

5.3

Probabilities Using Counting Methods

GOAL

Solve probability problems that involve counting techniques.

INVESTIGATE the Math

As a volunteer activity, 10 students want to put on a talent show at a retirement home. To organize the show, 3 of these students will be chosen at random to form a committee. Victoria really wants to be on this committee, since her grandmother lives at the home. Each student's name will be written on a slip of paper and placed in a hat. Then 3 names will be drawn.



? What is the likelihood that Victoria's name will be drawn from the hat?

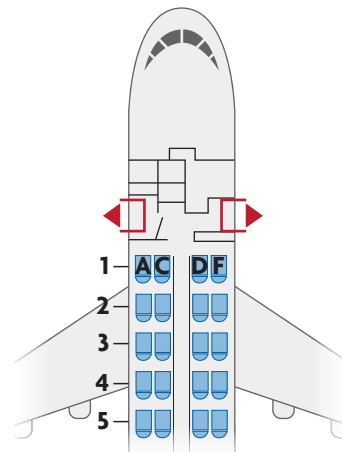
- Is Victoria's name just as likely to be drawn as any other name? Explain.
- Does the order in which the names are drawn matter? Explain.
- In how many different ways can 3 names be drawn from a hat with 10 names? Explain.
- In how many different ways can Victoria's name be drawn with 2 other names? Explain.
- What is the probability that Victoria's name will be drawn? Explain.

YOU WILL NEED

- calculator

EXPLORE...

- Ari, Ben, Cam, and Dan have booked a flight to Florida. The airplane has four seats in each row, and they are planning to sit in the same row. What is the probability that Cam and Dan will be sitting on the same side of the airplane?



Reflecting

- F. Did you assume that this problem involved a situation with or without replacement? Explain.
- G. Suppose that only 8 students, including Victoria, volunteered for the talent show. Would her name be more likely or less likely to be drawn from the hat? Justify your decision.
- H. Suppose that only 2 of the 10 names will be drawn from the hat, instead of 3 names. Is Victoria's name more likely or less likely to be drawn? Justify your decision.

APPLY the Math

EXAMPLE 1

Solving a probability problem using counting techniques

Jamaal, Ethan, and Alberto are competing with seven other boys to be on their school's cross-country team. All the boys have an equal chance of winning the trial race. Determine the probability that Jamaal, Ethan, and Alberto will place first, second, and third, in any order.

Brandi's Solution: Using permutations

Jamaal, Ethan, and Alberto can place first, second, or third, in any order. There are ${}_3P_3$ ways in which three runners can place in three positions.

I determined the number of ways in which Jamaal, Ethan, and Alberto can finish in the top three positions.

$${}_3P_3 = \frac{3!}{(3-3)!}$$

$${}_3P_3 = \frac{3 \cdot 2 \cdot 1}{1}$$

$${}_3P_3 = 6$$

There are 6 favourable outcomes.

There are ${}_{10}P_3$ ways that 10 runners can place first, second, or third.

I determined the total number of ways that 10 runners can place first, second, and third.

$${}_{10}P_3 = \frac{10!}{(10-3)!}$$

$${}_{10}P_3 = \frac{10!}{7!}$$

$${}_{10}P_3 = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$$

$${}_{10}P_3 = 10 \cdot 9 \cdot 8$$

$${}_{10}P_3 = 720$$

There are 720 possible outcomes.



$$P(\text{J, E, and A place 1, 2, or 3}) = \frac{{}_3P_3}{{}_{10}P_3}$$

$$P(\text{J, E, and A place 1, 2, or 3}) = \frac{6}{720} \text{ or } \frac{1}{120}$$

The probability that Jamaal, Ethan, and Alberto will place in the top three positions is $\frac{1}{120}$ or about 0.83%.

Davinder's Solution: Using combinations

Jamaal, Ethan, and Alberto can place first, second, or third, in any order.

There are ${}_3C_3$ ways in which only these three runners place in the top three positions.

$${}_3C_3 = \frac{3!}{3!(3-3)!}$$

$${}_3C_3 = \frac{3!}{3! \cdot 0!}$$

$${}_3C_3 = \frac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1}$$

$${}_3C_3 = 1$$

There is only 1 favourable outcome.

$${}_{10}C_3 = \frac{10!}{3!(10-3)!}$$

$${}_{10}C_3 = \frac{10!}{3! \cdot 7!}$$

$${}_{10}C_3 = \frac{10 \cdot 9 \cdot 8}{3!}$$

$${}_{10}C_3 = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1}$$

$${}_{10}C_3 = 10 \cdot 3 \cdot 4$$

$${}_{10}C_3 = 120$$

There are 120 possible outcomes.

$$P(\text{J, E, and A place 1st, 2nd, or 3rd}) = \frac{{}_3C_3}{{}_{10}C_3}$$

$$P(\text{J, E, and A place 1st, 2nd, or 3rd}) = \frac{1}{120}$$

The probability that Jamaal, Ethan, and Alberto will place in the top three positions is $\frac{1}{120}$ or about 0.83%.

I divided the number of favourable outcomes by the total number of outcomes.

I determined the number of combinations in which Jamaal, Ethan, and Alberto finish in the top three positions.

I determined the total number of combinations that are possible for all 10 runners placing in first, second, and third.

I simplified by dividing both the numerator and denominator by $7!$, then I divided 9 by 3 and 8 by 2.

To determine probability, I divided the number of favourable outcomes by the total number of outcomes.



Your Turn

Suppose that Zachary is also trying out for the team, so now there will be 11 runners in the trial race. What is the probability that three of Jamal, Ethan, Alberto, and Zachary will place in the top three positions?

EXAMPLE 2

Solving a probability problem with the Fundamental Counting Principle

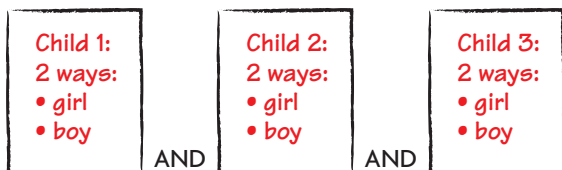
About 20 years after they graduated from high school, Blake, Mario, and Simon met in a mall. Blake had two daughters with him, and he said he had three other children at home. Determine the probability that at least one of Blake's children is a boy.

Mario's Solution: Using indirect reasoning

I considered the genders of the children at home.

I knew that the genders of the children at the mall had nothing to do with the genders of the children at home. I just needed to calculate the probability that the three children at home are girls or boys.

Each child is either a boy or a girl.



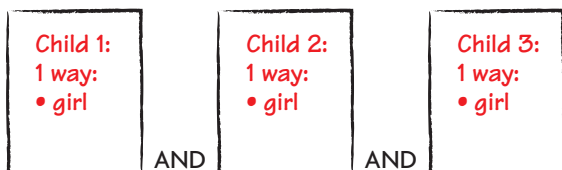
Let C represent the total possible outcomes.

I determined the total number of possible outcomes using the **Fundamental Counting Principle**.

$$C = 2 \cdot 2 \cdot 2$$

$$C = 8$$

There are 8 possible outcomes.



Let G represent the number of outcomes where each child is a girl.

My result made sense because there is only one way that all three children can be girls.

$$G = 1 \cdot 1 \cdot 1$$

$$G = 1$$

In only 1 outcome is each child a girl.



$$P(\text{all girls}) = \frac{n(\text{children at home all girls})}{n(\text{all possibilities})}$$

$$P(\text{all girls}) = \frac{1}{8}$$

The probability that all five children are girls is $\frac{1}{8}$ or 12.5%.

$$P(\text{at least one boy}) = 1 - P(\text{all girls})$$

$$P(\text{at least one boy}) = 1 - \frac{1}{8}$$

$$P(\text{at least one boy}) = \frac{7}{8}$$

The probability that at least one child is a boy is $\frac{7}{8}$ or 87.5%.

I wrote the probability.

The probability of at least one boy is the complement of the probability of all girls.

Simon's Solution: Using direct reasoning

I let U represent the total number of possible outcomes:

$$U = 2 \cdot 2 \cdot 2$$

$$U = 8$$

There are 8 possible outcomes for the genders of the children at home.

There are three cases in which at least one child is a boy. I wrote out the possible ways for each case.

I knew that the children at the mall were girls, so I needed to consider only the genders of the children at home.

Each child at home is either a girl or a boy.

In these cases, each child is unique, which means that order is important, so the problem involves permutations.

Case 1: exactly one boy

- boy, girl, girl
- girl, boy, girl
- girl, girl, boy

There can be one boy in 3 ways.

If there is one boy, he must be the eldest, middle, or youngest of the three children at home.

Case 2: exactly two boys

- girl, boy, boy
- boy, girl, boy
- boy, boy, girl

There can be two boys in 3 ways.

If there are two boys, then the girl must be the eldest, middle, or youngest of the three children at home.

Case 3: exactly three boys

- boy, boy, boy

There can be three boys in 1 way.



I let B represent the number of ways in which at least one child is a boy:

$$B = 3 + 3 + 1$$

$$B = 7$$

There are 7 ways in which at least one child is a boy.

$$P(\text{at least one boy}) = \frac{B}{U}$$

$$P(\text{at least one boy}) = \frac{7}{8}$$

The probability that at least one child is a boy

is $\frac{7}{8}$ or 87.5%.

I determined the total number of ways in which at least one child is a boy.

I wrote the probability.

Your Turn

Suppose that Blake had had one daughter with him at the mall and four children at home. Determine the probability of each event.

- All five of Blake's children are girls.
- At least one of Blake's children is a boy.

EXAMPLE 3 Solving a probability problem using reasoning

Beau hosts a morning radio show in Saskatoon. To advertise his show, he is holding a contest at a local mall. He spells out SASKATCHEWAN with letter tiles. Then he turns the tiles face down and mixes them up. He asks Sally to arrange the tiles in a row and turn them face up. If the row of tiles spells SASKATCHEWAN, Sally will win a new car. Determine the probability that Sally will win the car.



Sally's Solution

S A S K A T C H E W A N

There are 12 letters in total: 2 Ss, 3 As, and 7 other letters.

I let L represent the total number of ways in which I can arrange the letters:

$$L = \frac{12!}{2! \cdot 3!}$$

$$L = 39\,916\,800$$

This is the total number of outcomes.

I examined the letters in the word.

Since the order must be correct, this problem involves permutations.

I could arrange the 12 letters in $12!$ ways if they were all different. But there are 2 Ss and 3 As. I divided by $2!$ and by $3!$ to eliminate the arrangements that would be the same.



I let R represent the number of ways that I can spell SASKATCHEWAN.

$$R = 1$$

I can spell SASKATCHEWAN in just 1 way.

So, there is only 1 favourable outcome.

$$P(\text{winning the car}) = \frac{R}{L}$$

$$P(\text{winning the car}) = \frac{1}{39\,916\,800}$$

The probability that I will win the car is

$$\frac{1}{39\,916\,800}$$

Since I considered only the unique arrangements of the 12 letters to determine the total number of outcomes, there is only 1 correct way to spell SASKATCHEWAN.

I determined the probability that I will win.

Winning the car is very unlikely.

Your Turn

Suppose that the contest word was SASKATOON. Would Sally's probability of winning be greater or less? Explain.

EXAMPLE 4

Solving a probability problem with conditions

There are 18 bikes in Marnie's spinning class. The bikes are arranged in 3 rows, with 6 bikes in each row. Allison, Brett, Carol, Doug, Erica, and Franco each call the gym to reserve a bike. They hope to be in the same row, but they cannot request a specific bike. Determine the probability that all 6 friends will be in the same row, with Allison and Franco at either end.



Marnie's Solution

There is an equal likelihood that any of the 18 bikes will be assigned to a participant, since specific bikes cannot be requested.

Let F represent the event that the 6 friends are seated in any row:

2	4	3	2	1	1
12	11	10	9	8	7
6	5	4	3	2	1

I assumed that the bikes are assigned randomly. I also assumed that all 18 bikes will be used.

I determined the number of ways in which the 6 friends can be assigned bikes in the first row, if Allison and Franco are at either end.

Since order is important, this problem involves permutations.

- The number of ways to seat Allison and Franco at either end is $2!$ or ${}_2P_2$.
- The number of ways to seat the other 4 friends is $4!$ or ${}_4P_4$.
- The number of ways to seat the other 12 people in the class is $12!$ or ${}_{12}P_{12}$.

The total number of ways to seat the 6 friends in the first row is $({}_2P_2)({}_4P_4)({}_{12}P_{12})$.

I knew that I could multiply here, since I was seating Allison and Frank on the ends AND the other 4 friends in between AND the remaining 12 people.

The total number of ways to seat the 6 friends in any row is $3({}_2P_2)({}_4P_4)({}_{12}P_{12})$.

Since the 6 friends can be seated in the same row 3 different ways (row 1, 2, or 3), I multiplied the number of ways they can sit in the first row by 3.

The total number of ways to assign 18 people to 18 bikes is $18!$ or ${}_{18}P_{18}$.

The number of ways that the 18 bikes can be assigned is equivalent to the number of possible permutations for a set of 18 objects.

$$P(F) = \frac{3({}_2P_2)({}_4P_4)({}_{12}P_{12})}{{}_{18}P_{18}}$$

I determined the probability that the 6 friends will be in the same row by dividing the number of favourable outcomes by the total number of outcomes.

$$P(F) = \frac{3 \cdot 2! \cdot 4! \cdot 12!}{18!}$$

$$P(F) = \frac{3 \cdot 2! \cdot 4! \cdot 12!}{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}$$

I used the fact that $\frac{12!}{12!} = 1$ to simplify.

$$P(F) = \frac{3 \cdot 2! \cdot 4!}{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}$$

$$P(F) = \frac{144}{13\,366\,080}$$

$$P(F) = \frac{1}{92\,820}$$

I used my calculator to multiply since the numbers were large.

The probability that all 6 friends will be in the same row, with Allison and Franco at either end,

is $\frac{1}{92\,820}$.

Your Turn

Franco determined the solution to this problem by calculating $\frac{3 \cdot 2 \cdot {}_4P_4}{{}_{18}P_6}$.

Would this calculation give the correct answer? Explain.

In Summary

Key Idea

- You may be able to use the Fundamental Counting Principle and techniques involving permutations and combinations to solve probability problems with many possible outcomes. The context of each particular problem will determine which counting techniques you will use.

Need to Know

- Use permutations when order is important in the outcomes.
- Use combinations when order is not important in the outcomes.

CHECK Your Understanding

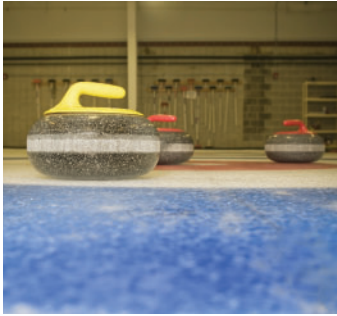
1. A credit card company randomly generates temporary four-digit pass codes for cardholders. Suri is expecting her credit card to arrive in the mail. Determine the probability that her pass code will consist of four different even digits.
2. In a card game called Crazy Eights, players are dealt 8 cards from a standard deck of 52 playing cards. Determine the probability that a hand will consist of 8 hearts.
3. From a committee of 12 people, 2 of these people are randomly chosen to be president and secretary. Determine the probability that Ben and Jen will be chosen.

PRACTISING

4. Five boys and six girls have signed up for a trip to see Francophone artists compete at Festival International de la Chanson de Granby. Only four students will be selected to go on the trip. Determine the probability for the following:
 - a) Only boys will be on the trip.
 - b) There will equal numbers of boys and girls on the trip.
 - c) There will be more girls than boys on the trip.
5. Access to a particular online game is password protected. Every player must create a password that consists of two capital letters followed by three digits. For each condition below, determine the probability that a password chosen at random will contain the letters S and Q.
 - a) Repetitions are not allowed in a password.
 - b) Repetitions are allowed in a password.



Festival International de la Chanson de Granby is the leading French language song competition in the country. Previous winners have included Senaya, of Montréal, and Steeve Thomas, of Vancouver.



6. A high-school athletics department is forming a beginners curling team to play in a social tournament. Nine students, including you and your three friends, have signed up for the four positions of skip, third, second, and lead. The positions will be filled randomly, so every student has an equal chance of being chosen for any position.
 - a) Determine the probability that you and your three friends will be chosen.
 - b) How would this probability change if only eight students had signed up for the team?
7. There are nine players on a baseball team, all with roughly equal athletic ability. The coach has decided to choose the players who will play the four infield positions (first base, second base, third base, and shortstop) randomly. Tara and Laura are on the team. Determine the odds in favour of Tara and Laura being chosen to play in the infield.
8. A student council has 15 members, including Yuko, Luigi, and Justin.
 - a) The staff advisor will select three members at random to be treasurer, secretary, and liaison to the principal. Determine the probability that the staff advisor will select Yuko to be treasurer, Luigi to be secretary, and Justin to be liaison.
 - b) The staff advisor will also select three members at random to clean up after the pep rally. Determine the probability that the staff advisor will select Yuko, Luigi, and Justin to do this.
9. Lesley needs to create a four-digit password to access her voice mail. She can repeat some of the digits, but all four digits cannot be the same.
 - a) Determine the probability that her password will be greater than 5000.
 - b) Determine the probability that the first and last digits of her password will be 4.
 - c) Determine the probability that the first digit of her password will be odd and the last digit will be even.
10. A student council consists of 16 girls and 7 boys. To form a subcommittee, 5 students are randomly selected from the council. Determine the odds in favour of 3 girls and 2 boys being on the subcommittee.
11. Larysa tosses four coins. Determine the probability that at least one coin will land as tails.
12. Five friends, including Bilyana and Bojana, are sitting in a row in a theatre.
 - a) Determine the probability that Bilyana and Bojana are sitting together.
 - b) Determine the probability that they are not sitting together.

13. Tanya is planning her schedule for university. She wants to take the following courses during her first two terms: biology, English, psychology, religion, linear algebra, political studies, economics, and philosophy. She is equally likely to take any of these courses in either term, since they are all introductory courses.
- Suppose that Tanya decides to take four courses in her first term. Determine the probability that three of them will be psychology, linear algebra, and English.
 - Suppose that Tanya decides to take five courses in her first term. Determine the probability that three of them will be religion, political studies, and biology.
14. Doc deals you eight cards at random from a standard deck of 52 playing cards. Determine the probability that you have the following hands.
- A, 2, 3, 4, 5, 6, 7, 8 of the same suit
 - Eight cards of the same colour
 - Four face cards and four other cards
15. Erynn has letter tiles that spell CABINET. She has selected three of these tiles at random. Determine the probability that the tiles she selected are two vowels and one consonant.
16. At a local dog show, dogs compete in eight different categories. The eight winners of these categories are all different breeds, including a sheltie and a bearded collie. The organizers randomly line up the eight winners for the “Best in Show” competition. Determine the probability that the bearded collie and the sheltie will be next to each other in the lineup.



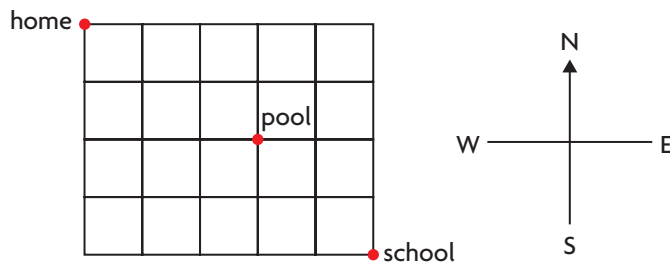
17. The starting lineup for a basketball team consists of two guards and three forwards. On the team that sisters Maggie and Tanya play for, there are seven forwards and five guards from which the coach can choose a starting lineup. Maggie is a guard and Tanya is a forward. For the first exhibition game of the year, the coach will select the starting players at random. What is the probability that both Maggie and Tanya will be in the starting lineup?

Closing

18. Explain when you would use permutations to solve a probability problem and when you would use combinations. Give an example.

Extending

19. Tuyet likes to vary her walk to school. At each intersection, she randomly walks either south or east. Determine the probability that she will pass the pool on her way to school.



20. In Chapter 4, page 257, you learned about birthday permutations. Determine how many people would need to be in a room for the probability of two people having the same birthday to be 80%.

History | Connection

Counter Intuition

Which game would you rather play?

Game 1: Roll a single standard die 4 times. If a 6 comes up, you win a point. If not, you lose a point.

Game 2: Roll two standard dice 24 times. If a double 6 comes up, you win a point. If not, you lose a point.

Around 1650, the Chevalier de Méré, Antoine Gombaud, said that these two games gave you the same chance of winning. His reasoning was as follows:

- With game 1, the chance of rolling a 6 in 4 rolls is $4\left(\frac{1}{6}\right)$ or $\frac{2}{3}$.
- With game 2, the chance of rolling two 6s in 24 rolls is $24\left(\frac{1}{36}\right)$ or $\frac{2}{3}$.

But when the Chevalier de Méré played game 2, he lost quite often. He asked the mathematician Blaise Pascal to explain why. After much analysis, Pascal realized that the probability of winning game 2 is actually less than 50%. Pascal's analysis of this situation laid the groundwork for the study of probability.

- A.** Determine why you have a better chance of winning game 1 than game 2.



Blaise Pascal (1623–1662) was a brilliant mathematician, physicist, and inventor. At 18, he invented a calculating device. He is also said to have been the first to wear a wrist watch.