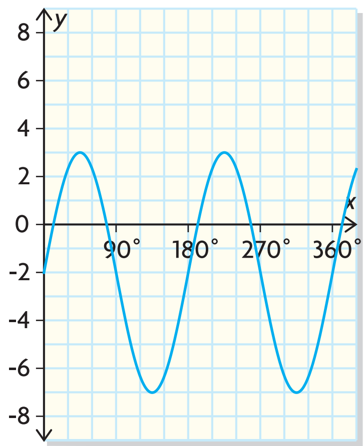
b)



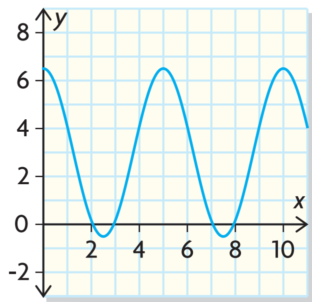
PRACTISING

4. Determine the range, amplitude, equation of the midline, and period of each graph.

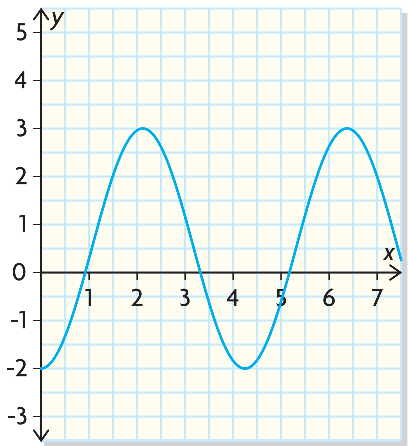
a)



b)



5. Determine the range, amplitude, equation of the midline, and period of the graph below.



6. Sketch a possible graph of a sinusoidal function with each set of characteristics.

a) Domain: $\{x \mid 0° \le x \le 180°, x \in R\}$
Range: $\{y \mid 2 \le y \le 6, y \in R\}$
Period: 90°
y-intercept: 4

b) Domain: $\{x \mid 0 \le x \le 16, x \in R\}$
Maximum value: 3
Minimum value: -3
Period: 8
y-intercept: -3

7. In 2011, the London Eye was the largest Ferris wheel in the western hemisphere. It rises 135 m above the ground and takes the same amount of time to make one rotation as the Star of Nanchang. How would the graph of a passenger's ride on the London Eye be the same as Simone's graph? How would it be different?

The London Eye was completed in 1999.

The West Edmonton Mall, which opened in 1981, was the largest mall in North America in 2011.

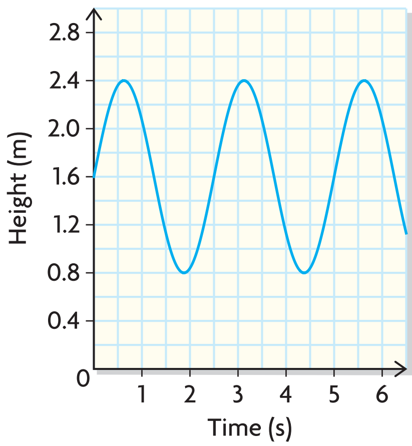
8. Mussab is sitting in an inner tube in the wave pool at West Edmonton Mall. The depth of the water below him, in terms of time, during a series of waves can be represented by the graph shown.

a) What is the depth of the water below Mussab when no waves are being generated?

b) How high is each wave?

c) How long does it take for one complete wave to pass?

d) What is the approximate depth of the water below Mussab after 4 s? What is the depth of the water below Mussab at 7.5 s? Assume that the waves continue at the same rate.

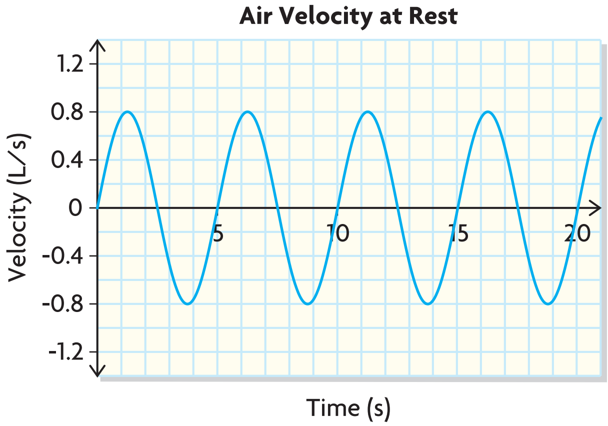


9. When you breathe, the air entering your lungs has a positive velocity and the air exiting your lungs has a negative velocity. The relationship between velocity, in litres of air per second (L/s), and time, in seconds, for an adult at rest, can be modelled by the graph shown.

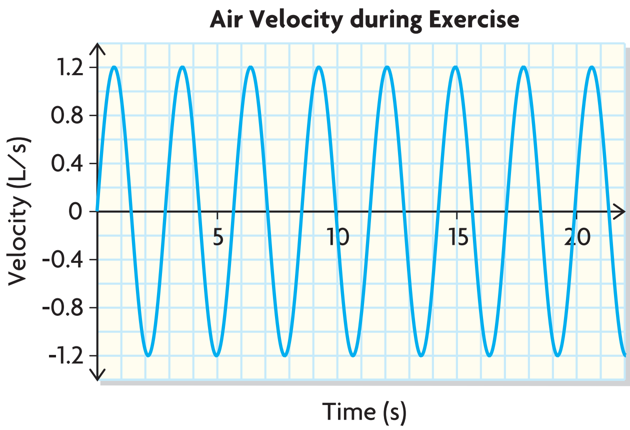
a) What is the equation of the midline? What does it represent in this situation?

b) What is the amplitude of the function?

c) What is the period of the function? What does it represent in this situation?



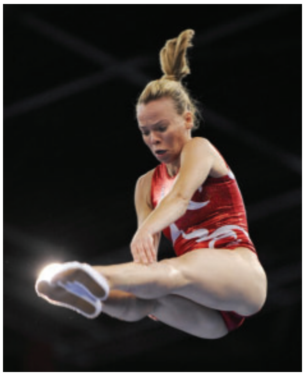
10. When you exercise, the velocity of the air entering and exiting your lungs, measured in litres per second, changes in terms of time, measured in seconds. The following graph models the relationship between velocity and time for an adult who is exercising.



a) According to this model, does an adult take more breaths per minute when exercising, or just deeper breaths, than an adult at rest (modelled in question 9)? How do you know?

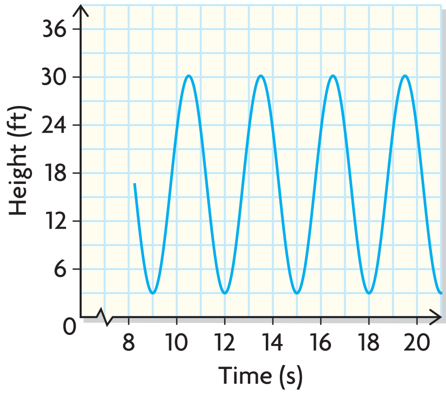
b) What characteristic (period, equation of midline, or amplitude) of this graph has changed, compared with the graph in question 9?

c) What is the maximum velocity of the air entering the lungs? Include the appropriate units of measure.

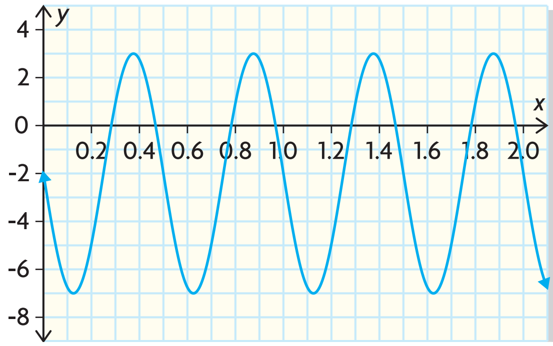


Karen Cockburn is the only trampoline athlete to have won a medal at three consecutive Olympic Games: 2000, 2004, and 2008.

11. A competitive gymnast's coach graphs one particular series of jumps. Describe the gymnast's jumps using the graph.



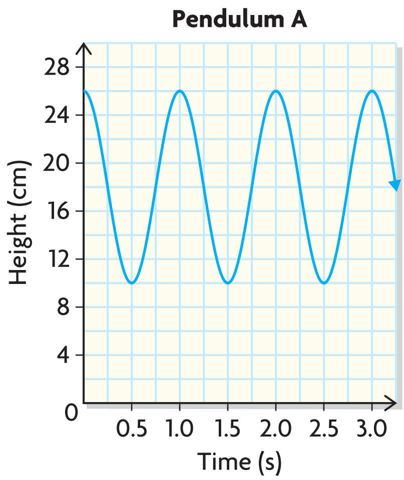
12. Determine the characteristics of this graph.

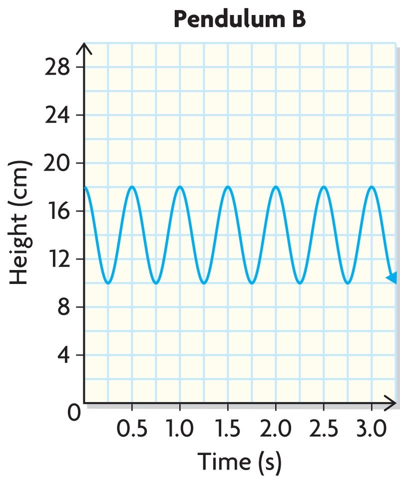


13. Caitlin and Rahim conducted an experiment in physics class. They swung two different pendulums above a table and recorded the motion of the pendulums in graphs.

a) Compare the periods, minimum values, maximum values, and amplitudes of the two pendulums.

b) Which pendulum is longer? Explain.





14. Takoda and his sister Talula are hoop dancers. In part of a dance, they spin hoops about their arms. Each of the following graphs indicates the height of a point on a hoop, measured from the ground, that Takoda or Talula is spinning vertically.

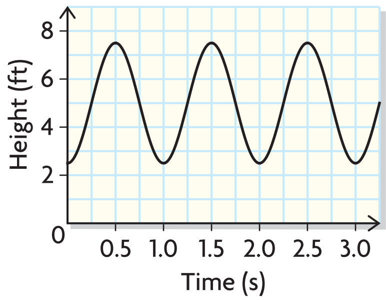
a) What does the amplitude represent?

b) Which hoop is the smallest?

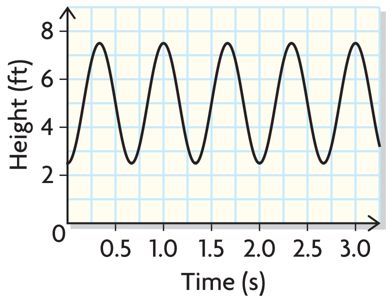
c) Which hoop is being spun at the slowest rate? Which is being spun at the fastest rate?

d) Which hoop do you think the shorter person is spinning? Explain.

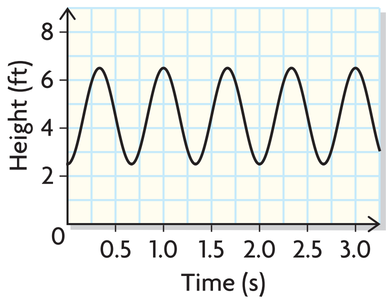
Hoop 1



Hoop 2



Hoop 3



Hoop dancers form shapes using hoops to tell stories in their dances. The hoop dance, or some form of it, has existed among First Nations across North America for hundreds of years. The modern form of hoop dancing came into being in the 1930s. Hoop dance competitions are often part of Pow Wows. Modern hoop dances contain spectacular moves and shapes, and are performed by both men and women.

Closing

15. How can you determine the amplitude, range, and equation of the midline of a given sinusoidal graph?

Extending

16. In high winds, the top of a flagpole sways back and forth. The distance that the tip of the flagpole vibrates away from its resting position can be modelled by the function

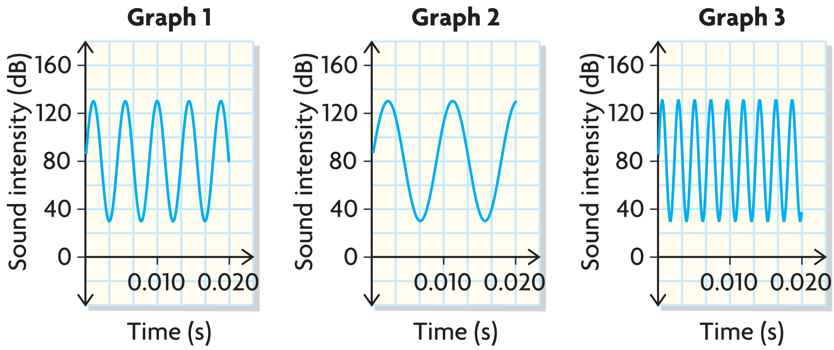
$$d(t) = 2 \sin (4\pi t)$$

where $d(t)$ represents the distance in centimetres and t represents the time in seconds. If the wind speed decreases by 20 km/h, the distance can be modelled by the following function:

$$d(t) = 1.5 \sin (4\pi t)$$

Plot the two graphs using technology in radian mode. How are the period, midline, and amplitude affected when the wind speed decreases?

17. Music is composed of sound waves, which can be modelled using sinusoidal functions. For example, these graphs show three consecutive A pitches, but not in increasing order, for $\{x \mid 0 \leq x \leq 0.02 \text{ s}, x \in \mathrm{R}\}$. A higher **frequency** is equivalent to a higher pitch.



a) Arrange the graphs in order, from the lowest pitch to the highest pitch.
b) Compare the graphs. What do you notice about the frequencies of consecutive A pitches?
c) Graph 1 has a frequency of 220 Hz, or 220 cycles per second. What is the frequency of each of the other two graphs?
d) Research the sound waves that are produced by musical instruments. For example, you could consider the following questions:
- Are all sound waves sinusoidal?
- What impact does the size of an instrument have on its sound waves?
- What other elements have an impact on its sound waves? Give some examples.

frequency
The number of times that a cycle occurs in a given time period. For example, the fourth A note on a piano has a frequency of 440 Hz or 440 cycles per second.

18. Electrical outlets provide power using an alternating current (AC) that alternates according to a sinusoidal function. The typical voltage in Canada oscillates between −170 V (volts) and +170 V, with a frequency of 60 Hz (hertz), or 60 times per second.
a) How many milliseconds does it take for one complete cycle of electricity?
b) Draw a graph that represents the alternating current for three complete cycles. Label all the axes.

Communication | *Tip*
A millisecond is 0.001 s.