7.4 Characteristics of Logarithmic Functions with Base 10 and Base *e*

YOU WILL NEED

- graphing technology

EXPLORE...

- Use benchmarks to estimate the solution to this equation:

$$120 = 10^y$$

logarithmic function

A function of the form

$$y = a \log_b x$$

where $b > 0$, $b \neq 1$, and $a \neq 0$, and a and b are real numbers.

GOAL

Investigate the characteristics of logarithmic functions with base 10 and base e from graphs and equations.

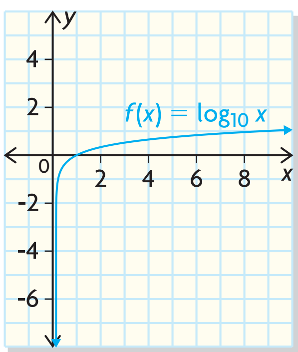
INVESTIGATE the Math

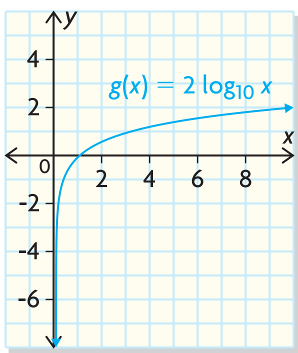
Randy came across a new type of function in his chemistry class. He learned that the pH of a solution is determined by a **logarithmic function**. He wondered how the characteristics of these functions differ from the characteristics of polynomial and exponential functions. Randy noticed that he had a log button on his calculator, so he looked in his user manual and discovered that this was a base 10 logarithm. Using this button, he created a table of values and a graph for each of the functions shown below to investigate the similarities and differences.

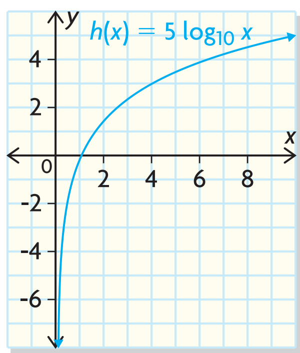
x	$f(x) = \log_{10} x$
−1	undefined
0	undefined
1	0
2	0.301...
3	0.477...
4	0.602...
5	0.698...
6	0.778...
7	0.845...
8	0.903...
9	0.954...
10	1

x	$g(x) = 2 \log_{10} x$
−1	undefined
0	undefined
1	0
2	0.602...
3	0.954...
4	1.204...
5	1.397...
6	1.556...
7	1.690...
8	1.806...
9	1.908...
10	2

x	$h(x) = 5 \log_{10} x$
−1	undefined
0	undefined
1	0
2	1.505...
3	2.385...
4	3.010...
5	3.494...
6	3.890...
7	4.225...
8	4.515...
9	4.771...
10	5







What are the characteristics of logarithmic functions of the form $y = a \log_b x$, where $b = 10$ or $b = e$, and a is a real number?

A. Use a graphing calculator to graph the logarithmic function $y = \log_{10} x$. On the same axes, graph $y = 10^x$. How are these two graphs related?

B. Randy choose three functions of the form $y = a \log_{10} x$, where $a > 0$. On a new screen, graph Randy's functions, shown on the previous page.

C. Examine the graph of each function, and state the following characteristics:
- the number of x-intercepts
- the y-intercept
- the end behaviour
- the domain
- the range

D. Examine the graphs you created in part B. As you move from left to right along the x-axis, do the y-values of each function increase or decrease? Then examine the table of values for each function on your graphing calculator. What happens to the y-values as the x-values increase?

E. Choose three new functions of the form $y = a \log_{10} x$, where $a < 0$. Repeat parts C and D for these new functions.

F. On a new screen, graph the natural logarithmic function $y = \ln x$. On the same axes, graph $y = e^x$. How are these two graphs related?

G. On a new screen, graph the function $y = \ln x$ and two other functions of the form $y = a \ln x$, where $a > 0$. Examine the graph of each function, and state the following characteristics:
- the number of x-intercepts
- the y-intercept
- the end behaviour
- the domain
- the range

H. Examine the graphs you created in part G. As you move from left to right along the x-axis, do the y-values of each function increase or decrease? Then examine the table of values for each function on your graphing calculator. What happens to the y-values as the x-values increase?

I. Choose three new functions of the form $y = a \ln x$, where $a < 0$. Repeat parts G and H for these new functions.

Communication *Tip*

The expression $\log_{10} x$ is known as the common logarithm or a logarithm with a base of 10. The expression is often written without the 10, so the two functions $y = \log_{10} x$ and $y = \log x$ are equivalent.

Communication *Tip*

The function $y = \log_{10} x$ is equivalent to $x = 10^y$, so a logarithm is an exponent. The meaning of $\log_{10} x$ is "the exponent that must be applied to base 10 to get the value of x." For example, $\log_{10} 100 = 2$.

Communication *Tip*

A logarithm with base e is called the natural logarithm and is written as $\ln x$. The functions $y = \log_e x$, $y = \ln x$, and $x = e^y$ are equivalent.

Reflecting

J. Based on your observations, which characteristics of the graphs of logarithmic functions are similar to the characteristics of graphs of exponential functions you have studied?

K. Based on your observations, which characteristics of the graphs of logarithmic functions differ from the characteristics of graphs of exponential functions you have studied?

L. Can you predict the end behaviour of functions of the form $y = a \log x$ or $y = a \ln x$ based on the parameter a? Explain.

M. In logarithmic functions of the form $y = a \log_b x$, did changing the base from $b = 10$ to $b = e$ cause any of the characteristics to change? Explain.

APPLY the Math

EXAMPLE 1 Connecting the characteristics of an increasing logarithmic function to its equation and graph

Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of the following function:

$$y = 15 \log x$$

Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

Sid's Solution

The function $y = 15 \log x$ is a logarithmic function of the form $y = a \log_b x$, with $a = 15$ and $b = 10$.

> I knew that this is a common logarithmic function since the base is not written. Its base is 10.
>
> I determined the values of the parameters a and b.

Common Characteristics

x-intercept: 1
Number of y-intercepts: 0
Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
Range: $\{y \mid y \in \mathrm{R}\}$

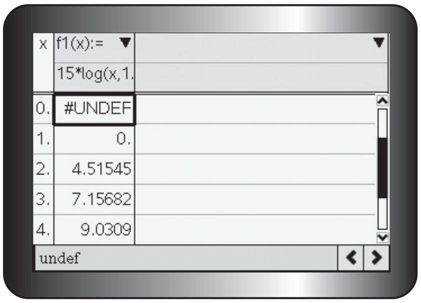
> I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for all logarithmic functions of the form
>
> $y = a \log_b x$, where $b = 10$.

Unique Characteristics

End behaviour: The curve extends from quadrant IV to quadrant 1.

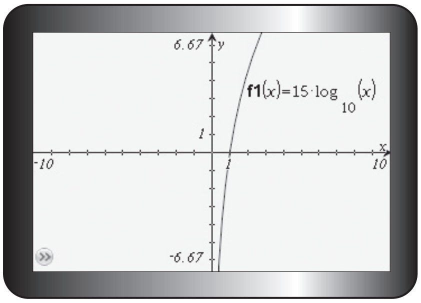
Since the parameter a is greater than 0, I knew that the function increases as you move from left to right along the x-axis.

I used a graphing calculator to create a table of values and a graph to verify my predictions.



x	f1(x):= 15*log(x,1.
0.	#UNDEF
1.	0.
2.	4.51545
3.	7.15682
4.	9.0309

undef



My predictions were correct.

Your Turn

Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of this function:

$$y = -5 \log x$$

Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

EXAMPLE 2 Connecting the characteristics of a decreasing natural logarithmic function to its equation and graph

Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of the following function:

$$y = -4 \ln x$$

Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

Anne's Solution

The function $y = -4 \ln x$ is a logarithmic function of the form $y = a \log_b x$ with a base of e. In this function, $a = -4$ and $b = e$.

I knew that $y = \ln x$ is the natural logarithmic function, which is equivalent to $y = \log_e x$, so I knew the base of the logarithmic function is 2.718.... I determined the values of the parameters a and b.

Common Characteristics

x-intercept: 1

Number of y-intercepts: 0

Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$

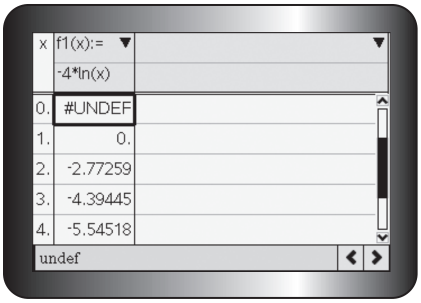
Range: $\{y \mid y \in \mathrm{R}\}$

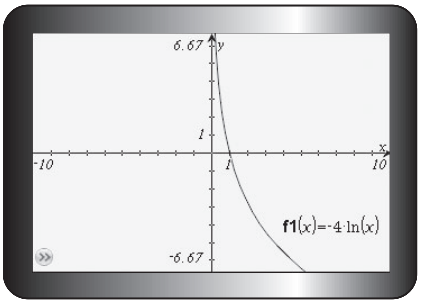
I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for all logarithmic functions of the form $y = a \log_b x$, where $b = e$.

Unique Characteristics

End behaviour: The curve extends from quadrant I to quadrant IV.

Since the paameter a is less than 0, I knew that the function decreases as you move from left to right along the x-axis.





I used a graphing calculator to create a table of values and a graph.

My predictions were correct.

Your Turn

Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of this function:

$$y = 12 \ln x$$

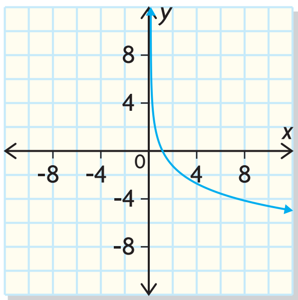
Use the equation of the function to make your predictions. Verify your predictions using graphing technology.

EXAMPLE 3 Matching equations of exponential and logarithmic functions with their graphs

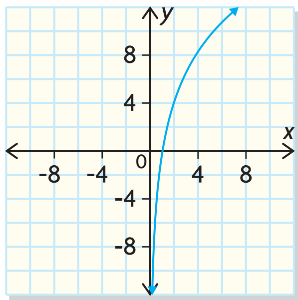
Which function matches each graph below? Provide your reasoning.

i) $y = 5(2)^x$ **ii)** $y = 2(0.1)^x$ **iii)** $y = 6 \log x$ **iv)** $y = -2 \ln x$

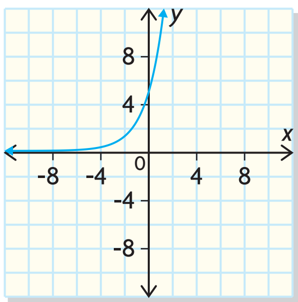
a)



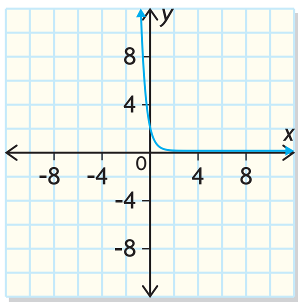
c)



b)



d)

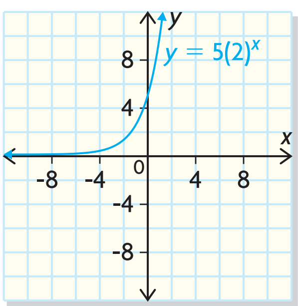


Erryn's Solution

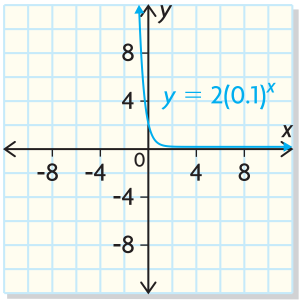
Exponential functions:

I know that graphs b) and d) are exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, since both extend from quadrant II to quadrant I.

I also know that functions i) and ii) are exponential, since the exponent in each equation is a variable.



Since $b > 0$ in the function $y = 5(2)^x$, I knew that this function matches the graph of the increasing function.



Since $b < 0$ in the function $y = 2(0.1)^x$, I knew that this function matches the graph of the decreasing function.

I knew that I was right since the y-intercepts for the equations match the y-intercepts on the graphs.

$y = 5(2)^x$

y-intercept: 5

$y = 2(0.1)^x$

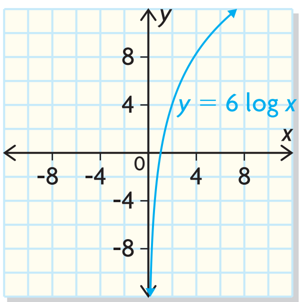
y-intercept: 2

Function i) matches graph b) and function ii) matches graph d).

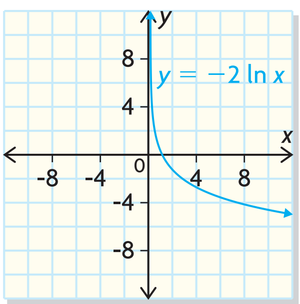
Logarithmic functions:

I know that graphs a) and c) are logarithmic, since they extend from quadrant I to quadrant IV and from quadrant IV to quadrant I, respectively.

I also know that functions iii) and iv) are logarithmic.



Since $a > 0$ in the function $y = 6 \log x$, I knew that this function matches the graph of the increasing function.



Since $a < 0$ in the function $y = -2 \ln x$, I knew that this function matches the graph of the decreasing function.

Function iii) matches graph c) and function iv) matches graph a).

Your Turn

Jordan claims that she can tell which graphs are exponential and which graphs are logarithmic by examining the domain of each function. Do you agree or disagree? Explain.

In Summary

Key Ideas

- A logarithmic function has the form $f(x) = a \log_b x$, where $b > 0$, $b \neq 1$, and $a \neq 0$, and a and b are real numbers.
- All logarithmic functions of the form $f(x) = a \log x$ and $f(x) = a \ln x$ have these characteristics:

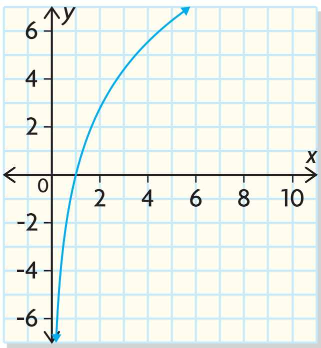
x-Intercept	1
Number of y-Intercepts	0
End Behaviour	The curve extends from quadrant IV to quadrant I or quadrant I to quadrant IV.
Domain	$\{x \mid x > 0, x \in \mathbb{R}\}$
Range	$\{y \mid y \in \mathbb{R}\}$

- All logarithmic functions of the form $f(x) = a \log x$ and $f(x) = a \ln x$ have these unique characteristics:
  - If $a > 0$, the function increases.
  - If $a < 0$, the function decreases.

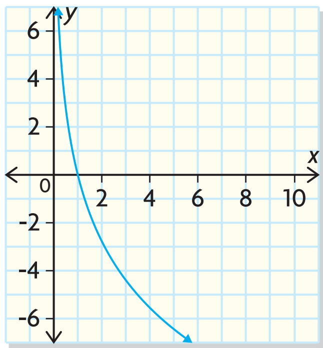
Need to Know

- The graph of a logarithmic function of the form $f(x) = a \log x$ or $f(x) = a \ln x$ will look like one of the following cases:

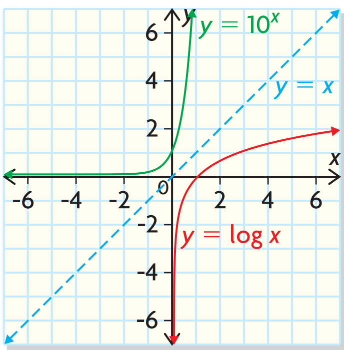
Case 1: an increasing function, where $a > 0$



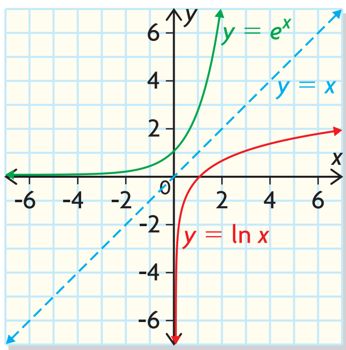
Case 2: a decreasing function, where $a < 0$

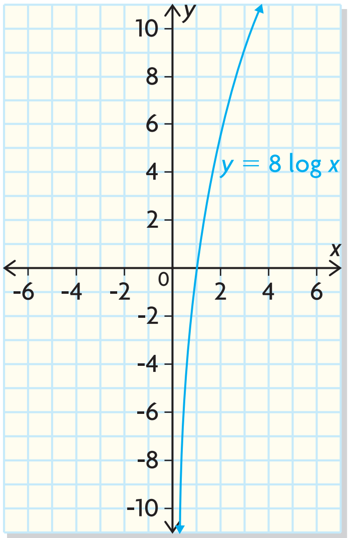


- The graph of $y = \log x$ is a reflection of the graph of $y = 10^x$ about the line $y = x$.



- The graph of $y = \ln x$ is a reflection of the graph of $y = e^x$ about the line $y = x$.



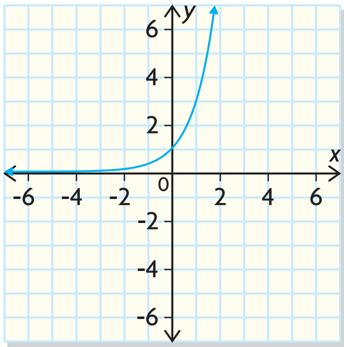


CHECK *Your Understanding*

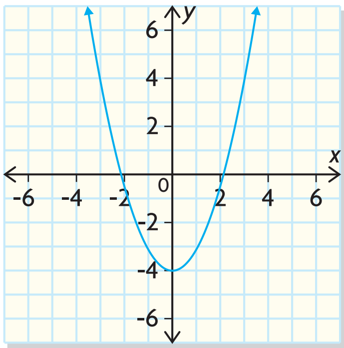
1. State the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range of the function shown on the left.

2. Identify which of these graphs are logarithmic functions. Explain how you know.

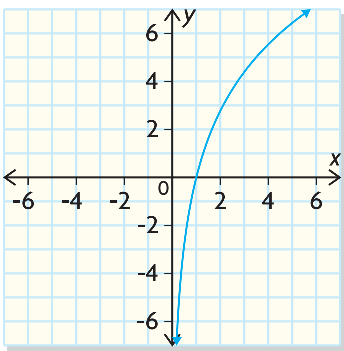
a)



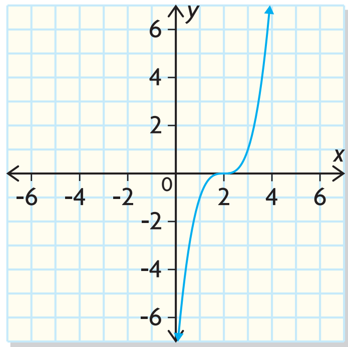
b)



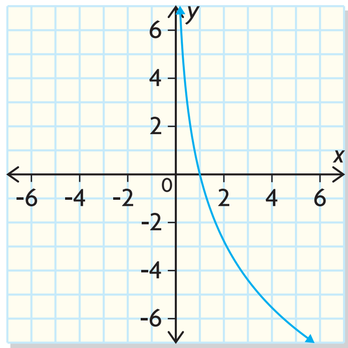
c)



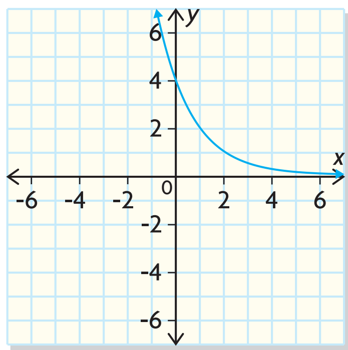
d)



e)



f)



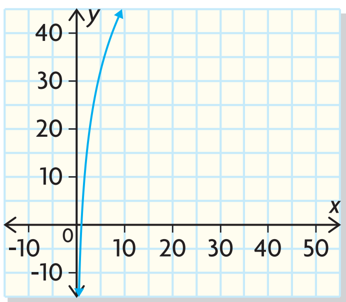
3. For the functions in question 2 that are logarithmic, state the x-intercept, the number of y-intercepts, the end behaviour, the domain, and the range. Also state whether the parameter a in the equation of the function is greater than zero or less than zero, and explain how you know.

PRACTISING

4. Use graphing technology to graph the functions $y = \log x$ and $y = -\log x$. Compare the graphs of the functions. Consider the x-intercept, the number of y-intercepts, the end behaviour, the domain, the range, and whether they are increasing or decreasing.

5. Predict the x-intercept, the number of y-intercepts, the end behaviour, the domain, the range, and whether each logarithmic function is increasing or decreasing, based on its equation. Verify your predictions using graphing technology.
   a) $f(x) = 3 \log x$
   b) $f(x) = -9 \log x$
   c) $f(x) = 12 \ln x$
   d) $f(x) = 0.5 \ln x$
   e) $f(x) = -0.2 \log x$
   f) $f(x) = 100 \ln x$

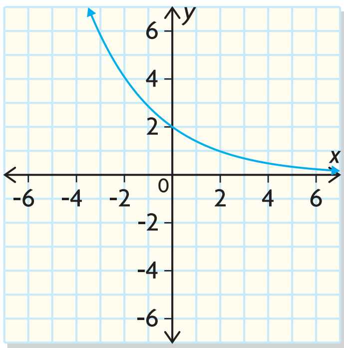
6. State three reasons why you know that the following graph represents a logarithmic function of the form $y = a \log x$.



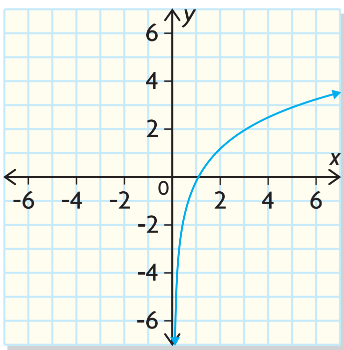
7. Sketch the graph of the logarithmic function with each set of characteristics.
   a) x-intercept: 1
   Number of y-intercepts: 0
   End behaviour: The curve extends from quadrant IV to quadrant I.
   Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
   Range: $\{y \mid y \in \mathrm{R}\}$
   b) x-intercept: 1
   Number of y-intercepts: 0
   End behaviour: The curve extends from quadrant I to quadrant IV.
   Domain: $\{x \mid x > 0, x \in \mathrm{R}\}$
   Range: $\{y \mid y \in \mathrm{R}\}$

8. Match each function with its corresponding graph. Provide your reasoning.
   i) $y = 4.2 \log x$
   ii) $y = -3 \log x$
   iii) $y = 6^x$
   iv) $y = 2(0.7)^x$

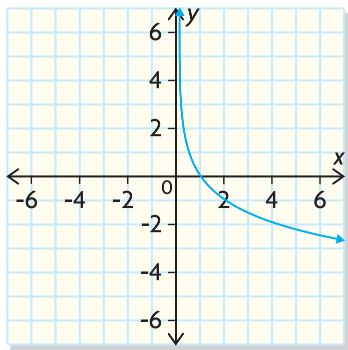
a)



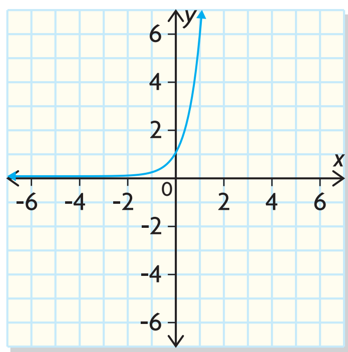
b)



c)



d)



9. Holly claims that she can distinguish between the graph of an exponential function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, and the graph of a logarithmic function of the form $y = a \log x$, where $a \neq 0$, based entirely on the number of x-intercepts shown. Do you agree or disagree? Explain.

10. The pH of a solution, $p(x)$, can be determined using the function

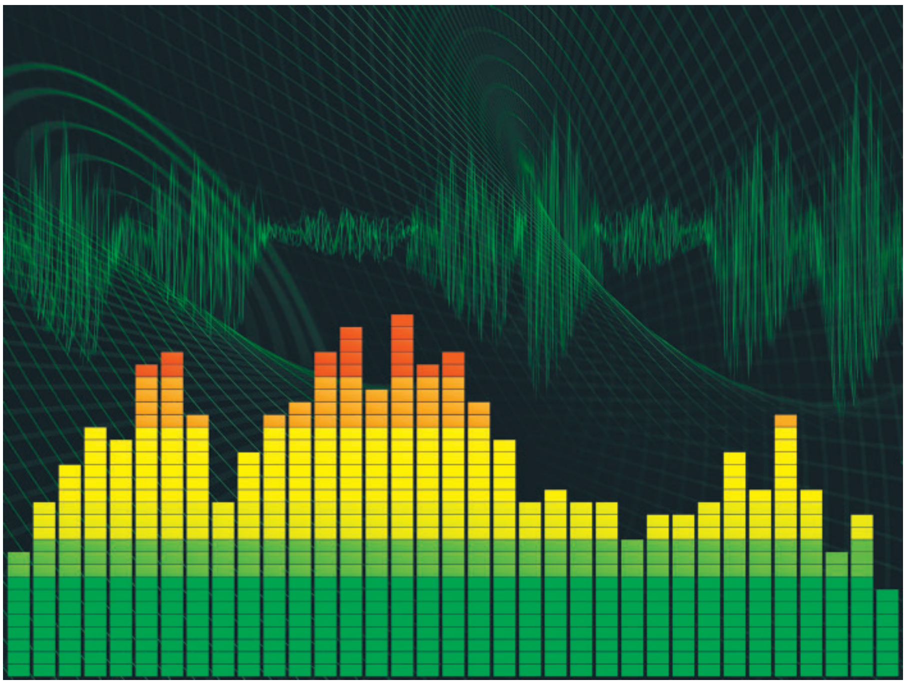
$$p(x) = -\log x$$

where x represents the concentration of hydrogen ions in the solution, in moles per litre. Use the equation of the function to predict what happens to the pH of a solution as the concentration of hydrogen ions increases. Verify your prediction using graphing technology.

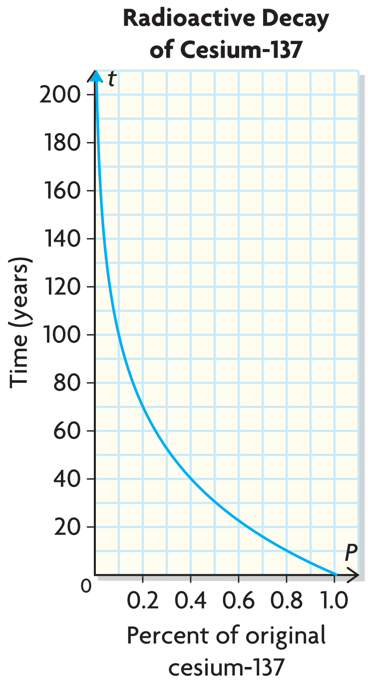
11. Sound intensity is measured on the decibel scale. A decibel is $\frac{1}{10}$ of a bel, named after Alexander Graham Bell. The intensity of a sound (number of decibels) can be determined using the function

$$D(x) = 10 \log (10^{16} \cdot x)$$

where x represents the energy of the sound in watts per square centimetre (watts/cm^2). Use the equation of the function to predict what happens to the number of decibels, $D(x)$, as the energy of the sound, x, increases. Verify your prediction using graphing technology.



A recording of speech. Like other sounds, speech can be measured by pitch (top graph) and by intensity (bottom graph).



12. Cesium-137 is created in nuclear reactors. It decays naturally over time. The approximate time for cesium-137 to decay is given by the function

$$t = -100.422 \log P$$

where t represents the time in years and P represents the percent of the original amount of cesium-137, as a decimal, that remains. The function is graphed at the left.

a) State the following characteristics of the graph:
- the intercepts
- the domain and range within the context of this problem
- whether the function is increasing or decreasing
- the value of the function at $P = 1$

b) A sample of cesium-137 is taken from a reactor. Estimate the time when 5% of the sample will remain.

c) Estimate the time when 50% of the sample will remain.

Closing

13. Consider the following functions:

$$y = a \log x \quad \text{and} \quad y = a \ln x$$

a) For what values of a is each function decreasing?

b) Do these functions have an unrestricted domain? Explain.

c) Do these functions have an unrestricted range? Explain.

Extending

14. The largest known earthquake in Saskatchewan occurred in 1909. Based on recorded observations, the earthquake is believed to have measured approximately 5.5 on the Richter scale. The magnitude, M, of an earthquake that is T times more intense than an earthquake registering 5.5 on the Richter scale is given by the following function:

$$M = \log T + 5.5$$

For every 1 unit increase on the Richter scale, the intensity of an earthquake increases by a factor of 10.

a) Use technology to graph the function, and state the following characteristics of the graph:
- the intercepts
- the domain and range within the context of this problem
- whether the function is increasing or decreasing
- the value of the function at $T = 1$

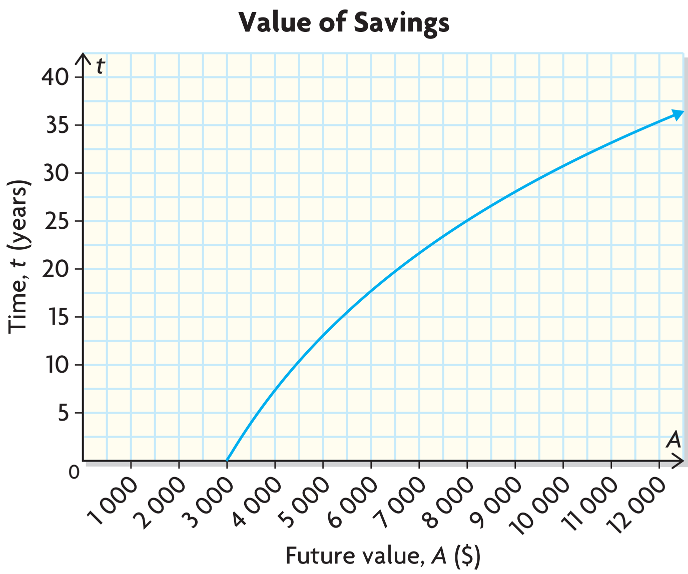
b) Determine the magnitude of an earthquake that is 50 times more intense than the Saskatchewan earthquake.

c) In 2011, an earthquake that measured 9.0 on the Richter scale occurred off the coast of Japan, creating a devastating tsunami. About how many times more intense was this earthquake than the Saskatchewan earthquake? Use your graph to estimate.

15. Christie has \$3000 invested in a savings account that earns 4%/a. The approximate time for her investment to reach a given future value is modelled by the function

$$t = 58.7 \log A - 204.1$$

where t is the time in years and A is the future value. The function is graphed below.



a) State the following characteristics of this function:
- the intercepts
- the domain and range within the context of this problem
- whether the function is increasing or decreasing

b) How long, in years, will it take for Christie's investment to reach \$10 000?

c) How long, in years, will it take for Christie's investment to double?