3.3 Intersection and Union of Two Sets

EXPLORE…

Given: $n(A) = x$, $n(B) = y$, $n(A \text{ and } B)' = \{\ \}$, and $n(U) = z$, where $U =$ the universal set, and sets A and B are subsets of U.

- How can you determine whether sets A and B are disjoint or overlap?

Communication *Tip*

The 24-hour clock goes from 0 to 24:00 each day; for example, 2:00 a.m. is 02:00 and 5:00 p.m. is 17:00. The times 00:00 and 24:00 both represent midnight.

GOAL

Understand and represent the intersection and union of two sets.

INVESTIGATE the Math

Jacquie is a zookeeper. She is responsible for feeding any baby animal that cannot be fed by its mother. She needs to feed a baby raccoon every 2 h and a baby lemur every 3 h. She uses the 24-hour clock to plan their feeding times. She starts to feed the raccoon at 02:00 and the lemur at 03:00.

? At what times will Jacquie need help to feed the raccoon and the lemur?

A. Let set T represent all the hourly times from 01:00 to 24:00.
Let set R represent times for every 2 h, up to 24 h.
Let set L represent times for every 3 h, up to 24 h.
Describe the universal set T and the subsets R and L in set notation.

B. List the elements in each subset.

C. Represent the sets T, R, and L using a Venn diagram. Place the feeding times in the correct regions. Explain why you placed the sets and elements where you did.

D. List the elements in the **intersection** of sets R and L, $R \cap L$.

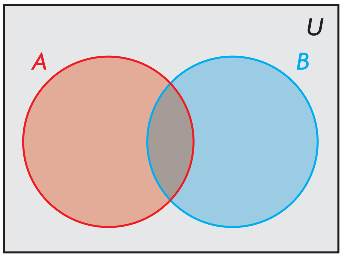
E. List the elements in set $R \setminus L$ and in set $L \setminus R$ using set notation.

intersection

The set of elements that are common to two or more sets. In set notation, $A \cap B$ denotes the intersection of sets A and B; for example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

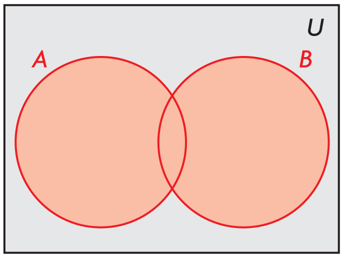
Communication | *Notation*

In set notation, $A \cap B$ is read as "intersection of A and B." It denotes the elements that are common to A and B. The intersection is the region where the two sets overlap in the Venn diagram below.



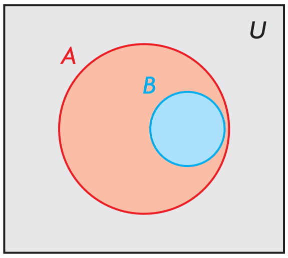
$\boldsymbol{A \cap B}$

$A \cup B$ is read as "union of A and B." It denotes all elements that belong to at least one of A or B. The union is the red region in the Venn diagram below.

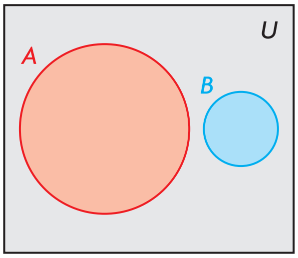


$\boldsymbol{A \cup B}$

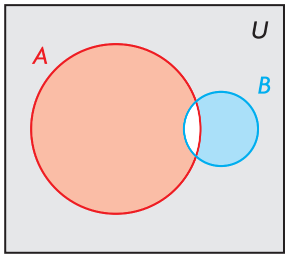
$A \setminus B$ is read as "A minus B." It denotes the set of elements that are in set A but not in set B. It is the red region in each Venn diagram below.



$A \setminus B$ when $B \subset A$



$A \setminus B$ when they are disjoint



$A \setminus B$ when they intersect

F. The set of times that belong to **either** a feeding schedule of every 2 h **or** a feeding schedule of every 3 h forms the **union** of sets R and L, or $R \cup L$. List the elements in $R \cup L$ using set notation.

union

The set of all the elements in two or more sets; in set notation, $A \cup B$ denotes the union of sets A and B; for example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

G. List the elements in $(R \cup L)'$, the complement of the union of R and L. Include these elements in your Venn diagram.

H. Complete each statement with "and" or "or."
- The set $R \cap L$ consists of the elements in set R _____ set L.
- The set $R \cup L$ consists of the elements in set R _____ set L.

I. How does your Venn diagram show the times when Jacquie will need help to feed the two baby animals?

Reflecting

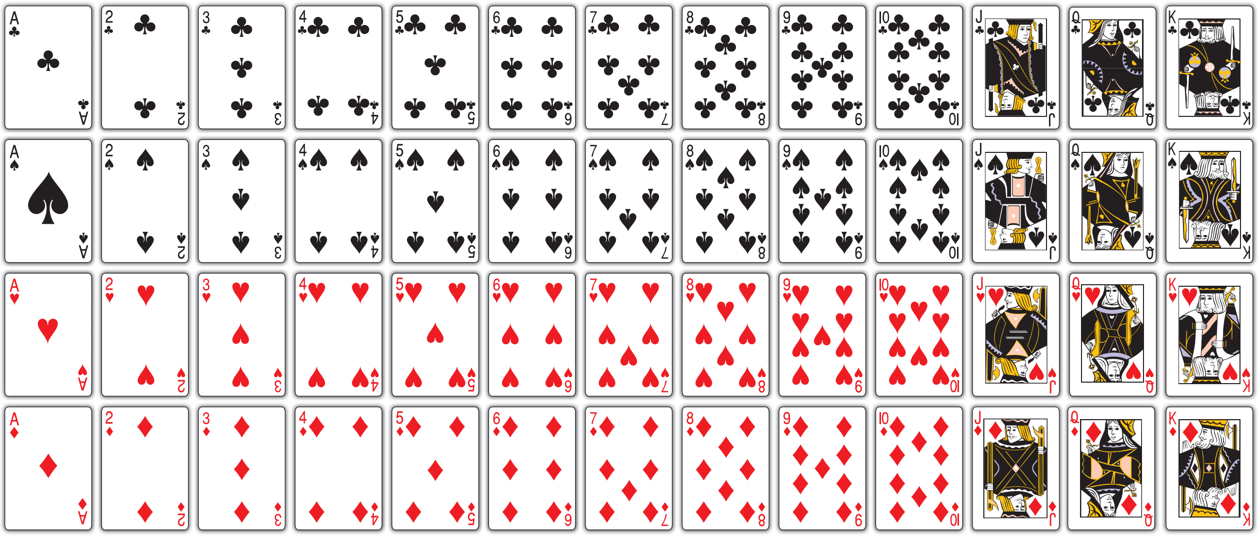
J. Explain whether you agree or disagree with the following statement: The union of any two sets is like the addition of two numbers, so $n(R \cup L) = n(R) + n(L)$. If you disagree with this statement, write the correct formula for $n(R \cup L)$.

K. Write a formula for $n(L \setminus R)$, the number of elements that are in set L but not in set R. Explain your formula. Will your formula also work for disjoint sets?

APPLY the Math

EXAMPLE 1 Determining the union and intersection of disjoint sets

If you draw a card at random from a standard deck of cards, you will draw a card from one of four suits: clubs (C), spades (S), hearts (H), or diamonds (D).



a) Describe sets C, S, H, and D, and the universal set U for this situation.
b) Determine $n(C)$, $n(S)$, $n(H)$, $n(D)$, and $n(U)$.
c) Describe the union of S and H. Determine $n(S \cup H)$.
d) Describe the intersection of S and H. Determine $n(S \cap H)$.
e) Determine whether the events that are described by sets S and H are mutually exclusive, and whether sets S and H are disjoint.
f) Describe the complement of $S \cup H$.

Petra's Solution

a) U = {drawing a card from a deck of 52 cards}
S = {drawing a spade ♠}
H = {drawing a heart ♥}
C = {drawing a club ♣}
D = {drawing a diamond ♦}

I described the sets.

b) $n(U) = 52$
$n(S) = 13$
$n(H) = 13$
$n(C) = 13$
$n(D) = 13$

I used the deck of cards to count the number of elements in each set.

c) $S \cup H = \{$the set of 13 spades and of 13 hearts$\}$
$n(S \cup H) = 26$

The union of S and H consists of 26 cards, either spades or hearts.

d) $S \cap H = \{ \}$
The events described by S and H are mutually exclusive.
$n(S \cap H) = 0$

A card cannot be a spade and a heart, so sets S and H have no common elements. Their intersection is the empty set.

The events that are described by sets S and H (drawing a heart and drawing a spade) must be mutually exclusive.

e) Since the events described by sets S and H are mutually exclusive, these sets are disjoint.

There is no intersection of sets S and H, so these sets are disjoint.

f) $(S \cup H)' = \{$the set of cards that are not hearts or spades, or the set of clubs and diamonds$\}$
$(S \cup H)' = (C \cup D)$

The complement of $S \cup H$ contains the cards in the deck that are not in S or H. These are clubs and diamonds.

Since clubs and diamonds are also disjoint, I can write $(S \cup H)'$ as the union of C and D.

Your Turn

Petra thinks that $n(S) + n(H) = n(S \cup H)$. Is she correct? Explain.

EXAMPLE 2 Determining the number of elements in a set using a formula

The athletics department at a large high school offers 16 different sports:

badminton	hockey	tennis
basketball	lacrosse	ultimate
cross-country running	rugby	volleyball
curling	cross-country skiing	wrestling
football	soccer	
golf	softball	

Determine the number of sports that require the following types of equipment:

a) a ball and an implement, such as a stick, a club, or a racquet
b) only a ball
c) an implement but not a ball
d) either a ball or an implement
e) neither a ball nor an implement

Terry's Solution

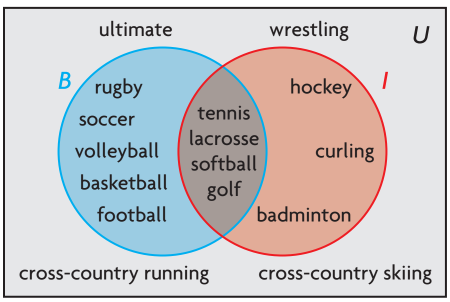
a) U = {sports offered by the athletics department}
B = {sports that use a ball}
I = {sports that use an implement}

I defined the universal set, U, and the subsets I would use.

$n(B) = 9$ $n(I) = 7$

I could have just counted the sports, but I wanted to use an organized method to make sure that I accounted for each sport.

I started to draw a Venn diagram. I drew one circle, B, for sports that use a ball. I drew another circle, I, for sports that use an implement.



Number of elements in the intersection:

$n(B \cap I) = 4$

Therefore, 4 sports use a ball and an implement.

I realized my first diagram could be improved since golf, lacrosse, softball, and tennis were in both circles, so I overlapped the circles and put these sports in the overlap.

Sports that use neither a ball nor an implement are in the box outside the circles. U is the universal set of all sports offered at this school.

I counted. There are 4 sports in the intersection.

b) Number of elements in B minus I:

$$n(B \setminus I) = n(B) - n(B \cap I)$$
$$n(B \setminus I) = 9 - 4$$
$$n(B \setminus I) = 5$$

Therefore, 5 sports use only a ball.

Sports that use only a ball are in the blue circle, but not in the intersection.

c) Number of elements in I minus B:

$$n(I \setminus B) = n(I) - n(B \cap I)$$
$$n(I \setminus B) = 7 - 4$$
$$n(I \setminus B) = 3$$

Therefore, 3 sports use an implement but not a ball.

Sports that use an implement but not a ball are in the red circle, but not in the intersection.

d) Number of elements in union of I and B

$$n(I \cup B) = n(B) + n(I) - n(B \cap I)$$
$$n(I \cup B) = 9 + 7 - 4$$
$$n(I \cup B) = 12$$

Therefore, 12 sports use either a ball or an implement.

Sports that use either a ball or an implement are in the union of the blue and red circles.

I used the **Principle of Inclusion and Exclusion** to determine the number of elements in the union of I and B.

e) Number of elements in the union of the complement of B and I:

$$n((B \cup I)') = n(U) - n(B \cup I)$$
$$n((B \cup I)') = 16 - 12$$
$$n((B \cup I)') = 4$$

Therefore, 4 sports use neither a ball nor an implement.

Sports that use neither a ball nor an implement lie outside the circles. The number of these sports is the complement of $n(I \cup B)$.

I confirmed all my results by counting on my Venn diagram.

Principle of Inclusion and Exclusion

The number of elements in the union of two sets is equal to the sum of the number of elements in each set, less the number of elements in both sets; using set notation, this is written as $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Your Turn

The high school in Example 2 now offers water polo. Organize all the sports in a Venn diagram according to the special headgear and footwear they require. What questions can you answer using your Venn diagram?

EXAMPLE 3 Determining the number of elements in a set by reasoning

Jamaal surveyed 34 people at his gym. He learned that 16 people do weight training three times a week, 21 people do cardio training three times a week, and 6 people train fewer than three times a week.
How can Jamaal interpret his results?

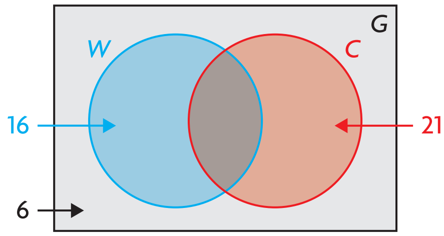
Jamaal's Solution

Number surveyed = 34
Sum of survey data = 16 + 21 + 6 or 43

Since there are more replies than people surveyed, I knew that some people do both types of exercise three times a week.

G = {all the people surveyed at the gym}
W = {people who do weight training}
C = {people who do cardio training}

I defined the universal set G and subsets W and C.

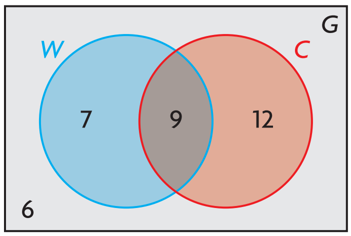


I drew a Venn diagram. I drew two circles for people who exercise three times a week.

I knew that the 6 people who train fewer than three times a week are in neither circle.

Of the 34 people I surveyed, 6 train less than three times a week. So, 34 – 6 or 28 people exercise three times a week. Of these people, 16 do weight training and 21 do cardio training.

I represented the number of people in each region in my Venn diagram.



I learned that 16 + 21 or 37 people do either one or both types of training. So, 37 − 28 or 9 people must do both types of training. I wrote 9 where the circles intersect.

Since 16 people are in set W, 16 − 9 or 7 people must be only in set W.

Similarly, 21 − 9 or 12 people are only in set C.

9 people do cardio training three times a week and do weight training three times a week.
7 people do only weight training three times a week.
12 people do only cardio training three times a week.
6 people train fewer than three times a week.

I summarized my results.

Your Turn

Jamaal surveyed 50 other gym members. Of these members, 9 train fewer than three times a week, 11 do cardio training three times a week, and 16 do both cardio and weight training three times a week. Determine how many of these members do weight training three times a week.

EXAMPLE 4 Correcting errors in Venn diagrams

Morgan surveyed the 30 students in her mathematics class about their eating habits.

- 18 of these students eat breakfast.
- 5 of the 18 students also eat a healthy lunch.
- 3 students do not eat breakfast and do not eat a healthy lunch.

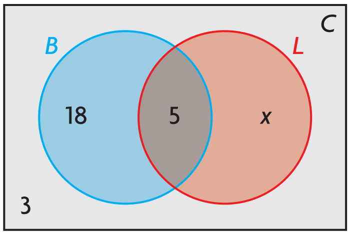
How many students eat a healthy lunch?

Tyler solved this problem, as shown below, but made an error. What error did Tyler make? Determine the correct solution.

Tyler's Solution
Let C represent the universal set, the students in Morgan's mathematics class. Let B represent those who eat breakfast, and let L represent those who eat a healthy lunch.

There are 30 students in total.

I drew a Venn diagram showing the number of elements in each region.



There are 18 students in set B. I put the 5 students who are in sets B and L in the overlap.

There are 3 students who do not belong in either circle. This means there are $30 - 3$ or 27 people in the coloured regions.

$$18 + 5 + x = 27$$
$$x = 4$$

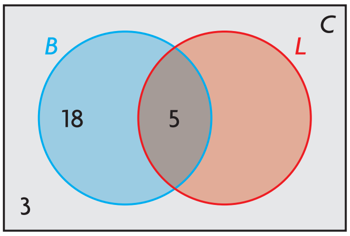
I determined the total number of elements in set L.

$$n(L) = 5 + 4$$
$$n(L) = 9$$

Therefore, 9 students eat a healthy lunch.

Vanessa's Solution

Tyler erred in interpreting the data and placing it on the Venn diagram. He assumed that 18 students ate breakfast but not a healthy lunch.



$n(C) = 30$
$n(B \cup L) = n(C) - n(B \cup L)'$
$n(B \cup L) = 30 - 3$
$n(B \cup L) = 27$

I knew the number of students surveyed.

The set of students who do not eat breakfast or a healthy lunch is $(B \cup L)'$.

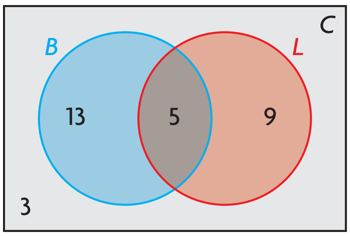
I determined $n(B \cup L)$.

$n(B) = 18$
$n(B \cap L) = 5$
$n(B \setminus L) = n(B) - n(B \cap L)$
$n(B \setminus L) = 18 - 5$
$n(B \setminus L) = 13$

I knew that 18 students eat breakfast and 5 of them also eat a healthy lunch. I determined the number of students who eat breakfast but not a healthy lunch.

$n(B \cup L) = n(B \setminus L) + n(L \setminus B) + n(B \cap L)$
$27 = 13 + n(L \setminus B) + 5$
$9 = n(L \setminus B)$

I determined the number of students who eat a healthy lunch but do not eat breakfast by adding the number of elements in the three regions of the Venn diagram that involve sets *B* and *L*.



I redrew Tyler's Venn using the correct numbers of elements.

$n(L) = n(L \setminus B) + n(B \cap L)$
$n(L) = 9 + 5$
$n(L) = 14$

I determined the total in set *L*.

Therefore, 14 students eat a healthy lunch.

Your Turn

Susan surveyed the 34 students in her science class.

- 14 students eat breakfast.
- 16 students eat a healthy lunch.
- 4 of the 30 students above eat breakfast and a healthy lunch.

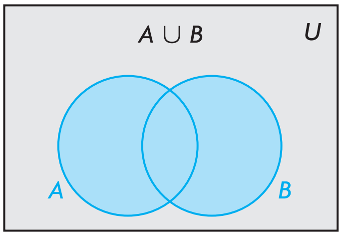
Since 14 + 16 + 4 = 34, Susan concluded that everyone eats either breakfast or a healthy lunch, or both.

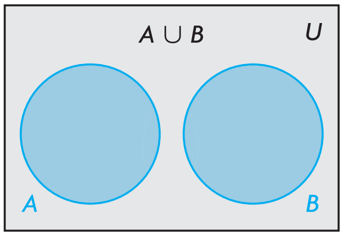
What error did Susan make? How many students do not eat either meal?

In Summary

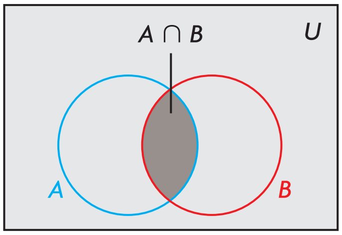
Key Ideas

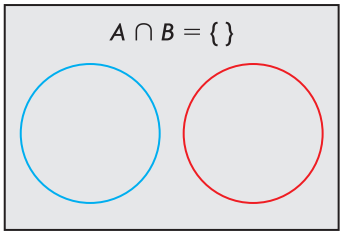
- The **union** of two or more sets, for example, $\boldsymbol{A \cup B}$, consists of all the elements that are in at least one of the sets. It is represented by the entire region of these sets on a Venn diagram. It is indicated by the word "**or**."





- The **intersection** of two or more sets, for example, $\boldsymbol{A \cap B}$, consists of all the elements that are common to these sets. It is represented by the region of overlap on a Venn diagram. It is indicated by the word "**and**."



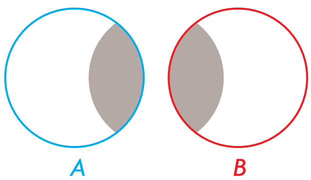


Need to Know

- If two sets, A and B, contain common elements, the number of elements in A or B, $n(A \cup B)$, is:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

This is called the Principle of Inclusion and Exclusion. To calculate $n(A \cup B)$, subtract the elements in the intersection so they are not counted twice, once in $n(A)$ and once in $n(B)$.



- If two sets, A and B, are disjoint, they contain no common elements:

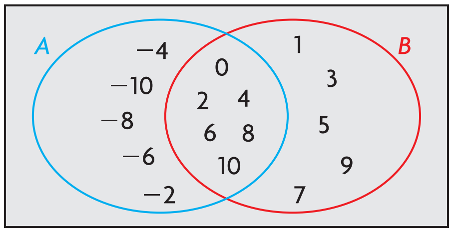
$$n(A \cap B) = 0 \text{ and}$$
$$n(A \cup B) = n(A) + n(B)$$

- Elements that are in set A but not in set B are expressed as $A \setminus B$. The number of elements in A or B, $n(A \cup B)$, can also be determined as follows:

$$n(A \cup B) = n(A \setminus B) + n(B \setminus A) + n(A \cap B)$$

CHECK *Your Understanding*

1. Consider the following Venn diagram:
   a) Determine $A \cup B$.
   b) Determine $n(A \cup B)$.
   c) Determine $A \cap B$.
   d) Determine $n(A \cap B)$.



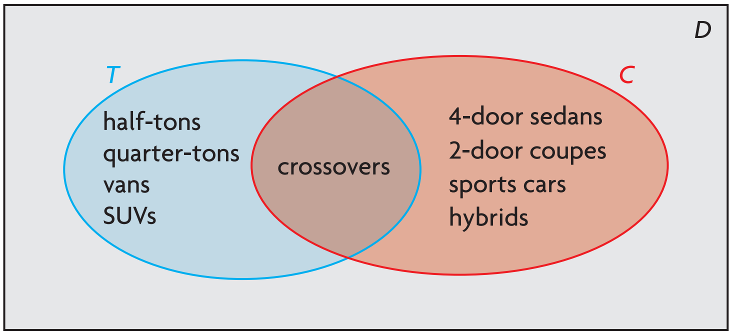

The Canadian lynx prefers dense forest.

2. Animals that are native to the tundra, the ecosystems of Canada's Far North, include the Arctic fox, caribou, ermine, grizzly bear, muskox, and polar bear. The taiga, ecosystems south of the tundra, consists primarily of coniferous forest. Animals that are native to the taiga include the bald eagle, Canadian lynx, grey wolf, grizzly bear, long-eared owl, and wolverine.
   a) Determine the union and intersection of these two sets of animals.
   b) Draw a Venn diagram of these two sets.
3. Consider the following two sets:
   - $A = \{-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10\}$
   - $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$

   a) Determine $A \cup C$, $n(A \cup C)$, $A \cap C$, and $n(A \cap C)$.
   b) Draw a Venn diagram to show these two sets.

PRACTISING

4. The following Venn diagram shows the types of vehicles at a car dealership:



   a) Determine $T \cup C$.
   b) Determine $n(T \cup C)$.
   c) Determine $T \cap C$.


The takin is the national animal of Bhutan.

5. Animals that are native to Africa include the lion, camel, giraffe, hippopotamus, and elephant. Animals that are native to Asia include the elephant, tiger, takin, and camel.
   a) Draw a Venn diagram to show these two sets of animals.
   b) Determine the union and intersection of these two sets.

6. Consider the following two sets:
- $A = \{j \mid j = 2x, -3 \le x \le 6, x \in I\}$
- $B = \{k \mid k = 3x, -4 \le x \le 5, x \in I\}$

a) Draw a Venn diagram to show these two sets.
b) Determine $A \cup B$, $n(A \cup B)$, $A \cap B$, and $n(A \cap B)$.


Sherlock Holmes and Hercule Poirot are fictional characters.

7. Rosie asked 25 people at a mystery convention if they liked Sherlock Holmes or Hercule Poirot.
- 4 people did not like either detective.
- 16 people liked Sherlock Holmes.
- 11 people liked Hercule Poirot.

Determine how many people liked both detectives, how many liked only Sherlock Holmes, and how many liked only Hercule Poirot.

8. Tashi asked 80 people if they liked vanilla or chocolate ice cream.
- 9 people did not like either flavour.
- 11 people liked both vanilla and chocolate.
- 20 people liked only vanilla.

Determine how many people liked only chocolate.

9. John asked 26 people at a gym if they liked to ski or swim.
- 5 people did not like to do either sport.
- 19 people liked to ski.
- 14 people liked to swim.

Determine how many people liked to ski and swim.

10. Tiffany volunteers in an elementary classroom. She is helping students understand multiples of 2 and 3 in mathematics. The students are working with the numbers 1 to 30. How can Tiffany use a Venn diagram to show the students how the multiples relate to one another?

11. Mark surveyed 100 people at a local doughnut shop.
- 65 people ordered coffee.
- 45 people ordered a doughnut.
- 10 people ordered something else.

Mark wants to determine how many people ordered coffee and a doughnut.

a) Model this situation with sets. Identify the universal set, and explain what subsets you will use.
b) Draw a Venn diagram to model this situation. Explain what each part of your diagram represents.
c) Determine how many people ordered coffee and a doughnut.

12. At a retirement home, 100 seniors were interviewed.
- 16 seniors like to watch television and listen to the radio.
- 67 seniors like to watch television.

Determine how many seniors prefer to listen to the radio only.

13. In Edmonton, Anita asked 56 people if they had been to the Calgary Stampede or the Pacific National Exhibition (PNE).
- 14 people had not been to either.
- 30 people had been to the Calgary Stampede.
- 22 people had been to the PNE.

Determine how many people had been to both events.

14. Armour is a real estate agent. He asked 54 clients where they live now.
- 31 people own their home.
- 30 people live in a condominium.
- 9 people rent their house.

Determine how many people own the condominium in which they live.

15. Jamal asked 32 people what type of television shows they like.
- 13 people like reality shows but not contest shows.
- 9 people like contest shows but not reality shows.
- 4 people like neither type of show.

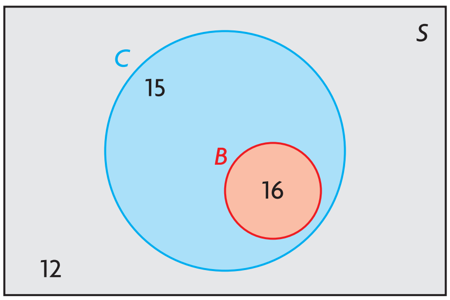
Determine how many people like both types of shows.

16. Beyondé solved the following problem:

A total of 48 students were asked how they got to school.
- 31 students drive a car.
- 16 students take a bus.
- 12 students do not drive a car or take a bus.
- Some students drive a car or take a bus.

Determine how many students do not take a bus to school.

Beyondé's Solution



15 students drive a car but do not take a bus, 12 students do neither. So, 27 students do not take a bus.

The total of the three numbers is 59. So, I knew that the region for students who take a bus overlaps the region for students who drive a car.

I drew a Venn diagram with 31 students in the car region and 16 students in the bus region.

Is Beyondé correct? Justify your answer.

17. Given:

$n(A) + n(B) = n(A \cup B)$ and

$n(A) + n(C) > n(A \cup C)$

a) Which sets do you know are disjoint?

b) Which sets do you know intersect?

c) Are there any sets that could either be disjoint or intersect? If so, which sets? Explain.

Closing

18. Which is more like the addition of two numbers: the union of two sets or the intersection of two sets? Explain.

Extending

19. The Arctic Winter Games include alpine skiing, cross-country skiing, free-style skiing, badminton, basketball, snowshoe biathlon, ski biathlon, curling, dog mushing, figure skating, gymnastics, hockey, indoor soccer, snowboarding, snowshoeing, speed skating, table tennis, volleyball, and wrestling. There are also two categories of sports that are unique to the Arctic, called Arctic Sports and Dene Games.

a) Determine a way to sort the games into sets and subsets.

b) List each set and subset.

c) Draw a Venn diagram to illustrate the sets.

d) Compare your results with your classmates' results. Is there more than one way to sort the games?

The Arctic Winter Games give youth who live in the North the opportunity to meet each other. Historically, participants come from Yukon, Northwest Territories, Nunavut, Northern Alberta, Northern Québec, Alaska, Greenland, Russia, and the Sami people of northern Europe.

History | Connection

Unexpected Infinities

Things do not always turn out the way you might expect. For example, most people would expect two infinite sets to be the same "size." But Georg Cantor, who developed set theory, showed that the set of natural numbers and the set of real numbers contain a *different* number of elements.

A. Set theory gives rise to other situations that are paradoxical or counterintuitive (not what you would expect). For example, explore the "barber paradox" on the Internet.

B. Research other paradoxes and counterintuitive ideas associated with set theory.

Georg Cantor (1845–1918) was born in Russia and spent his career teaching and researching at German universities.