7.2 Relating the Characteristics of an Exponential Function to Its Equation

YOU WILL NEED

- graphing technology

GOAL

Predict the characteristics of an exponential function by examining its equation.

EXPLORE…

- Tables of values for three different functions are shown below. How can you use the values to decide which functions are exponential?

x	$f(x)$
1	1
2	4
3	9
4	16
5	25

x	$g(x)$
1	2
2	4
3	8
4	16
5	32

x	$h(x)$
1	1
2	8
3	27
4	64
5	125

It was thought to be impossible to fold a piece of paper, of any size or shape, in half more than eight times. Britney Gallivan proved this incorrect in 2005, when she folded a very long piece of toilet paper 12 times.

INVESTIGATE the Math

Recall the origami activity in Getting Started. Sacha completed a table of values to compare the number of folds, x, to the number of layers of paper created, $f(x)$.

Number of Folds, x	Number of Layers Created, $f(x)$
0	1
1	2
2	4
3	8
4	16
5	32
6	64

He determined that the function

$$f(x) = 1(2)^x$$

can be used to predict the number of layers that will be created for a given number of folds.

? How can you predict the characteristics of an exponential function of the form $f(x) = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, by looking at its equation?

A. Graph Sacha's function:

$$f(x) = 1(2)^x$$

B. Examine the values of $f(x)$ in a table of values, where the x-values increase by 1 each time. How are the values of $f(x)$ related? Which parameter in the equation is associated with this relationship?

C. Examine the graph and table of values of your function. Determine the y-intercept, and state whether the graph is increasing or decreasing as you move from left to right along the x-axis. Record your results in a table like the one below.

Characteristics	$f(x) = 1(2)^x$
y-intercept	
Increasing or decreasing?	

D. Choose a new exponential function where $a > 0$, $b > 0$, and $b \neq 1$. Graph the new function, and repeat parts B and C. Add this information to your table.

E. Repeat part D for several more exponential functions where $a > 0$, $b > 1$, and $b \neq 1$.

F. Consider an exponential function where $a > 0$ and $0 < b < 1$, such as

$$f(x) = 2\left(\frac{1}{3}\right)^x$$

Repeat parts B and C. Add this information to your table.

G. Repeat part F for several more exponential functions where $a > 0$ and $0 < b < 1$.

H. Examine the table you have created and the parameters a and b in each function.

i) Can you predict the y-intercept from the parameters in the function? Explain.

ii) Can you predict whether the graph of the function will increase or decrease? Explain.

Reflecting

I. Consider a table of values for an exponential function, where values of the independent variable increase by 1. What relationship exists between consecutive values of the dependent variable? Why does this relationship occur?

J. Suppose that you are given the equation of an exponential function of the form

$$y = a(b)^x$$

where $a > 0$, $b > 0$, and $b \neq 1$.
Explain how you can determine:

i) the y-intercept

ii) whether the function is increasing or decreasing

K. Consider the general exponential function given in part J. Does changing either of the parameters a or b affect the end behaviour, domain, or range? Explain.

APPLY the Math

EXAMPLE 1 Connecting the characteristics of an increasing exponential function to its equation and graph

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, and the range of the following function:

$$y = e^x$$

Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing of the function.

Communication *Tip*

The symbol e is a constant known as Euler's number. It is an irrational number that equals 2.718.... This number occurs naturally in some situations where a quantity increases continuously, such as increasing populations.

Sharon's Solution

The function

$y = e^x$

can also be written as

$y = 1(2.718...)^x$

This function is an exponential function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$.

In this function, $a = 1$ and $b = e$ or 2.718....

I determined that this is an exponential function, since the variable x is an exponent. I also determined the values of parameters a and b.

Common Characteristics

Number of x-intercepts: 0

End behaviour: The curve extends from quadrant II to quadrant I.

Domain: $\{x \mid x \in \mathrm{R}\}$

Range: $\{y \mid y > 0, y \in \mathrm{R}\}$

I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for exponential functions of this form. These functions are defined for all real numbers, so this gave me the domain. I also knew that these functions have a restricted range bounded by the x-axis. This means the graph only occurs in quadrants II and I and has no x-intercepts.

Unique Characteristics

$y = 1(2.718...)^0$

$y = 1(1)$

$y = 1$

y-intercept: 1

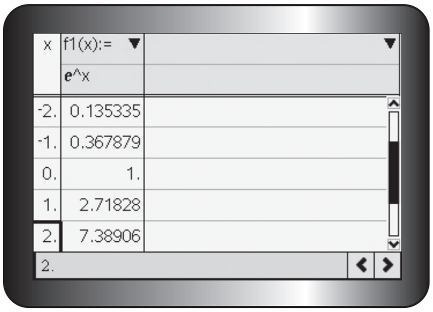
I determined the y-intercept by substituting $x = 0$ into the function and determining y.

I noticed that the parameter a in the equation is also equal to 1. I predict that a is always the y-intercept of the function, since any base to the exponent 0 is always 1.

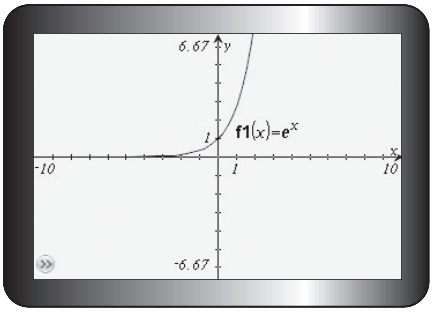
The function increases as you move from left to right along the x-axis.

I predicted that the function increases, since parameter b in the equation is greater than 1.

I can use a graphing calculator to create a table of values and a graph to verify my predictions.



x	f1(x):= e^x
-2.	0.135335
-1.	0.367879
0.	1.
1.	2.71828
2.	7.38906



My predictions were correct.

Your Turn

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, and the range of this function:

$$f(x) = 2(5)^x$$

Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing the function.

EXAMPLE 2 Connecting the characteristics of a decreasing exponential function to its equation and graph

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, the range, and whether this function is increasing or decreasing:

$$y = 9\left(\frac{2}{3}\right)^x$$

Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing the function.

Pat's Solution

The function $y = 9\left(\frac{2}{3}\right)^x$ is an exponential function of the form $y = a(b)^x$, where $a > 0$ and $0 < b < 1$.

I determined that this is an exponential function, since the variable x is an exponent. I also determined the values of parameters a and b.

In this function, $a = 9$ and $b = \frac{2}{3}$.

Common Characteristics

Number of x-intercepts: 0

End behaviour: The curve extends from quadrant II to quadrant I.

Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y > 0, y \in \mathbb{R}\}$

I predicted the characteristics of the function from its equation. I started with the common characteristics I knew for exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$.

Unique Characteristics

$y = 9\left(\frac{2}{3}\right)^0$

$y = 9(1)$

$y = 9$

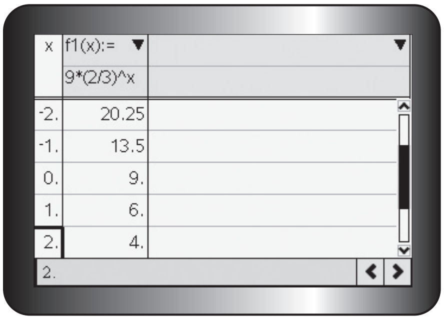
y-intercept: 9

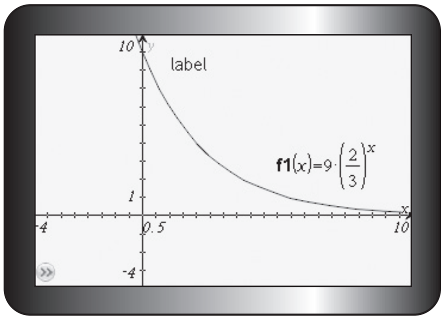
I determined the y-intercept by substituting $x = 0$ into the function and determining y. The parameter a in the equation has the same value as the y-intercept.

The function decreases as you move from left to right along the x-axis.

I predicted that the function decreases, since parameter b in the equation is greater than 0 and less than 1.

I can use a graphing calculator to create a table of values and a graph to verify my predictions.





My predictions were correct.

Your Turn

Predict the number of x-intercepts, the y-intercept, the end behaviour, the domain, the range, and whether this function is increasing or decreasing:

$$f(x) = 8\left(\frac{3}{4}\right)^x$$

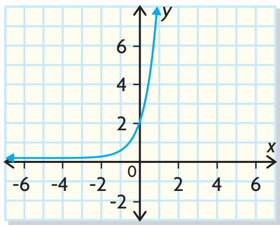
Use the equation of the function to make your predictions. Verify your predictions by creating a table of values and graphing the function.

EXAMPLE 3 Matching an exponential equation with its corresponding graph

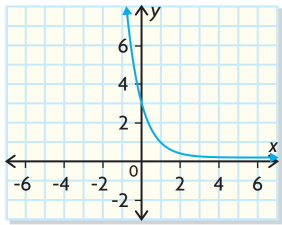
Which exponential function matches each graph below? Provide your reasoning.

i) $y = 3(0.2)^x$ **ii)** $y = 4(3)^x$ **iii)** $y = 4(0.5)^x$ **iv)** $y = 2(4)^x$

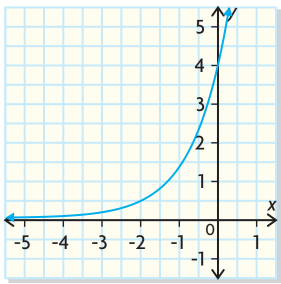
a)



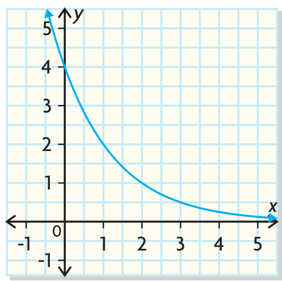
c)



b)

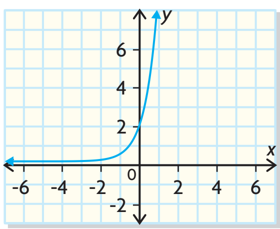


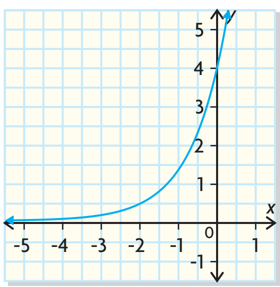
d)



Arthur's Solution

Graphs a) and b) are of increasing functions. Since they are increasing, the functions for these graphs must have $b > 1$.

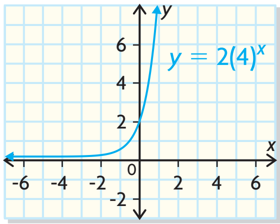




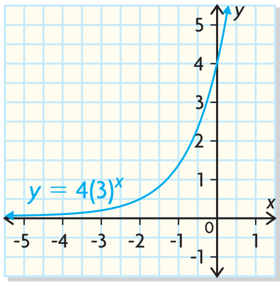
When parameter b is greater than 1, I knew that the graph of a function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, increases as you move from left to right along the x-axis.

There are two possible functions for these two graphs:
$y = 4(3)^x$ and $y = 2(4)^x$
y-intercept: 4 y-intercept: 2

I knew that parameter a, in a function of the form $y = a(b)^x$, represents the y-intercept of the function. So in the function $y = 4(3)^x$, the y-intercept is 4, and in the function $y = 2(4)^x$, the y-intercept is 2. I can use the y-intercepts of the graphs of the two increasing functions to match the graphs to the equations.



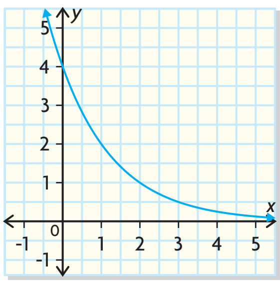
This graph passes through 2 on the y-axis, so its function must be $y = 2(4)^x$.

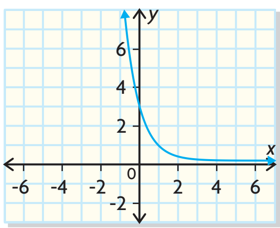


This graph passes through 4 on the y-axis, so its function must be $y = 4(3)^x$.

The remaining two graphs are of decreasing functions. Since they are decreasing, the functions for these graphs must have $0 < b < 1$.

When parameter b is greater than 0 and less than 1, I knew that the graph of a function of the form $y = a(b)^x$, where $a > 0$, decreases as you move from left to right along the x-axis.



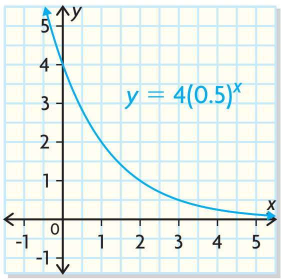


There are two possible equations for these two graphs:

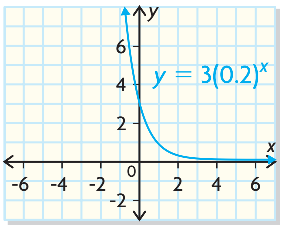
$y = 3(0.2)^x$ and $y = 4(0.5)^x$

y-intercept: 3 y-intercept: 4

I knew that parameter a, in a function of the form $y = a(b)^x$, represents the y-intercept of the function. So in the function $y = 3(0.2)^x$, the y-intercept is 3, and in the function $y = 4(0.5)^x$, the y-intercept is 4. I used the y-intercepts of the graphs of the two decreasing functions to match the graphs to the equations.



This graph passes through 4 on the y-axis, so its function must be $y = 4(0.5)^x$.



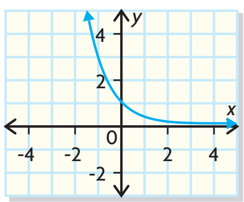
This graph passes through 3 on the y-axis, so its function must be $y = 3(0.2)^x$.

Your Turn

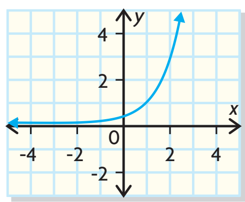
Match each function with the corresponding graph below. Provide your reasoning.

i) $y = (3)^x$ **ii)** $y = \frac{1}{3}(3)^x$ **iii)** $y = 3\left(\frac{1}{3}\right)^x$ **iv)** $y = \left(\frac{1}{3}\right)^x$

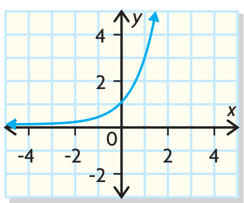
a)



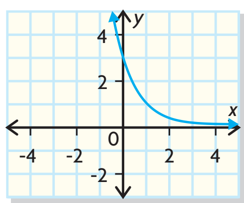
c)



b)



d)



In Summary

Key Ideas

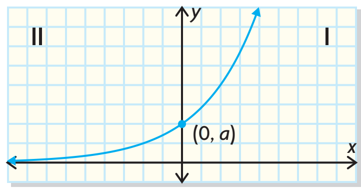
- In a table of values for an exponential function, there is a constant ratio between consecutive y-values when the x-values increase by the same amount. The value of this ratio is equal to the parameter b in the function

$$y = a(b)^x, \text{ where } b \neq 1.$$

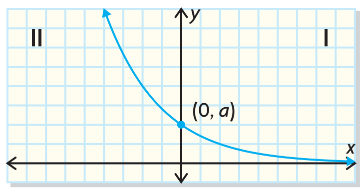
- In an exponential function of the form $y = a(b)^x$, a is a non-zero multiplier and b is the base (where $b > 0$ and $b \neq 1$). The value of a is the y-intercept of the graph of the function.

Need to Know

- An exponential function is an increasing function if $a > 0$ and $b > 1$.
- An exponential function is a decreasing function if $a > 0$ and $0 < b < 1$.
- Changing the parameters a and b in exponential functions of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, does not change the number of x-intercepts, the end behaviour, the domain, or the range of the function. These characteristics are identical in all exponential functions of this form.



$a > 0, b > 1$



$a > 0, 0 < b < 1$

CHECK Your Understanding

1. Determine if the data in each table represents an exponential function. Provide your reasoning.

a)

x	0	1	2	3	4	5
y	7	14	28	56	112	224

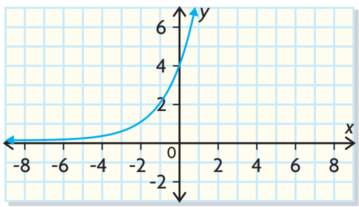
b)

x	0	1	2	3	4	5
y	3072	768	192	48	12	3

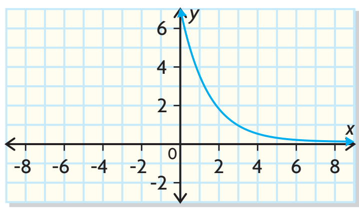
2. Describe the characteristics of the following exponential graphs by completing a table like the one below.

Number of x-Intercepts	y-Intercept	End Behaviour	Domain	Range

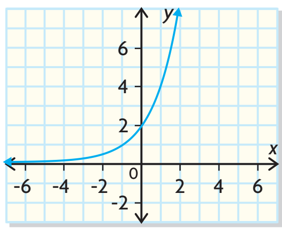
a)



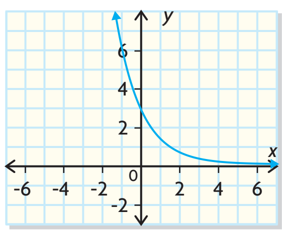
c)



b)



d)



3. What is the difference between an increasing exponential function and a decreasing exponential function? Sketch a graph of each type of function to illustrate your answer.

4. Determine the y-intercept of each exponential function, and state whether the function is increasing or decreasing.

a) $y = 5(2)^x$ b) $y = 2\left(\frac{1}{2}\right)^x$ c) $y = 10(1.5)^x$ d) $y = (0.4)^x$

PRACTISING

5. For each set of data:
   i) Determine if an exponential function could be used to model the data, and explain how you know;
   ii) Identify the y-intercept, and state if the function is increasing or decreasing or both.

a)

x	y
0	1
1	2
2	4
3	8
4	16
5	32

b)

x	y
−4	−5
−3	−3
−2	−1
−1	1
0	3
1	5

c)

x	y
−1	256
0	64
1	16
2	4
3	1
4	0.25

d)

x	y
−2	5
−1	2
0	1
1	2
2	5
3	−10

6. Predict the characteristics of each function, based on the given equation. Record your predictions in a table like the one below. Then verify your predictions by graphing each function.

Number of *x*-Intercepts	*y*-Intercept	End Behaviour	Domain	Range

a) $y = 3(5)^x$

b) $y = 4(0.3)^x$

c) $y = 2\left(\frac{1}{5}\right)^x$

d) $y = 3.5\left(\frac{1}{8}\right)^x$

e) $y = 25(1.7)^x$

f) $y = 12(0.8)^x$

7. Explain how you could use the base of each function to determine if the function increases or decreases.

a) $y = \frac{1}{3}(8)^x$ b) $y = 4\left(\frac{3}{5}\right)^x$ c) $y = 0.5(e)^x$

8. Predict the characteristics of each exponential function. Record your predictions in a table like the one below.

Number of *x*-Intercepts	*y*-Intercept	End Behaviour	Domain	Range

a) $y = 4(2)^x$

b) $y = 8\left(\frac{1}{4}\right)^x$

c) $y = 3(3)^x$

d) $y = 10\left(\frac{1}{2}\right)^x$

e) $y = 30(1.8)^x$

f) $y = (0.5)^x$

g) $y = 3(e)^x$

h) $y = 45\left(\frac{2}{5}\right)^x$

9. Determine if any of these are decreasing exponential functions without graphing them. Explain how you know.

a) $f(x) = 7\left(\frac{2}{3}\right)^x$

b) $g(x) = 0.6(2.5)^x$

c) $h(x) = 0.4(0.5)^x$

d) $k(x) = 5(2)^x$

10. Predict which functions are increasing and which are decreasing, without graphing them. Verify your predictions by graphing the functions.

a) $y = 5(2)^x$

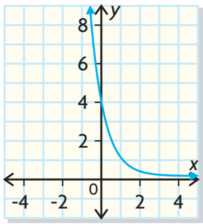
b) $y = 20\left(\frac{1}{4}\right)^x$

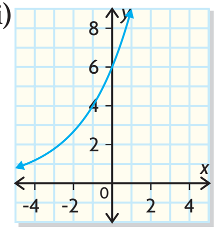
c) $y = 6(8)^x$

d) $y = 12\left(\frac{1}{3}\right)^x$

11. Examine each function to determine if it increases or decreases. Then match each function with the corresponding graph below.

a) $y = 6(1.5)^x$
b) $y = 4(0.25)^x$
c) $y = 2(0.1)^x$
d) $y = 3(1.15)^x$

i)


iii)


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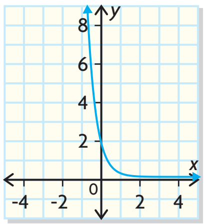
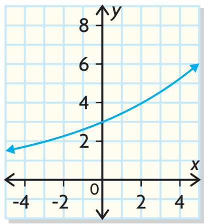


iv)


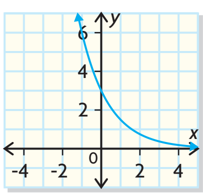
12. Copy and complete this table. Verify your answers by graphing.

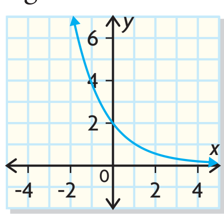
	Function	y-Intercept	Base	Domain	Range	Increasing or Decreasing
a)	$y = 9(7)^x$					
b)	$y = 7(4)^x$					
c)	$y = 6\left(\frac{1}{7}\right)^x$					
d)	$y = 2(0.35)^x$					
e)	$y = 2(e)^x$					

13. a) Determine the range of each function, and state the values of *a* and *b*. Use this information to decide if the function increases or decreases.

i) $y = 2(0.5)^x$
ii) $y = 1(3)^x$
iii) $y = 3\left(\frac{1}{2}\right)^x$
iv) $y = 2(4)^x$

b) Match each function in part a) with the corresponding graph below. Provide your reasoning.

A.


B.


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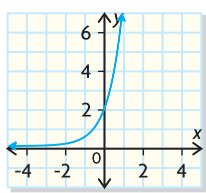

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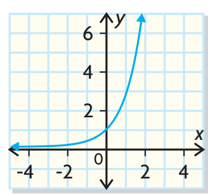


14. a) Write the equations of an increasing exponential function and a decreasing exponential function that each have a y-intercept of 5.

b) Which characteristics of the functions would be the same? Which characteristics would be different?

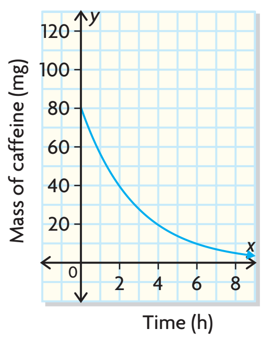
15. Natasha claims that if she is given the equation of a polynomial function and the equation of an exponential function she can identify which is which. Do you agree or disagree? Justify your decision.

16. Linear functions are defined by $f(x) = ax + b$, where $a \in \mathrm{R}$ and $b \in \mathrm{R}$. The exponential functions you have studied are defined by $f(x) = a(b)^x$, where $a > 0$, $b > 0$, $b \neq 1$, and $a \in \mathrm{R}$ and $b \in \mathrm{R}$. Compare the number of x-intercepts, y-intercept, end behaviour, domain, range, increasing or decreasing for both types of functions. Create a list of characteristics that are similar and a list of characteristics that are different.

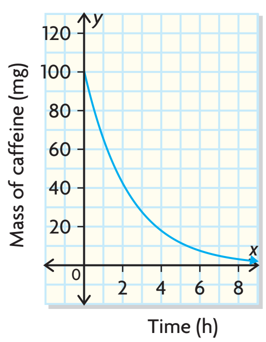
Closing

17. You are given an exponential function of the form $y = a(b)^x$, where $a > 0$, $b > 0$, and $b \neq 1$, and are asked to identify the number of x-intercepts, the y-intercept, end behaviour, domain, range, and whether the function increases or decreases. Which of these characteristics are unique to the function you have been given, and which of these characteristics are common to all exponential functions of this form?

Student A



Student B



Extending

18. Two students consumed energy drinks that contained caffeine. The mass of caffeine, y, in milligrams, in each student's body was then measured over an 8 h period. The data is shown in the graphs at left, where x represents time in hours.

a) Use each graph to determine the y-intercept, domain, and range of the function.

b) Why are both functions decreasing?

c) Based on the graphs, which student consumed more caffeine? Which student's body processed the caffeine more quickly?

d) Estimate the amount of caffeine in each student's body 4 h after consuming the energy drink.

e) Explain the meaning of the y-intercept in this context. Why are the values of the y-intercept different?

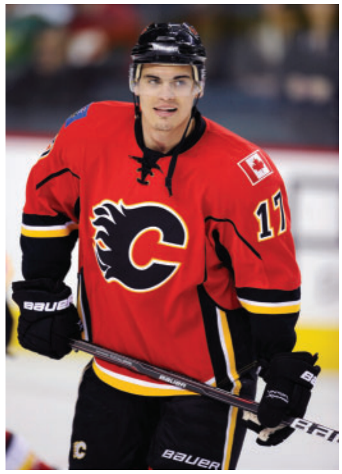
19. Three student council members booked professional hockey player René Bourque to make a presentation at their school. By the end of the first hour after they made the booking, each student had told the news to two of their friends. By the end of the second hour, each of their friends had told the news to two more friends. The pattern continued for several hours.

a) Create a table of values to model the spread of the news during the first 5 h.

b) Can an exponential function be used to model the data? Explain.

c) Use your table of values to define a function that models the data. What is the domain and range of your function?

d) Graph your function. From your graph, extrapolate the number of people who would have heard the news if the pattern had continued through 8 h.



René Bourque, a Franco-Albertan of Métis heritage, was acquired by the Calgary Flames of the NHL in 2008.

20. Philippe Sly was nominated for the music award Jeune soliste 2012. When the news of his nomination broke, bloggers, newspapers, and television stations wanted to interview him. By 9:00 a.m. on Monday morning, Philippe had received four requests for an interview. The number of interview requests he received then doubled every 3 h.

a) Kendra claims that an exponential function can be used to model this situation. Is she correct? Explain.

b) Define an exponential function of the form $y = a(b)^x$ to model Philippe's situation. State what a, b, x, and y represent in this context.

c) What is the domain and range of the function in this context?

d) Extend the following table to determine how many interview requests Philippe had received by 4:00 p.m. on Monday afternoon. Explain your process.

Time	x	y
9:00 a.m.		
10:00 a.m.		

Philippe Sly, of Ottawa, Ontario, is a member of the Canadian Opera Company Ensemble Studio.