

## 7.5

Solving Quadratic Equations  
by Factoring

## GOAL

Solve quadratic equations by factoring.

## LEARN ABOUT the Math

The entry to the main exhibit hall in an art gallery is a parabolic arch. The arch can be modelled by the function

$$h(w) = -0.625w^2 + 5w$$

where the height,  $h(w)$ , and width,  $w$ , are measured in feet. Several sculptures are going to be delivered to the exhibit hall in crates. Each crate is a square-based rectangular prism that is 7.5 ft high, including the wheels. The crates must be handled as shown, to avoid damaging the fragile contents.

## YOU WILL NEED

- calculator

## EXPLORE...

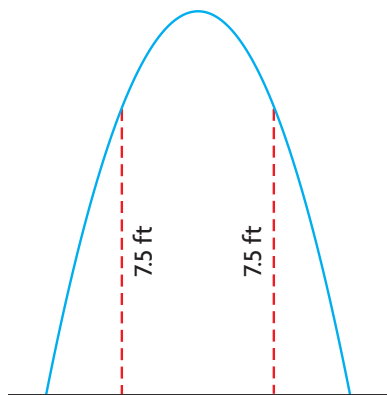
- What values could you substitute for  $n$  and  $x$  to make this equation true?  
 $(3x + n)(7x - 7) = 0$



- ? What is the maximum width of a 7.5 ft high crate that can enter the exhibit hall through the arch?

**EXAMPLE 1****Solving a quadratic equation by factoring**

Determine the distance between the two points on the arch that are 7.5 ft high.

**Brooke's Solution**

I sketched the situation.

The crate can only fit through the part of the arch that is at least 7.5 ft high. The arch is exactly 7.5 ft high at two points.

The following function describes the arch:

$$h(w) = -0.625w^2 + 5w$$

The height of the crate is 7.5 ft.

$$7.5 = -0.625w^2 + 5w$$

I wrote an equation, substituting 7.5 for  $h(w)$ .

$$0.625w^2 - 5w + 7.5 = 0$$

I rewrote the equation in standard form.

I decided to subtract  $-0.625w^2 + 5w$  from both sides so the coefficient of  $w^2$  would be positive.

$$\frac{0.625w^2}{0.625} - \frac{5w}{0.625} + \frac{7.5}{0.625} = \frac{0}{0.625}$$

I divided by 0.625 to simplify the equation.

$$w^2 - 8w + 12 = 0$$

$$(w - 2)(w - 6) = 0$$

I factored the equation.

$$w - 2 = 0 \quad \text{or} \quad w - 6 = 0$$

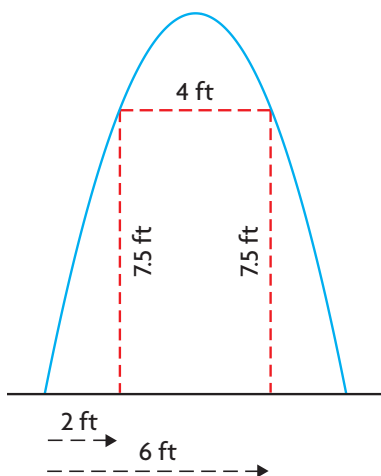
$$w = 2$$

$$w = 6$$

If the product of two factors is 0, then at least one factor must equal 0.

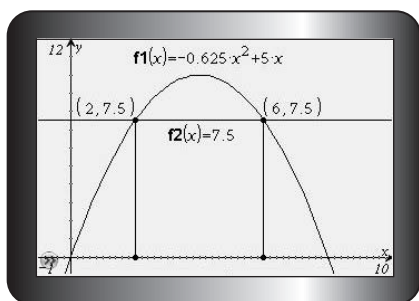
The parabola reaches a height of exactly 7.5 ft at widths of 2 ft and 6 ft.





I determined the difference between the widths to determine the maximum width of the crate.

To fit through the archway, the crate cannot be more than 4 ft wide.



I checked my solution by graphing  $y = -0.625x^2 + 5x$  and  $y = 7.5$ .

The x-coordinates of the points of intersection are 2 and 6, so my solution is correct.

## Reflecting

- How did rewriting the equation in standard form and then factoring it help Brooke determine the roots?
- Was Brooke's decision to divide both sides of the equation by 0.625 (the coefficient of  $w^2$ ) reasonable? Explain.
- Describe another way that Brooke could verify the solutions to her equation.
- Tim says that if you know the roots of an equation, you can use factors to determine the equation. How could Tim use the roots 2 and 6 to determine the equation that Brooke solved?
- Can you always use factoring to solve a quadratic equation? Explain.

## APPLY the Math

### EXAMPLE 2

### Solving a quadratic equation using a difference of squares

Determine the roots of the following equation:

$$75p^2 - 192 = 0$$

Verify your solution.

#### Alberto's Solution

$$75p^2 - 192 = 0$$

$$\frac{75p^2}{3} - \frac{192}{3} = \frac{0}{3}$$

$$25p^2 - 64 = 0$$

$$(5p - 8)(5p + 8) = 0$$

$$5p - 8 = 0 \quad \text{or} \quad 5p + 8 = 0$$

$$5p = 8 \qquad 5p = -8$$

$$p = \frac{8}{5} \qquad p = -\frac{8}{5}$$

The roots are  $\frac{8}{5}$  and  $-\frac{8}{5}$ .

$$75p^2 - 192 = 0$$

$$75p^2 = 192$$

$$p^2 = \frac{192}{75}$$

$$p^2 = \frac{64}{25}$$

$$p = \pm\sqrt{\frac{64}{25}}$$

$$p = \pm\frac{8}{5}$$

I noticed that 3 is a factor of both 75 and 192.

I noticed that  $25p^2$  and 64 are both perfect squares, so  $25p^2 - 64$  is a difference of squares.

I determined the roots.

I decided to verify my solutions by solving the equation using a different method.

I isolated  $p^2$  and then took the square root of each side. I knew that  $p^2$  has two possible square roots, one positive and the other negative.

My solution matched the solution I obtained by factoring.

#### Your Turn

How can you tell that any equation with a difference of squares is factorable?

What can you predict about the roots?

**EXAMPLE 3****Solving a quadratic equation with only one root**

Solve and verify the following equation:

$$4x^2 + 28x + 49 = 0$$

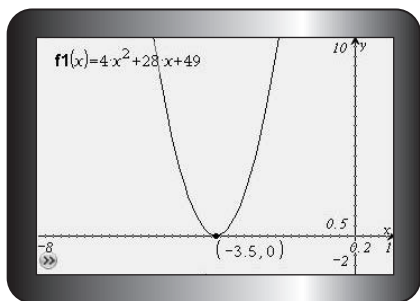
**Arya's Solution**

$$4x^2 + 28x + 49 = 0$$

$$(2x + 7)(2x + 7) = 0$$

$$2x + 7 = 0$$

$$x = -3.5$$



I factored the trinomial. I noticed that both factors are the same, so there is only one root.

I decided to verify my solution by graphing the corresponding quadratic function.

I noticed that the vertex of the function is on the x-axis at  $-3.5$ , so my solution makes sense.

**Your Turn**

How can factoring an equation help you determine whether the equation has two roots or one root?

**EXAMPLE 4****Using reasoning to write an equation from its roots**

Tori says she solved a quadratic equation by graphing. She says the roots were  $-5$  and  $7$ . How can you determine an equation that she might have solved?

**Philip's Solution**

$$x = -5 \quad \text{or} \quad x = 7$$

$$x + 5 = 0 \quad x - 7 = 0$$

One factor is  $x + 5$ .

The other factor is  $x - 7$ .

$$(x + 5)(x - 7) = 0$$

$$x^2 + 5x - 7x - 35 = 0$$

$$x^2 - 2x - 35 = 0$$

The x-intercepts of the quadratic function are the roots of the equation.

I decided to use the roots to help me write the factors of the equation.

I wrote the factors as a product. Since each root is equal to 0, their product is also equal to 0.

I simplified to write the equation in standard form.

## Your Turn

The  $x$ -intercepts of the graph of a quadratic function are 3 and  $-2.5$ . Write a quadratic equation that has these roots.

### EXAMPLE 5 Describing errors in a solution

Matthew solved a quadratic equation as shown. Identify and correct the error in Matthew's solution.

$$\begin{aligned}4x^2 &= 9x \\ \frac{4x^2}{x} &= \frac{9x}{x} \\ 4x &= 9 \\ x &= 2.25\end{aligned}$$

### Raj's Solution

Matthew made an error in the second line of his solution. When he divided both sides by  $x$ , he eliminated a possible factor,  $x = 0$ .

$\frac{4x^2}{x}$  and  $\frac{9x}{x}$  are not defined when  $x = 0$ , so Matthew cannot divide by  $x$ .

Correctly solving the equation:

$$\begin{aligned}4x^2 - 9x &= 0 \\ x(4x - 9) &= 0\end{aligned}$$

To solve the equation, I rewrote it in standard form and then factored the left side.

$$\begin{aligned}x = 0 \quad \text{or} \quad 4x - 9 &= 0 \\ x = 0 \quad \text{or} \quad 4x &= 9 \\ & \quad \quad \quad x = 2.25\end{aligned}$$

For my equation to be true, either  $x$  or  $4x - 9$  must equal 0.

Verify:

$$4x^2 - 9x = 0$$

$$x = 0$$

$$x = 2.25$$

I verified each solution by substituting it into the original equation. For both solutions, the left side is equal to the right side. Therefore, both solutions are correct.

LS	RS	LS	RS
$4x^2 - 9x$	0	$4x^2 - 9x$	0
$4(0)^2 - 9(0)$		$4(2.25)^2 - 9(2.25)$	
$0 - 0$		$20.25 - 20.25$	
0		0	
LS = RS		LS = RS	

## Your Turn

What number will always be a root of an equation that can be written in standard form as  $ax^2 + bx = 0$ ? Explain how you know.

## In Summary

### Key Idea

- Some quadratic equations can be solved by factoring.

### Need to Know

- To factor an equation, start by writing the equation in standard form.
- You can set each factor equal to zero and solve the resulting linear equations. Each solution is a solution to the original equation.
- If the two roots of a quadratic equation are equal, then the quadratic equation is said to have one solution.

## CHECK Your Understanding

1. Solve by factoring. Verify each solution.  
a)  $x^2 - 11x + 28 = 0$       c)  $2y^2 + 11y + 5 = 0$   
b)  $x^2 - 7x - 30 = 0$       d)  $4t^2 + 7t - 15 = 0$
2. Solve by factoring.  
a)  $x^2 - 121 = 0$       e)  $s^2 - 12s + 36 = 0$   
b)  $9r^2 - 100 = 0$       f)  $16p^2 + 8p + 1 = 0$   
c)  $x^2 - 15x = 0$       g)  $-14z^2 + 35z = 0$   
d)  $3y^2 + 48y = 0$       h)  $5q^2 - 9q = 0$

## PRACTISING

3. Solve by factoring. Verify each solution.  
a)  $x^2 - 9x - 70 = 0$       c)  $3a^2 + 11a - 4 = 0$   
b)  $x^2 + 19x + 48 = 0$       d)  $6t^2 - 7t - 20 = 0$
4. Solve each equation.  
a)  $12 - 5x = 2x^2$       c)  $49d^2 + 9 = -42d$   
b)  $4x^2 = 9 - 9x$       d)  $169 = 81g^2$
5. Geeta solved this equation:  
$$20x^2 - 21x - 27 = 0$$
  
Her solutions were  $x = 0.75$  and  $x = -1.8$ .  
a) Factor and solve the equation.  
b) What error do you think Geeta made?
6. Determine the roots of each equation.  
a)  $5u^2 - 10u - 315 = 0$       c)  $1.4y^2 + 5.6y - 16.8 = 0$   
b)  $0.25x^2 + 1.5x + 2 = 0$       d)  $\frac{1}{2}k^2 + 5k + 12.5 = 0$

7. The graph of a quadratic function has  $x$ -intercepts  $-5$  and  $-12$ . Write a quadratic equation that has these roots.
8. A bus company charges \$2 per ticket but wants to raise the price. The daily revenue that could be generated is modelled by the function

$$R(x) = -40(x - 5)^2 + 25\,000$$

where  $x$  is the number of 10¢ price increases and  $R(x)$  is the revenue in dollars. What should the price per ticket be if the bus company wants to collect daily revenue of \$21 000?



9. Solve and verify the following equation:

$$5x - 8 = 20x^2 - 32x$$

10. Identify and correct any errors in the following solution:

$$\begin{aligned} 5a^2 - 100 &= 0 \\ 5a^2 &= 100 \\ a^2 &= 25 \\ \sqrt{a^2} &= \sqrt{25} \\ a &= 5 \end{aligned}$$

11. Identify and correct the errors in this solution:

$$\begin{aligned} 4r^2 - 9r &= 0 \\ (2r - 3)(2r + 3) &= 0 \\ 2r - 3 &= 0 \quad \text{or} \quad 2r + 3 = 0 \\ 2r &= 3 & 2r &= -3 \\ r &= 1.5 \quad \text{or} \quad r &= -1.5 \end{aligned}$$

12. a) Write a quadratic function with zeros at  $0.5$  and  $-0.75$ .  
 b) Compare your function with a classmate's function. Did you get the same function?  
 c) Working with a classmate, determine two other possible functions with the same zeros.

13. Sanela sells posters to stores. The profit function for her business is

$$P(n) = -0.25n^2 + 6n - 27$$

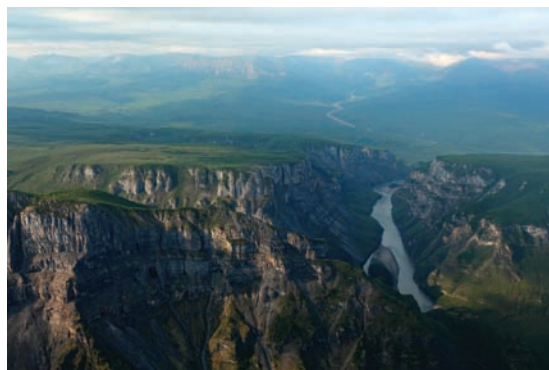
where  $n$  is the number of posters sold per month, in hundreds, and  $P(n)$  is the profit, in thousands of dollars.

- How many posters must Sanela sell per month to break even?
- If Sanela wants to earn a profit of \$5000 ( $P(n) = 5$ ), how many posters must she sell?
- If Sanela wants to earn a profit of \$9000, how many posters must she sell?
- What are the domain and range of the profit function?  
Explain your answer.

14. Samuel is hiking along the top of First Canyon on the South Nahanni River in the Northwest Territories. When he knocks a rock over the edge, it falls into the river, 1260 m below. The height of the rock,  $h(t)$ , at  $t$  seconds, can be modelled by the following function:

$$h(t) = -25t^2 - 5t + 1260$$

- How long will it take the rock to reach the water?
  - What is the domain of the function? Explain your answer.
15. a) Create a quadratic equation that can be solved by factoring. Exchange equations with a classmate, and solve each other's equations.  
b) Modify the equation you created so that it cannot be factored. Explain how you modified it. Then exchange equations with a classmate again, and solve each other's equations a different way.



The Nahanni River in the Northwest Territories is a popular area for wilderness tours and whitewater rafting.

## Closing

- a) Explain the steps you would follow to solve a quadratic equation by factoring.  
b) When does it make sense to solve a quadratic equation by factoring? When does it make sense to use graphing?

## Extending

- One root of an equation in the form  $ax^2 + c = 0$  is 6.
  - What can you predict about the factors if there is no  $bx$  term in the equation?
  - Determine the other root.
  - Write the equation in factored form.
  - Write the equation in standard form.
- The perimeter of this right triangle is 60 cm. Determine the lengths of all three sides.

