8.6 Scale Factors and 3-D Objects

YOU WILL NEED

- calculator
- ruler
- linking cubes
- pattern blocks
- rectangular blocks

EXPLORE...

- Suppose that you are trying to decide whether you should order an 8 in. pizza or a 16 in. pizza. How much more pizza will you have if you order the 16 in. pizza instead of the 8 in. pizza? What assumptions are you making?

GOAL

Solve problems that involve scale factor, surface area, and volume.

INVESTIGATE the Math

Rostrum blocks are used as props in drama productions. They are three-dimensional objects of various sizes, such as cubes, right rectangular prisms, right triangular prisms, and right cylinders. Many of these objects have a horizontal top surface to stand on, sit on, lean on, or rest other props on.

Kayley, a set carpenter, has been asked to create sets of similar rostrum blocks. She needs to know the surface areas of the blocks, so she can determine how much material she will require to build them. She also needs to know their volumes, so she can predict how much space they will take up in the storage area between shows.

? What is the relationship between the scale factor and the surface areas of two similar objects? What is the relationship between the scale factor and the volumes of two similar objects?

A. Suppose that Kayley wants to make a set of similar cubes. Use linking cubes to act as models.

Measure the dimensions of one linking cube, and determine its surface area and volume. Create a table like the one below, and record your findings.

Length (cm)	Width (cm)	Height (cm)	Surface Area (cm^2)	Volume (cm^3)

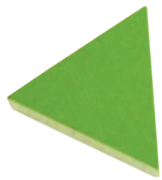
B. Starting with a single cube, add enough cubes to create a larger cube with dimensions that are double the dimensions of the single cube. Record the dimensions, surface area, and volume of the larger cube in your table.

C. Add more cubes to create a cube with dimensions that are three times greater than the dimensions of a single cube. Record the dimensions, surface area, and volume of the larger cube in your table.

D. Predict the surface area and volume of a cube with dimensions that are four times greater than those of a single cube. Check your predictions, and then record the dimensions, surface area, and volume in your table.

E. As the cube grows larger, how does its surface area relate to the scale factor k and the surface area of the original cube? How does its volume relate to the scale factor k and the volume of the original cube?

F. Suppose that Kayley wants to create a set of right triangular prisms. Repeat parts A to E using triangular pattern blocks and a table like the one below.

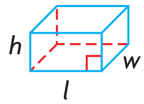
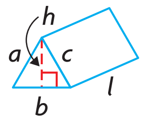
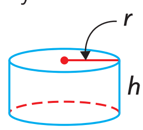
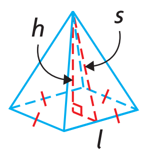
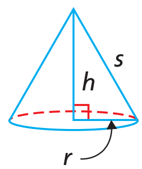
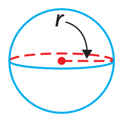


Side Length of Base (in.)	Height of Triangular Base (in.)	Height of Prism (in.)	Surface Area (in.^{2})	Volume (in.^{3})

G. Suppose that Kayley wants to create a set of right prisms with rectangular bases. Repeat parts A to E using rectangular blocks and a table like the one shown in part A.

H. **i)** Make a **conjecture** about the relationship among the surface area of the original object, the scale factor, and the surface area of a larger similar object.

ii) Make a conjecture about the relationship between the volume of the original object, the scale factor, and the volume of a larger similar object.

Formulas	
Object	**Surface Area and Volume**
rectangular prism 	$SA = 2(lw + lh + wh)$ $V = lwh$
right triangular prism 	$SA = bh + l(a + b + c)$ $V = \frac{1}{2}bhl$
right cylinder 	$SA = 2\pi r^2 + 2\pi rh$ $V = \pi r^2 h$
right pyramid 	$SA = l^2 + 2ls$ $V = \frac{1}{3}l^2 h$
right cone 	$SA = \pi r^2 + \pi rs$ $V = \frac{1}{3}\pi r^2 h$
sphere 	$SA = 4\pi r^2$ $V = \frac{4}{3}\pi r^3$

Reflecting

I. Do you think your conjectures will hold when you decrease the dimensions of an object by a specific scale factor? Explain.

J. Do you think your conjectures will hold for other similar objects, such as pyramids, cones, cylinders, or spheres? Explain.

K. Do you think your conjectures will hold for any pair of similar 3-D objects? Explain.

APPLY the Math

EXAMPLE 1 Reasoning about relationships among scale factor, surface area, and volume

Prove the scaling conjectures for the surface area and volume of a rectangular right prism with dimensions $l \times w \times h$.

Connor's Solution: Proving the conjecture for surface area

The surface area of the original rectangular right prism, $SA_{original}$, can be expressed as

I decided to prove the conjecture for surface area.

$$SA_{original} = 2(lw + lh + wh)$$

Let k be the scale factor.
The surface area of a new scaled rectangular right prism, SA_{new}, can be expressed as

The rectangular right prism will be scaled by a factor of k. To ensure that the scaled prism is similar to the original, each of the dimensions of the original prism must be multiplied by k.
$l_{new} = k \cdot l_{original}$
$w_{new} = k \cdot w_{original}$
$h_{new} = k \cdot h_{original}$

$$SA_{new} = 2[(kl)(kw) + (kl)(kh) + (kw)(kh)]$$

$$SA_{new} = 2[k^2(lw) + k^2(lh) + k^2(wh)]$$

$$SA_{new} = 2k^2[(lw) + (lh) + (wh)]$$

$$SA_{new} = k^2(SA_{original})$$

k^2 is a common factor.

The expression inside the brackets is the same as the expression for the surface area of the original prism.

The conjecture is valid for the surface area of a rectangular right prism.

Isabelle's Solution: Proving the conjecture for volume

The volume of the original rectangular right prism, $V_{original}$ can be expressed as

I decided to prove the conjecture for volume.

$$V_{original} = lwh$$

Let k be the scale factor.

The rectangular right prism will be scaled by a factor of k, so each of the dimensions will be multiplied by k.

The volume of a new scaled rectangular right prism, V_{new}, can be expressed as

$$V_{new} = (kl)(kw)(kh)$$

$$V_{new} = k^3lwh$$

$V_{new} = k^3(lwh)$
$V_{new} = k^3(V_{original})$

The expression inside the brackets is the same as the expression for the volume of the original prism.

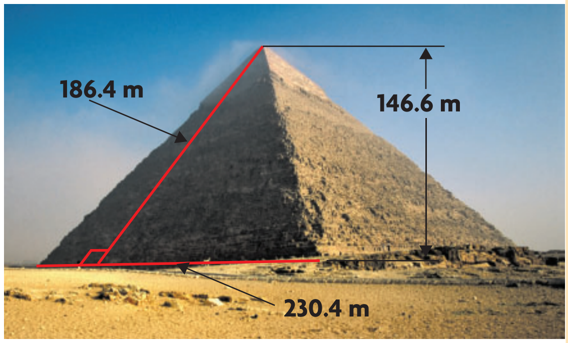
The conjecture is valid for the volume of a rectangular right prism.

Your Turn

Prove the scaling conjectures for the surface area and volume of a sphere.

EXAMPLE 2 Solving a surface area problem

The Great Pyramid of Giza in Egypt was built on a square base, with the dimensions shown.



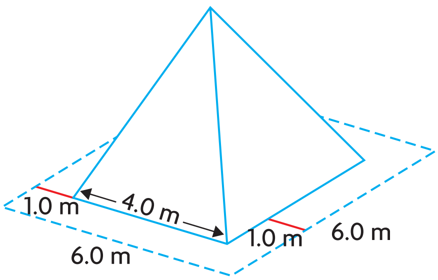
An artist who works with plate glass wants to build a replica of the pyramid for an installation at an art gallery. The artist is restricted by the floor dimensions, which are 6.0 m by 6.0 m, and the ceiling height of 3.5 m. As well, the glass sculpture must have room for a 1.0 m walkway around its base.

a) What scale factor might the artist use to build the sculpture?

b) How much glass will the artist need to build the sculpture?

Twila's Solution

a)



I drew a diagram to show the situation. The artist's sculpture will be a pyramid that is similar to the actual pyramid in Giza. Since 1.0 m of clearance is needed on each side of the sculpture's base, the longest possible side length is 4.0 m.

$$k = \frac{\text{Side length for base of original pyramid}}{\text{Side length for base of sculpture}}$$

$$k = \frac{230.4 \text{ m}}{4.0 \text{ m}}$$

$$k = 57.6$$

$$k \doteq 60$$

Since I knew the measures of the side lengths of both bases, I chose to determine a scale factor, k, that could be used to create the sculpture. Then I rounded up my answer to a convenient value. If I had rounded down, the sculpture would not have fit, as $\frac{1}{60}$ is less than $\frac{1}{57.6}$.

I'll test a scale factor of $\frac{1}{60}$ to determine the height of the sculpture from the height of the original pyramid.

The height of the sculpture must be less than 3.5 m.

Height of sculpture = (Scale factor)(Height of original pyramid)

$$\text{Height of sculpture} = \left(\frac{1}{60}\right)(146.6 \text{ m})$$

I checked that the height of the sculpture would fit in the display area.

Height of sculpture = 2.443... m

2.4 m is less than 3.5 m, so the sculpture will fit.

The artist could use a scale factor of $\frac{1}{60}$.

b) $$\frac{SA_{\text{sculpture}}}{SA_{\text{original}}} = k^2$$

I knew the dimensions of the original pyramid. I can determine the surface area of the sculpture, $SA_{\text{sculpture}}$, by calculating the surface area of the original, SA_{original}.

$$SA_{\text{sculpture}} = k^2(SA_{\text{original}})$$

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2 (SA_{\text{original}})$$

Since the base is a square, each of the triangular faces is congruent.

The expression in square brackets is the formula for the surface area of a square-based pyramid.

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2 [\text{Area of base} + 4(\text{Area of each triangular face})]$$

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2 \left[b^2 + 4\left(\frac{bs}{2}\right)\right]$$

In the original pyramid, b is the length of one of the sides of the square base, which is 230.4 m. In the original pyramid, s is the slant height (altitude) of each triangular face, which is 186.4 m.

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2 \left[(230.4 \text{ m})^2 + 4\left(\frac{(230.4 \text{ m})(186.4 \text{ m})}{2}\right)\right]$$

$$SA_{\text{sculpture}} = 38.604... \text{ m}^2$$

The artist will need about 38.6 m^2 of glass to build the sculpture in the given space.

Your Turn

The gift shop at the art gallery would like to sell miniature replicas of the artist's sculpture. A scale ratio of 1 : 50 will be used to make the replicas.

a) Determine the dimensions of the replicas.

b) Determine the amount of glass that will be needed to make each replica.

EXAMPLE 3 Solving a capacity problem

The smaller tank in the photograph has a capacity of 1400 m^3, and the larger tank has a capacity of 4725 m^3.

a) During the refining process, both tanks are filled with oil from a pumping station at the same rate. How many times longer will it take to fill the larger tank than it will take to fill the smaller tank?

b) How many times greater is the radius of the larger tank than the radius of the smaller tank?

Spherical tanks are often used to store oil and gas at refineries, since this shape is the most economical to build.

Esther's Solution

a)

$$\frac{V_{large}}{V_{small}} = \frac{4725 \cancel{m^3}}{1400 \cancel{m^3}}$$

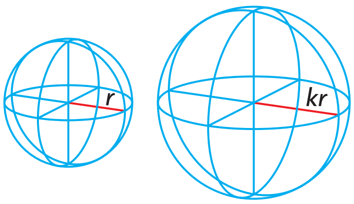
$$\frac{V_{large}}{V_{small}} = 3.375$$

It will take a little more than three times longer to fill the larger tank than it will take to fill the smaller tank.

Since both tanks are being filled at the same rate, the factor that relates the capacities of the tanks will tell me the relationship between the times needed to fill the tanks. I decided to use V_{large} to represent the capacity (interior volume) of the large tank and V_{small} to represent the capacity of the small tank.

The larger tank holds 3.375 times more oil than the smaller tank.

b)



$$V_{large} = k^3 \cdot V_{small}$$

$$\frac{V_{large}}{V_{small}} = k^3$$

$$3.375 = k^3$$

$$\sqrt[3]{3.375} = k$$

$$1.5 = k$$

The inner radius of the larger tank is 1.5 times greater than the inner radius of the smaller tank.

The only dimension that varies in a sphere is the radius. Therefore, any two spheres are similar objects, with radii related by a scale factor of k.

The capacities of the tanks are related by a factor of 3.375. Therefore, the scale factor that relates the radii of the tanks is the cube root of this number.

Your Turn

The larger tank is going to be reduced by a scale factor of 0.6 to build another small tank. Determine the capacity of the new small tank.

In Summary

Key Idea

- If two 3-D objects are similar and their dimensions are related by the scale factor k, then
  - Surface area of similar object $= k^2$(surface area of original object)
  - Volume of similar object $= k^3$(volume of original object)

Need to Know

- If you know the dimensions of a scale diagram or model of a 3-D object, as well as the scale factor used to enlarge/reduce from the diagram or model, you can determine the surface area and volume of the enlarged/reduced object, without determining its dimensions.

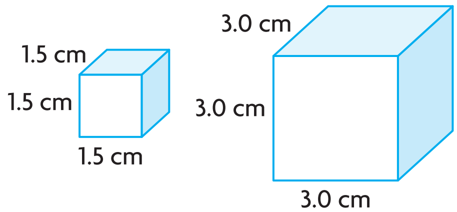
CHECK Your Understanding

1. Each pair of objects is similar.

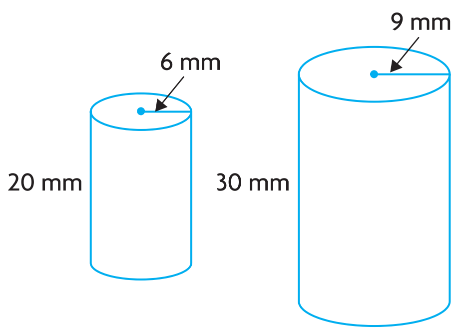
i) By what factor is the surface area of the larger object greater than the surface area of the smaller object?

ii) By what factor is the volume of the larger object greater than the volume of the smaller object?

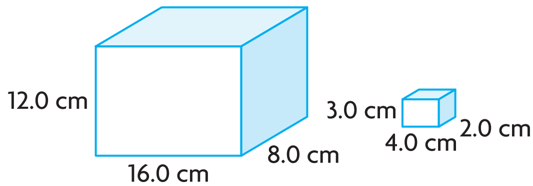
a)



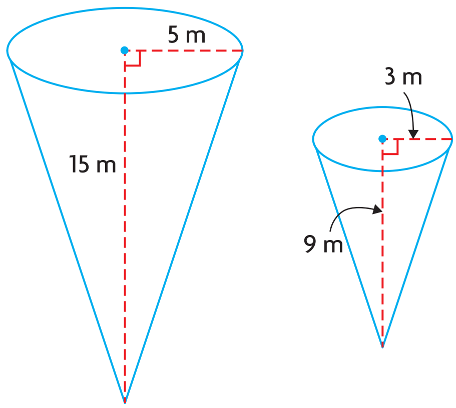
b)

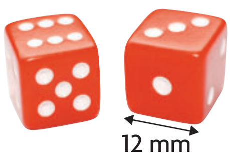


c)



d)





actual dice

2. A stage director needs a pair of large dice for a scene with children playing a board game. He estimates that the measure of each edge of each enlarged die must be 600 mm.

a) What scale factor must he apply to create the enlarged dice?

b) How many times greater will the surface area of each larger die be?

c) How many times greater will the volume of each larger die be?

3. A model of a ship is built to a scale ratio of 1 : 30. The model is 16 cm tall, and the area of one sail is 8.5 cm^2. What are the corresponding measurements of the actual ship?

PRACTISING

4. An oil tank has a capacity of 32 m^3. A similar oil tank has dimensions that are larger by a scale factor of 3. What is the capacity of the larger tank?

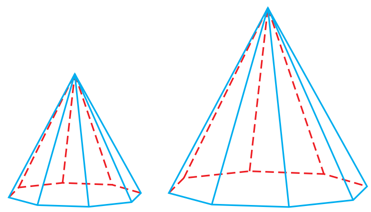
5. A soft-cover book will be modified so that it has large print for people who are visually impaired. To maintain the same number of pages, both the print size and page dimensions will be tripled.

a) The area of each page in the original book is 500 cm^2. Determine the area of each page in the large-print book.

b) The same type of paper will be used for the pages in the large-print book. By what factor will the volume of the paper change? Justify your answer.

6. Brenda is a potter. She is creating two similar vases, with their dimensions related by a scale factor of $\frac{3}{4}$. The larger vase has a volume of 9420 cm^3. Determine the volume of the smaller vase.

7. The dimensions of a right octagonal pyramid are enlarged by a scale factor of 1.5. Determine the value of each of the following ratios.



a) $\dfrac{\text{Volume of large pyramid}}{\text{Volume of small pyramid}}$

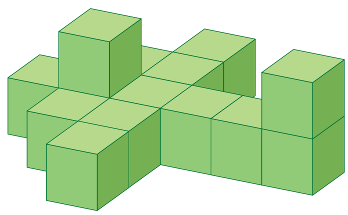
b) $\dfrac{\text{Surface area of large pyramid}}{\text{Surface area of small pyramid}}$

c) $\dfrac{\text{Base perimeter of large pyramid}}{\text{Base perimeter of small pyramid}}$

8. A jewellery box has a volume of 4500 cm^3. Its lid has a surface area of 375 cm^2. If each dimension of the jewellery box is tripled to create a prop for a theatre production, by what factors would the surface area of the lid and the volume of the box increase?

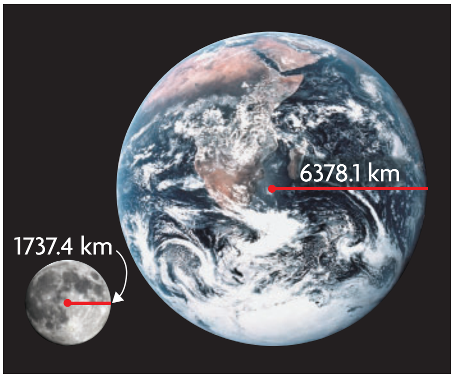
9. Celine's grandmother brought her a set of Russian dolls from St. Petersburg. The dolls stack inside each other and are similar to each other. The diameters of the two smallest dolls are 2.0 cm and 3.5 cm. The scale factor is the same from each doll to the next larger doll. Celine estimates the smallest doll has a volume of about 8 cm^3. Estimate the volume of the largest of the five dolls.

10. Mario made a scale model of an airplane using linking cubes.



a) How many linking cubes would he need to make a model with dimensions five times as large?

b) By what factor is the surface area of the new model greater than the surface area of the first model?



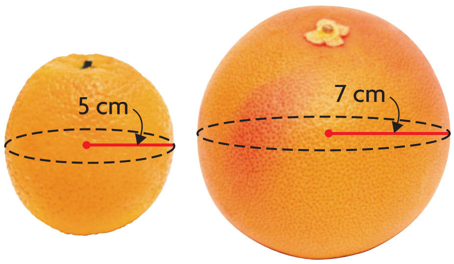
11. Adele wants to compare Earth and the Moon by creating spherical models. She has decided to represent Earth with a sphere that has a radius of 10.0 cm.

a) What is the radius of the sphere she should use to represent the Moon? Round your answer to the nearest tenth of a centimetre.

b) Determine the ratio that compares the circumference of the model of Earth to the circumference of the model of the Moon.

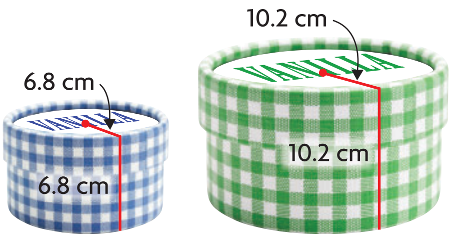
c) Determine the ratio that compares the surface area of the model of Earth to the surface area of the model of the Moon.

d) Determine the ratio that compares the volume of the model of Earth to the volume of the model of the Moon.



12. Markian likes both oranges and grapefruits. He wonders how much more fruit he gets in a grapefruit. Estimate how many times greater the volume of a grapefruit is, compared with the volume of an orange.

13. A baseball has a diameter of about 2.9 in. A softball has a diameter of about 3.8 in. By what percent is the amount of leather needed to cover the softball greater than the amount of leather needed to cover the baseball?



14. Josephine packages her ice cream in right cylindrical cardboard containers, in the two different sizes shown.

a) Determine the factor by which the height of the letters in "VANILLA" differs on the two containers.

b) Determine the factor by which the surface areas of the lids differ.

c) Determine the factor by which the capacities of the two containers differ.

d) How much ice cream does each container hold, to the nearest cubic centimetre?

15. Suppose that your class is sending shoeboxes filled with school supplies to schools in need after a devastating earthquake. A cardboard manufacturer has donated two sizes of shoeboxes. The small shoebox is 18.0 cm long, 11.5 cm wide, and 9 cm high. The large shoebox is 36.0 cm long, 23 cm wide, and 18 cm high.

a) Matty claims that it will take about twice as much paper to wrap the large shoebox for shipping. Do you agree? Justify your decision.

b) Is the volume of the small shoebox half the volume of the large shoebox? Explain how you know.

16. Show that when the dimensions of each given object are enlarged/reduced by a scale factor of k, the surface area of the resulting similar object has changed by a factor of k^2 and its volume has changed by a factor of k^3.

a) right cylinder

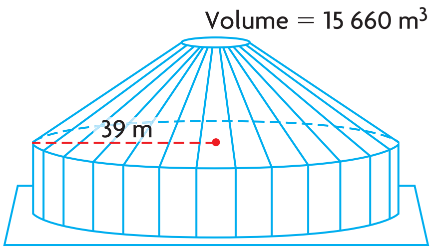
b) right cone

Closing

17. Two similar right rectangular prisms have the dimensions 3 m by 4 m by 2 m and 6 m by 8 m by 4 m. Explain how you can determine the number of smaller prisms that will fit in the larger prism.

Extending

18. A travelling circus holds performances under a large tent, as shown. A smaller version of this circus, which visits smaller communities, uses a similar tent with a volume of 580 m^3. How many times greater is the floor area of the larger tent, compared with the floor area of the smaller tent?



19. A manufacturer has created a spherical model of the Moon, using a scale ratio of 1 : 11 580 000. The model fits exactly into a cubic box with a volume of 27 000 cm^3.

a) Determine the surface area of the Moon.

b) Determine the volume of the Moon.

20. A bakery sells two sizes of birthday cakes. The small cake has a diameter of 10 in., and the large cake has a diameter of 12 in. The small cake sells for $14.00. What should the price of the large cake be? Justify your answer, and state any assumptions you are making.

Math in Action

Are Prices of TVs Scaled Appropriately?

Some products are available in different sizes. Sometimes these products are similar, like flat-screen TVs. TVs are measured by the diagonal of the viewing area. Some common sizes are 32 in., 40 in., and 52 in. The ratio of the length to the width of the viewing area is 16 : 9.

- Work with a partner or in a small group.
- Determine the dimensions of the viewing areas of the three sizes of TVs mentioned above.
- Research to find prices of these sizes of TVs from the same manufacturer.
- Determine if the prices are related by the scale factor.
- Do manufacturers consider scale factor when they set prices? Summarize the results of your findings and give possible reasons for what you found.