7.3 Modelling Data Using Exponential Functions

YOU WILL NEED

- large jar or plastic container
- pennies
- graphing technology

EXPLORE...

- A biologist was monitoring the growth in the number of fruit flies in a controlled environment.

The fruit fly population was recorded over a five-day period, as shown.

Day	Number of Flies
0	70
1	76
2	84
3	92
4	101
5	111

How can you estimate when the fruit fly population will exceed 160?

GOAL

Represent data using an exponential function, and interpret the graph to solve a problem.

INVESTIGATE the Math

Sabrina has been saving pennies in a jar. She decided to use these pennies to test an idea. She asked several of her friends to follow these steps:

- Take about 100 pennies from the jar, and place the pennies on a table.
- Put all the coins that have landed heads back in the jar.
- Pick up the remaining coins (the tails), shake them, and then place them on the table.
- Again, put all the coins that have landed heads back in the jar.
- Continue until only one coin or no coins are left.

Sabrina claimed that if all her friends graphed their results, all the graphs would look the same.

? What exponential function best models this situation?

A. Work in a group. Take a handful of pennies, and record the number in your handful. Predict how many pennies will land tails on the first toss. Record your data and predictions in a table like the one below.

Number of Tosses	Number of Tails	Prediction for Tails on Next Toss
0	initial sample size:	
1		
2		
3		

B. Shake the pennies, and then place them on a table. Remove all the pennies that landed heads, and count the remaining tails. Record the number of tails, and predict the number of coins that will land tails on the next toss. Repeat this process until your group has one penny left or runs out of coins.

C. Calculate the ratios for consecutive pairs of numbers in the second column of your table. What do you notice? Explain.

D. Should an increasing exponential function or a decreasing exponential function be used to model the data you collected? Explain.

E. Create a scatter plot for "Number of Tails" versus "Number of Tosses."

F. Perform an exponential regression on your data to determine the curve of best fit and its equation.

Reflecting

G. Sabrina claimed that if all her friends graphed their results, all the graphs would look the same. Do you agree with Sabrina? How is your graph the same as the other groups' graphs? How is it different?

H. How do the parameters in the exponential equation $y = a(b)^x$ relate to the parameters of the equation of your regression function?

I. What are the domain and range of your function in this context? Explain.

APPLY the Math

EXAMPLE 1 Creating graphical and algebraic models of given data

The population of Canada from 1871 to 1971 is shown in the table below. In the third column, the values have been rounded.

Year	Actual Population of Canada	Population of Canada (millions)
1871	2 436 297	2.44
1881	3 229 633	3.23
1891	3 737 257	3.74
1901	5 418 663	5.42
1911	7 221 662	7.22
1921	8 800 249	8.80
1931	10 376 379	10.38
1941	11 506 655	11.51
1951	14 009 429	14.01
1961	18 238 247	18.24
1971	21 568 305	21.57

Statistics Canada

Top: Main St., Winnipeg, in 1880. Bottom: Main St., Winnipeg, in 2010.

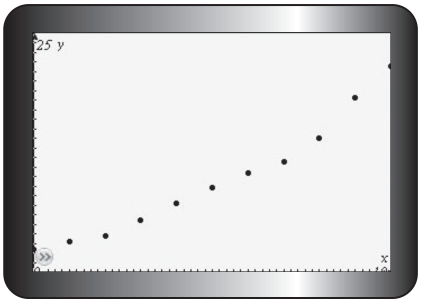
a) Using graphing technology, create a graphical model and an algebraic exponential model for the data.

b) Assuming that the population growth continued at the same rate to 2011, estimate the population in 2011. Round your answer to the nearest million.

Luba's Solution

a) Let x represent the number of 10-year intervals since 1871. Let y represent the population of Canada in millions.

I defined variables x and y.



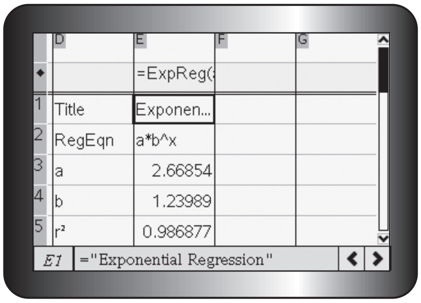
I entered the numbers 0 to 10, to represent the years 1871 to 1971, into list one of my graphing calculator. I entered the data from the third column of the data table into list two of my graphing calculator. Then I graphed the points in a scatter plot.

I noticed that the y-values grow as x gets larger, but not at a constant rate. I predicted that the data could be modelled by an **exponential growth function.**

exponential growth function

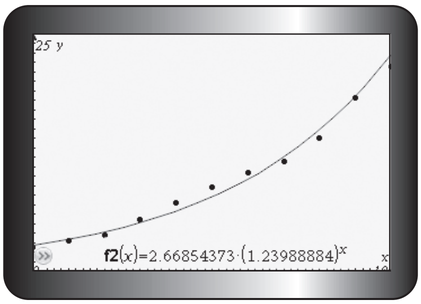
An exponential function whose y-values increase as you move from left to right along the x-axis; for an exponential function of the form $y = a(b)^x$, exponential growth occurs when $a > 0$ and $b > 1$.

I used my calculator to do an exponential regression on the data, to determine the equation of the curve of best fit.



The values of a and b agree with my prediction that the model is an exponential growth function, since $a > 0$ and $b > 1$.

I verified the equation by graphing it on the same grid as the given points.



I noticed that even though some points lie above or below the graph, the exponential regression function appears to be a good model for the data.

$y = a(b)^x$

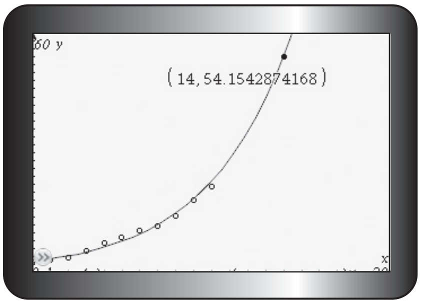
The exponential equation that models the data is

$y = 2.668...(1.239...)^x$

I used the regression result to determine the values of a and b, $a = 2.668...$ and $b = 1.239....$

b) $x = \frac{2011 - 1871}{10}$ or 14

To determine the population in 2011, I first needed to determine the number of 10-year intervals from 1871 to 2011.



I extrapolated the y-value from the graph at $x = 14$.

$y \doteq 54.15$

Verify:

$y = 2.668...(1.239...)^x$

$y = 2.668...(1.239...)^{14}$

$y = 54.154...$

Assuming that the population growth continued at the same rate, the population of Canada would be about 54 million in 2011.

I verified my extrapolated value by substituting $x = 14$ into the regression equation. My result is the same as the value extrapolated from the graph.

Your Turn

a) Determine an exponential model for the data in the original problem, if x represents the actual year and y represents the actual population.

b) How does your model differ from Luba's model?

c) Does your model predict the same population of Canada in 2011? Explain.

d) Which model do you prefer? Explain why.

EXAMPLE 2 Solving a problem using an exponential regression model

Sonja did an experiment to determine the cooling curve of water. She placed the same volume of hot water in three identical cups. Then she recorded the temperature of the water in each cup as it cooled over time. Her data for three trials is given as follows.

Trial 1

Time (min)	Temperature (°C)
0	80
5	69
10	61
20	45
30	34
40	26

Trial 2

Time (min)	Temperature (°C)
0	75
5	66
10	59
20	44
30	32
40	23

Trial 3

Time (min)	Temperature (°C)
0	78
5	68
10	61
20	44
30	33
40	25

a) Construct a scatter plot to display the data. Determine the equation of the exponential regression function that models Sonja's data.

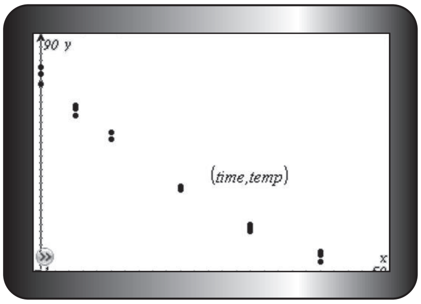
b) Estimate the temperature of the water 15 min after the experiment began. Round your answer to the nearest degree.

c) Estimate when the water reached a temperature of 30 °C. Round your answer to the nearest minute.

Clare's Solution

a) Let x represent the time, in minutes, since the experiment began.

Let y represent the temperature in degrees Celsius.



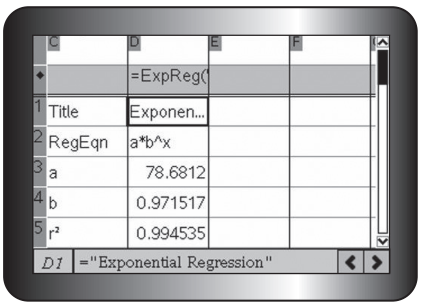
I defined variables for the data. Then I entered the data for all three trials into my graphing calculator and created a scatter plot.

Since the points are decreasing, I knew that the data can be modelled by an **exponential decay function.**

exponential decay function

An exponential function whose y-values decrease as you move from left to right along the x-axis; for an exponential function of the form $y = a(b)^x$, exponential decay occurs when $a > 0$ and $0 < b < 1$.

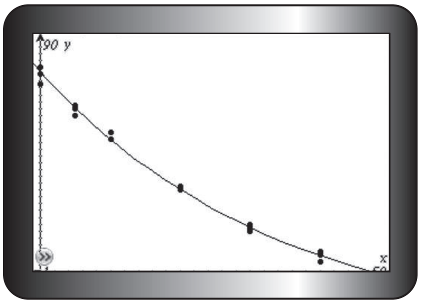
I used my calculator to do an exponential regression on the data, to determine the equation of the curve of best fit.



The values of a and b agree with my prediction that the model is an exponential decay function, because $a > 0$ and $0 < b < 1$.

The initial temperatures of the three samples were not the same: 80 °C, 75 °C, and 78 °C. The regression model defines $a = 78.68$, which is close to all three initial values.

I verified the regression function by graphing it on the same grid as the given data.

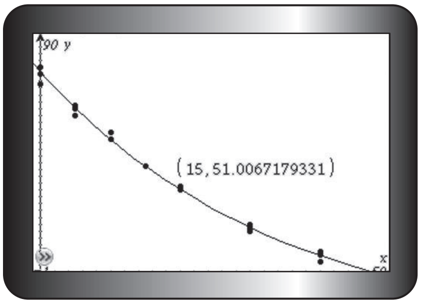


Even though some data points lie above or below the graph, the regression function appears to be a good model for the data.

The equation of the exponential function that models the given data is

$$y = 78.681...(0.971...)^x$$

b)



To determine the temperature of the water at 15 min, I interpolated the y-value from the graph at $x = 15$.

About 15 min after the experiment started, the temperature of the water should be about 51 °C.

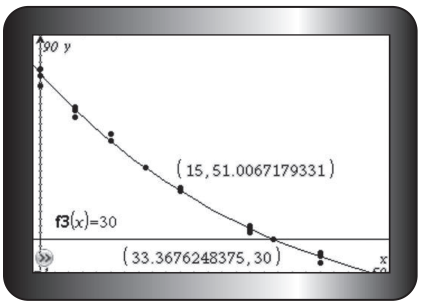
c) Substitute $y = 30$ into the exponential regression model.

$$y = 78.681...(0.971...)^x$$
$$30 = 78.681...(0.971...)^x$$

To determine the time when the water reached a temperature of 30 °C, I had to solve the exponential equation. I realized that I could not solve the equation algebraically, so I decided to use graphing technology.

$$y_1 = 78.681...(0.971...)^x$$
$$y_2 = 30$$

I knew that I could use the system of equations and point of intersection strategy I have used before with polynomial functions to solve this equation. This enables me to determine the value of the independent variable, x, when I know the value of the dependent variable, y, in any type of equation.



I already had the graph of the curve of best fit, so I entered $y = 30$ into my calculator to create the linear exponential system of equations that corresponded to the equation $30 = 78.681...(0.971...)^x$.

The point of intersection is (33.367..., 30).

$x = 33.367...$

The x-coordinate of this point is the solution to the equation $30 = 78.681...(0.971...)^x$.

After a little over 33 min from the start of the experiment, the water should reach a temperature of 30 °C.

Your Turn

Emma did the same experiment, but performed only one trial. Her data is given below.

Time (min)	0	5	10	20	30	45
Temperature (°C)	90	78	68	52	38	26

a) Construct a scatter plot to display Emma's data. Determine the equation of the exponential regression function that models her data.
b) Compare the characteristics of the exponential regression function for Emma's data with the characteristics of the function for Sonja's data. How are the graphs the same? How are they different?
c) At what time, to the nearest minute, did the water reach 51 °C in Emma's experiment? How much longer did it take than in Sonja's experiment?

In Summary

Key Idea

- Graphing technology can be used to create a scatter plot. When the scatter plot looks like an exponential function, exponential regression can be used to determine an algebraic model for the data.
- An exponential function models growth when $a > 0$ and $b > 1$. The y-values of an exponential growth function increase as you move from left to right along the x-axis.
- An exponential function models decay when $a > 0$ and $0 < b < 1$. The y-values of an exponential decay function decrease as you move from left to right along the x-axis.

Need to Know

- The exponential regression model is
$$f(x) = a(b)^x$$
where a represents the initial value, and b represents the growth factor if $b > 1$ or the decay factor if $0 < b < 1$.
- An exponential curve of best fit can be used to predict values that are not recorded or plotted. Predictions can be made by reading values from the curve of best fit on a scatter plot or by using the equation of the exponential regression function.

CHECK Your Understanding

1. Determine which of the following data sets involve exponential growth or decay. Explain how you know. Then graph each data set that involves exponential growth or decay to verify your answer, determine the equation of the exponential regression function, and graph the function on the same grid as the data set.

a)

x	y
0	0.0
1	5.1
2	10.2
3	15.3
4	20.4
5	25.5

b)

x	y
0	1.00
1	2.50
2	6.25
3	15.62
4	39.06
5	97.66

c)

x	y
−3	−2.25
−2	−1.00
−1	−0.25
0	0.00
1	−0.25
2	−1.00

d)

x	y
2	32
3	108
4	256
5	500
6	864
7	1372

2. Use exponential regression to determine the equation of the curve of best fit for each data set. Then state the characteristics of each function, including the domain and range.

a)

x	y
−3	1255.05
−2	248.50
−1	49.58
0	10.30
1	2.10
2	0.40
3	0.08

b)

x	y
1	2.93
2	3.09
3	3.26
4	3.44
5	3.63
6	3.83
7	4.02

PRACTISING

3. Four years ago, the Belands retired and decided to move into an apartment. Their annual rent for the four years is given below.

Years since Retirement	0	1	2	3
Rent ($)	9600	9960	10 344	10 752

a) Determine if the data can be represented by an exponential model. How do you know?

b) Explain how you can use the annual rate of increase in the rent to define an equation that models the data.

c) Assuming the same annual rate of increase, predict what the Belands will pay in 10 years. Round your answer to the nearest dollar.

4. Samuel and his grandfather recorded the length and mass of every rainbow trout they caught in the Red Deer River during the summer. Samuel plans to use the data for an upcoming science fair project.

Length (mm)	270	392	360	335	259	324	317	351	309	414	388	440	266	338
Mass (g)	210	622	480	474	203	357	338	504	392	750	661	843	225	459

a) Construct a scatter plot to display the data.
b) Determine the equation of the exponential regression function for the data. Graph the function on the same grid that you plotted the points.
c) Samuel caught a fish that measured 400 mm. Use your graph to estimate the mass of the fish, to the nearest gram. Describe your process.
d) Samuel's grandfather would like to catch a fish with a mass of 1000 g. How long, to the nearest millimetre, would you expect this fish to be? Describe your process.

5. The population of the world has been growing exponentially for about 150 years. The data table shows the global population in billions over a 40-year period.

Year	1970	1980	1990	2000	2010
Population (billions)	3.91	4.52	5.22	6.03	6.97

a) Construct a scatter plot to display the data.
b) Use exponential regression to define a function that models the data.
c) Assuming the same growth rate, estimate the population in the year 2020, to two decimal places. Describe your process.
d) If the growth rate remains the same, when would you expect the population to reach 9.50 billion? Describe your process.

6. The height of a sunflower was recorded every seven days as it grew.

Day	7	14	21	28	35	42
Height (cm)	17.9	36.4	67.8	98.0	131.1	169.7

a) Use exponential regression to model the growth of the sunflower.
b) Estimate the height of the sunflower, to the nearest tenth of a centimetre, on day 10.
c) Estimate the height of the sunflower, to the nearest tenth of a centimetre, on day 30. Does your answer make sense, based on the data? Explain.
d) On what day would you expect the sunflower to reach a height of 50 cm?

7. Computer chips use binary codes to operate. These codes are made up of binary numbers, which are combinations of zeros and ones. A binary number is a number written with a base of 2.

a) The following table shows the binary codes that can be made with one to four digits. Determine the equation of the exponential function that models the growth in the number of possible codes. Describe your process.

Number of Digits	Possible Codes	Number of Possible Codes
1	0, 1	2
2	00, 01, 10, 11	4
3	000, 001, 010, 011 100, 101, 110, 111	8
4	0000, 0001, 0010, 0011 0100, 0101, 0110, 0111 1111, 1110, 1101, 1100, 1011, 1010, 1001, 1000	16

b) State the domain and range of your function.
c) How many digits are needed to have a code for each of the 104 keys on a standard QWERTY keyboard?

8. Air pressure around a plane decreases as the plane's altitude increases. The data at the right shows the pressure, in millibars, that is exerted by the atmosphere on an object at different altitudes, in kilometres.

Altitude (km)	Air Pressure (mb)
1	898.7
2	795.0
3	701.2
4	616.0
5	540.5
6	472.2
7	411.1
8	356.5
9	307.0
10	264.9

a) Create a scatter plot to display the data.
b) Determine the equation of the exponential regression function that models the data. Graph the function on the same grid as your scatter plot.
c) Estimate the pressure at 15 km, to the nearest tenth of a millibar.
d) Estimate the altitude, to the nearest kilometre, where the pressure reaches 500.0 mb and 50.0 mb.

9. A biologist has been studying the effects of environmental stressors on a population of frogs. The observations are shown in the table.

Time (years)	0	1	2	3	4	5
Frog Population	575	460	368	294	235	188

a) By what percent is the frog population declining every year?
b) Determine the equation of the exponential regression function that models the decline in the frog population.
c) The biologist determines that the population is halved in about 3 years. Use interpolation to estimate when half of the population remains, and then use your regression equation to verify your estimate algebraically.

10. The Mauna Loa observatory in Hawaii has been collecting monthly data on the proportion of carbon dioxide (CO_2) in the atmosphere, in parts per million, since 1959. The data that was collected at the end of the year, every five years from 1960 to 2005, is shown in the table below.

Year	Atmospheric CO_2 (ppm)
1960	319
1965	322
1970	328
1975	333
1980	341
1985	348
1990	356
1995	364
2000	372
2005	381

a) Could a linear function model the data? Justify your answer.
b) Determine the equation of the exponential function that models the data.
c) Verify your model by doing an exponential regression.
d) According to your model, what is the predicted proportion of atmospheric CO_2 in 2010 and 2020, rounded to the nearest part per million?

11. Sebastian prepares hot sandwiches at his father's restaurant. He leaves the sandwiches on the counter to cool slightly before they are served. To prevent customers from burning their mouths, he conducts a test to determine how long he should leave the sandwiches to cool. He measures the internal temperature of a hot sandwich as it is cooling. His data is shown below.

Time (min)	0	1	2	3	4	5
Temperature (°C)	84.0	79.8	75.8	72.7	68.7	65.5

a) Determine the equation of the exponential regression function that models this situation.
b) How long, to the nearest minute, will it take for the sandwich to have an internal temperature that is half of its initial temperature?
c) How long, to the nearest minute, will the sandwich need to sit before its internal temperature is the same as the room temperature, 20 °C?

12. Tristan drank a cup of tea that contained 40 mg of caffeine. The amount of caffeine in his body was measured repeatedly after drinking the tea.

Time (h)	0	1	3	4	6
Amount of Caffeine (mg)	40	31	17	13	7

a) Using graphing technology, create a graphical exponential model and an algebraic exponential model for the data.
b) When will Tristan have less than 3 mg of caffeine in his body? Round your answer to the nearest hour.
c) Determine when half of the caffeine remained in Tristan's body. Round your answer to the nearest unit.

13. A sample of bacteria was taken from a patient. A lab technician then recorded the growth of the bacterial population at fixed time intervals. The first five observations are given below.

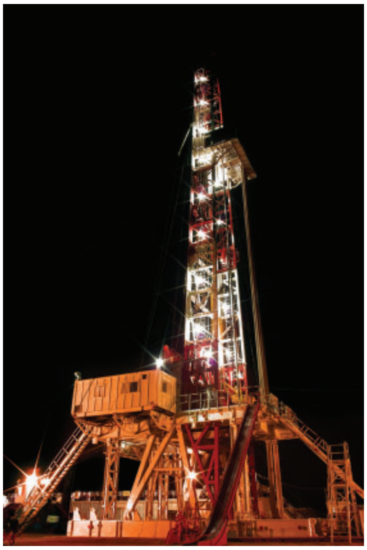
Observation	1	2	3	4	5
Number of Bacteria	6180	9270	13 905	20 857	31 285

a) Determine the equation of the exponential regression function that models the data.
b) If this growth rate continues, how many bacteria will there be when the lab technician makes the 10th observation?
c) A drug has been developed that affects the growth rate of the bacterial population. When studying a second identical sample that has been treated with the drug, the lab technician notices that the population now grows at a rate of 20% per hour. What function models the growth of the population in the second sample?

14. Researchers have been studying algae growth on small lakes and ponds in southern Manitoba. They measured the percentage of the surface of a pond that was covered by algae at fixed time intervals, as shown below.

Week	0	1	2	3	4	5
Pond Surface Covered by Algae (%)	25.0	27.5	30.3	33.3	36.6	40.3

a) By what percent are the algae growing every week?
b) Assuming the same growth rate, how many weeks will it take for the algae to cover 80% of the pond?



15. Graham, the lead geologist for an oil and gas company, has been analyzing data taken from a proposed drill site in southern Saskatchewan. He has determined that a well at this site should produce 200 barrels of oil per week over the next 10 years. At that point, however, the production of the well will begin to drop, according to the exponential regression function

$$P(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{48}}$$

where $P(t)$ represents the weekly amount of oil produced, in barrels, and t represents the number of weeks after production begins to drop.

a) Determine the expected production of this well, to the nearest barrel per week,

i) 26 weeks after production begins to drop

ii) 100 weeks after production begins to drop

b) The well will require mechanical assistance to bring up the oil once production drops to 30 barrels per week. During which week will production reach this level?

c) In his report, Graham has informed the company that the well will no longer be profitable once production drops to 5 barrels per week. During which week should the company cap the well?

16. The output of an oil well in Saskatchewan is initially 3000 barrels per week. Each week that the well operates, its output decreases by 10%.

a) Determine the equation of the exponential function that models the output of the well.

b) Assuming the same decrease in output, how much oil will be extracted from the well at the end of 20 weeks of operation?

c) The oil company will close the well when its output drops to less than five barrels per week. Assuming the same decrease in output, during which week will the well be closed?

Closing

17. The light intensity of a spotlight can be reduced by using gels, which are placed in front of the light. The table to the right represents the intensity of a spotlight after a specific number of gels have been used.

a) Explain how you could create an algebraic model of the data.

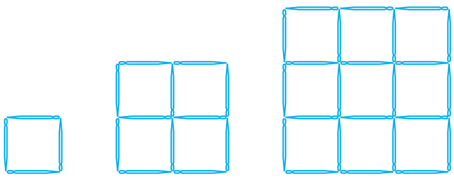
b) Determine the light intensity of a spotlight if seven gels are used, to two decimal places.

c) What is the minimum number of gels that would be needed to reduce the light intensity to less than 40.00%?

Number of Gels Used	Intensity (%)
0	100.00
1	97.00
2	94.09
3	91.27
4	88.53

Extending

18. Melinda used toothpicks to create a pattern of figures. The first three figures are shown below. She claims that this pattern can be modelled by an exponential function.



a) How many toothpicks will be needed to create the next figure in the pattern?

b) Is Melinda's claim true? Explain.

c) Assuming that this pattern continues, how many toothpicks will be needed to make the 7th figure?

19. A 100.0 g sample of radioactive tin-117 decayed to 16 g in 37 days, as shown below.

Number of Days	0	9	15	24	37
Mass of Tin-117 (g)	100.0	64.0	47.6	30.5	16.0

The function that models this decay can be written as

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

where $A(t)$ represents the mass of tin-117, A_0 represents the initial mass of tin-117, t represents the time in days, and h represents the time needed for half of the tin-117 to decay.

a) Use one of the points in the table to determine the value of h.

b) Use your answer from part a) to write a more precise exponential function that models the decay of tin-117.

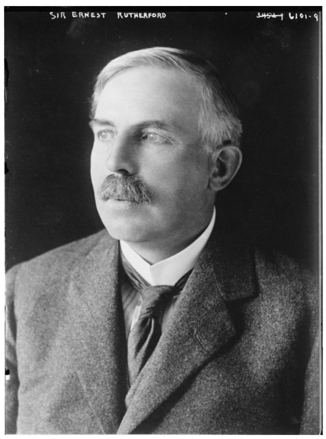
c) Determine the equation of the exponential regression function that models the decay of tin-117.

d) Verify that your equation from part b) matches your regression equation by using graphing technology.

History | Connection

Ernest Rutherford (1871–1937)

Many people consider Ernest Rutherford to be one of most important nuclear physicists. He was born in Nelson, New Zealand, and eventually came to Canada in 1898. He held the position of the MacDonald Chair of Physics at McGill University in Montreal. His work on the structure of the atom involved examining the decay of radioactive substances. He found that various particles are emitted as a radioactive element decays into different elements. The alpha particle, beta particle, and proton were named and described by him. Rutherford won the Nobel Prize in Chemistry in 1908.



Ernest Rutherford

A. Do research to determine the radioactive elements that Rutherford used in his work.

B. How could exponential functions be used to analyze the radioactive decay of an element? Provide exponential functions that could model the elements that Rutherford used.