7.4 Factored Form of a Quadratic Function

YOU WILL NEED

- graph paper and ruler OR graphing technology

GOAL

Relate the factors of a quadratic function to the characteristics of its graph.

EXPLORE...

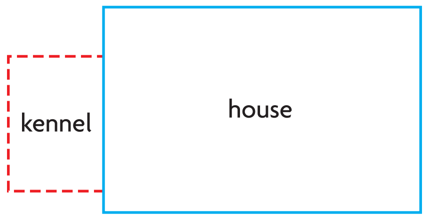
- John has made a catapult to launch baseballs. John positions the catapult and then launches a ball. The height of the ball, $h(t)$, in metres, over time, t, in seconds, can be modelled by the function $h(t) = -4.9t^2 + 14.7t$ From what height did John launch the ball? How long was the ball in the air?

Communication *Tip*

A quadratic function is in factored form when it is written in the form $y = a(x - r)(x - s)$

INVESTIGATE the Math

Ataneq takes tourists on dogsled rides. He needs to build a kennel to separate some of his dogs from the other dogs in his team. He has budgeted for 40 m of fence. He plans to place the kennel against part of his home, to save on materials.



? What dimensions should Ataneq use to maximize the area of the kennel?

A. Using x to represent the width of the kennel, create an expression for the length of the kennel.

B. Write a function, in terms of x, that defines the area of the kennel. Identify the factors in your function.

C. Create a table of values for the function, and then graph it.

D. Does the function contain a maximum or a minimum value? Explain.

E. Determine the x-intercepts of the parabola.

F. Determine the equation of the axis of symmetry of the parabola and the coordinates of the vertex.

G. What are the dimensions that maximize the area of the kennel?

Reflecting

H. How are the x-intercepts of the parabola related to the factors of your function?

I. Explain why having a quadratic function in factored form is useful when graphing the parabola.

APPLY the Math

EXAMPLE 1 Graphing a quadratic function given in standard form

Sketch the graph of the quadratic function:

$$f(x) = 2x^2 + 14x + 12$$

State the domain and range of the function.

Arvin's Solution

$f(x) = 2x^2 + 14x + 12$
The coefficient of x^2 is 2,
so the parabola opens upward.

The parabola opens upward when a is positive in the standard form of the function.

$f(x) = 2(x^2 + 7x + 6)$
$f(x) = 2(x + 1)(x + 6)$

I factored the expression on the right side so that I could determine the zeros of the function.

Zeros:
$0 = 2(x + 1)(x + 6)$
$x + 1 = 0$ or $x + 6 = 0$
$x = -1$ $x = -6$
The x-intercepts are $x = -1$ and $x = -6$.

To determine the zeros, I set $f(x)$ equal to zero. I knew that a product is zero only when one or more of its factors are zero, so I set each factor equal to zero and solved each equation.

The values of x at the zeros of the function are also the x-intercepts.

y-intercept:
$f(0) = 2(0 + 1)(0 + 6)$
$f(0) = 2(1)(6)$
$f(0) = 12$
The y-intercept is 12.

I knew that the y-intercept is 12 from the standard form of the quadratic function. However, I decided to verify that my factoring was correct.

I noticed that this value can be obtained by multiplying the values of a, r, and s from the factored form of the function: $f(x) = a(x - r)(x - s)$

Axis of symmetry:

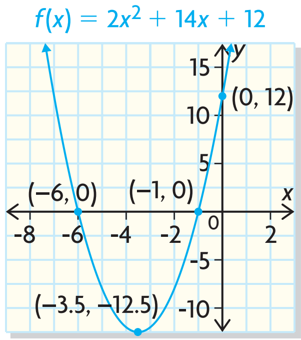
$$x = \frac{-6 + (-1)}{2}$$
$$x = -3.5$$

The axis of symmetry passes through the midpoint of the line segment that joins the x-intercepts. I calculated the mean of the two x-intercepts to determine the equation of the axis of symmetry.

$$f(x) = 2(x + 1)(x + 6)$$
$$f(-3.5) = 2(-3.5 + 1)(-3.5 + 6)$$
$$f(-3.5) = 2(-2.5)(2.5)$$
$$f(-3.5) = -12.5$$

The vertex of the parabola is $(-3.5, -12.5)$.

The vertex lies on the axis of symmetry, so its x-coordinate is -3.5. I substituted -3.5 into the equation to determine the y-coordinate of the vertex.



I plotted the x-intercepts, y-intercept, and vertex and then joined these points with a smooth curve.

Domain and range:
$\{(x, y) \mid x \in R, y \geq -12.5, y \in R\}$

The only restriction on the variables is that y must be greater than or equal to -12.5, the minimum value of the function.

Your Turn

Sketch the graph of the following function:

$$f(x) = -3x^2 + 6x - 3$$

a) How does the graph of this function differ from the graph in *Example 1*?

b) How are the x-intercepts related to the vertex? Explain.

EXAMPLE 2 Using a partial factoring strategy to sketch the graph of a quadratic function

Sketch the graph of the following quadratic function:

$$f(x) = -x^2 + 6x + 10$$

State the domain and range of the function.

Elliot's Solution

$f(x) = -x^2 + 6x + 10$
$f(x) = -x(x - 6) + 10$

I couldn't identify two integers with a product of 10 and a sum of 6, so I couldn't factor the expression. I decided to remove a partial factor of $-x$ from the first two terms. I did this so that I could determine the x-coordinates of the points that have 10 as their y-coordinate.

$-x = 0 \qquad x - 6 = 0$
$x = 0 \qquad x = 6$
$f(0) = 10 \qquad f(6) = 10$

The points (0, 10) and (6, 10) belong to the given quadratic function.

I determined two points in the function by setting each partial factor equal to zero.

When either factor is zero, the product of the factors is zero, so the value of the function is 10.

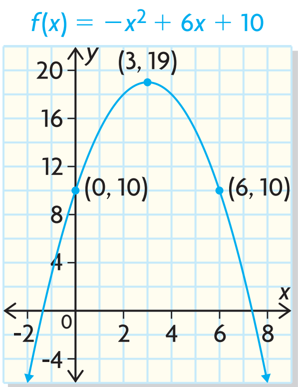
$$x = \frac{0 + 6}{2}$$
$x = 3$

Because (0, 10) and (6, 10) have the same y-coordinate, they are the same horizontal distance from the axis of symmetry. I determined the equation of the axis of symmetry by calculating the mean of the x-coordinates of these two points.

$f(3) = -(3)^2 + 6(3) + 10$
$f(3) = -9 + 18 + 10$
$f(3) = 19$

I determined the y-coordinate of the vertex.

The vertex is (3, 19).



The coefficient of the x^2 term is negative, so the parabola opens downward.

I used the vertex, as well as (0, 10) and (6, 10), to sketch the parabola.

Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y \leq 19, y \in \mathbb{R}\}$

The only restriction on the variables is that y must be less than or equal to 19, the maximum value of the function.

Your Turn

a) **i)** Apply the partial factoring strategy to locate two points that have the same y-coordinate on the following function:

$$f(x) = -x^2 - 3x + 12$$

ii) Determine the axis of symmetry and the location of the vertex of the function from part i).

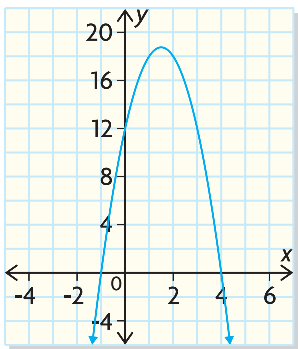
iii) Explain how the process you used in parts i) and ii) is different from factoring a quadratic function.

b) Explain whether you would use partial factoring to graph the function

$$g(x) = -x^2 - 4x + 12$$

EXAMPLE 3 Determining the equation of a quadratic function, given its graph

Determine the function that defines this parabola. Write the function in standard form.



Indira's Solution

The x-intercepts are $x = -1$ and $x = 4$. The zeros of the function occur when x has values of -1 and 4.

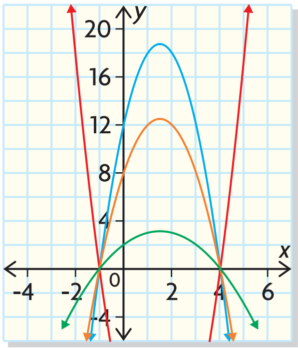
$y = a(x - r)(x - s)$

$y = a[x - (-1)][x - (4)]$

$y = a(x + 1)(x - 4)$

The graph is a parabola, so it is defined by a quadratic function.

I located the x-intercepts and used them to determine the zeros of the function. I wrote the factored form of the quadratic function, substituting -1 and 4 for r and s.



I knew that there are infinitely many quadratic functions that have these two zeros, depending on the value of *a*. I had to determine the value of *a* for the function that defines the blue graph.

The y-intercept is 12.

$$y = a(x + 1)(x - 4)$$
$$(12) = a[(0) + 1][(0) - 4]$$
$$12 = a(1)(-4)$$
$$12 = -4a$$
$$-3 = a$$

From the graph, I determined the coordinates of the *y*-intercept.

Because these coordinates are integers, I decided to use the *y*-intercept to solve for *a*.

In factored form, the quadratic function is
$y = -3(x + 1)(x - 4)$

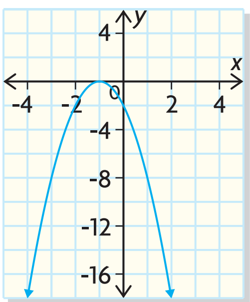
I substituted the value of *a* into my equation.

In standard form, the quadratic function is
$y = -3(x^2 - 3x - 4)$
$y = -3x^2 + 9x + 12$

My equation seems reasonable, because it defines a graph with a *y*-intercept of 12 and a parabola that opens downward.

Your Turn

If a parabola has only one x-intercept, how could you determine the quadratic function that defines it, written in factored form? Explain using the given graph.



EXAMPLE 4 Solving a problem modelled by a quadratic function in factored form

The members of a Ukrainian church hold a fundraiser every Friday night in the summer. They usually charge \$6 for a plate of perogies. They know, from previous Fridays, that 120 plates of perogies can be sold at the \$6 price but, for each \$1 price increase, 10 fewer plates will be sold. What should the members charge if they want to raise as much money as they can for the church?

Krystina's Solution: Using the properties of the function

Let y represent the total revenue.
$y = (\text{Number of plates})(\text{Price})$

Let x represent the number of \$1 price increases.
$y = (120 - 10x)(6 + x)$

For each price increase, x, I knew that $10x$ fewer plates will be sold.

If I expanded the factors in my function, I would create an x^2 term. This means that the function I have defined is quadratic and its graph is a parabola.

$0 = (120 - 10x)(6 + x)$

$120 - 10x = 0$ or $6 + x = 0$
$-10x = -120$ $\quad x = -6$
$x = 12$

The x-intercepts are $x = -6$ and $x = 12$.

To determine the zeros of the function, I substituted zero for y. A product is zero only when one or both of its factors are zero, so I set each factor equal to zero and solved each equation.

$x = \frac{12 + (-6)}{2}$
$x = 3$

$y = (120 - 10x)(6 + x)$
$y = [120 - 10(3)][6 + (3)]$
$y = (90)(9)$
$y = 810$

The coordinates of the vertex are (3, 810).

I determined the equation of the axis of symmetry for the parabola by calculating the mean distance between the x-intercepts.

I determined the y-coordinate of the vertex by substituting into my initial equation.

To generate as much revenue as possible, the members of the church should charge \$6 + \$3 or \$9 for a plate of perogies. This will provide revenue of \$810.

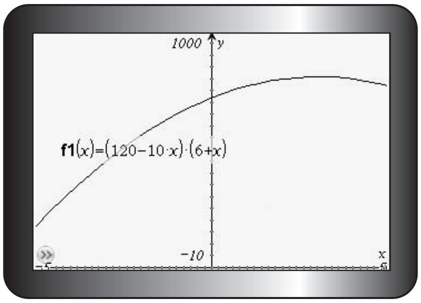
The vertex describes the maximum value of the function. Maximum sales of \$810 occur when the price is raised by \$3.

Jennifer's Solution: Using graphing technology

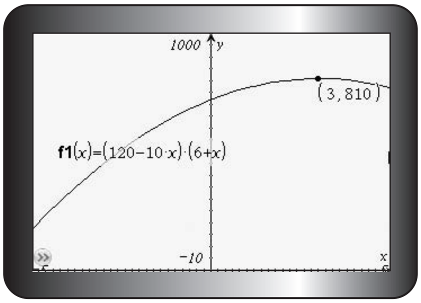
Revenue = (Number of plates)(Price)

$$y = (120 - 10x)(6 + x)$$

I let y represent Revenue and I let x represent the number of \$1 price increases. For each \$1 price increase, I knew that 10 fewer plates will be sold.



I graphed the equation on a calculator. Since a reduced price may result in maximum revenue, I set my domain to a minimum value of −5 and a maximum value of 5.



I used the calculator to locate the vertex of the parabola.

The members of the church should charge \$3 more than the current price of \$6 for a plate of perogies. If they charge \$9, they will reach the maximum revenue of \$810.

Your Turn

A career and technology class at a high school in Langley, British Columbia, operates a small T-shirt business out of the school. Over the last few years, the shop has had monthly sales of 300 T-shirts at a price of \$15 per T-shirt. The students have learned that for every \$2 increase in price, they will sell 20 fewer T-shirts each month. What should they charge for their T-shirts to maximize their monthly revenue?

In Summary

Key Ideas

- When a quadratic function is written in factored form

$$y = a(x - r)(x - s)$$

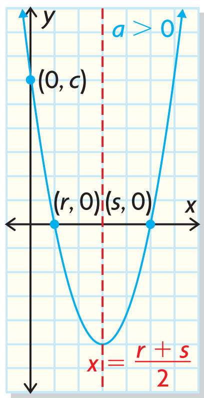
each factor can be used to determine a zero of the function by setting each factor equal to zero and solving.
- If a parabola has one or two x-intercepts, the equation of the parabola can be written in factored form using the x-intercept(s) and the coordinates of one other point on the parabola.
- Quadratic functions without any zeros cannot be written in factored form.

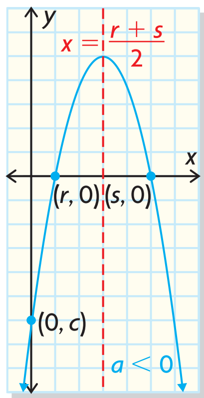
Need to Know

- A quadratic function that is written in the form

$$f(x) = a(x - r)(x - s)$$

has the following characteristics:
  - The x-intercepts of the graph of the function are $x = r$ and $x = s$.
  - The linear equation of the axis of symmetry is $x = \frac{r + s}{2}$.
  - The y-intercept, c, is $c = a \cdot r \cdot s$.





- If a quadratic function has only one x-intercept, the factored form can be written as follows:

$$f(x) = a(x - r)(x - r)$$
$$f(x) = a(x - r)^2$$

CHECK Your Understanding

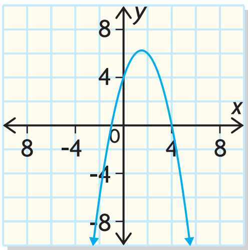
1. Match each quadratic function with its corresponding parabola.

a) $f(x) = (x - 1)(x + 4)$ **d)** $f(x) = (x - 1)(x - 4)$

b) $f(x) = (x + 1)(x - 4)$ **e)** $f(x) = (1 - x)(x + 4)$

c) $f(x) = (x + 1)(x + 4)$ **f)** $f(x) = (x + 1)(4 - x)$

i)



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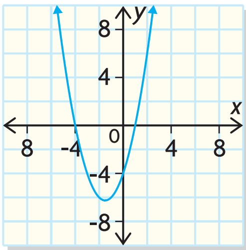

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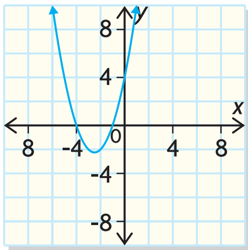

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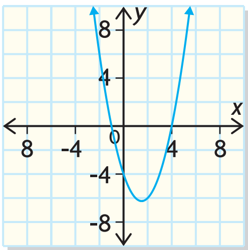

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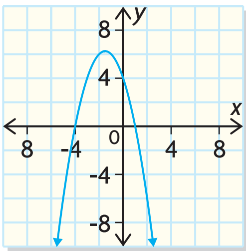

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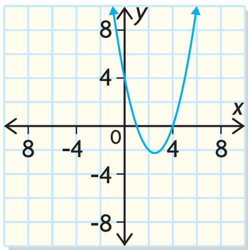


2. For each quadratic function below

i) determine the x-intercepts of the graph

ii) determine the y-intercept of the graph

iii) determine the equation of the axis of symmetry

iv) determine the coordinates of the vertex

v) sketch the graph

a) $f(x) = (x + 4)(x - 2)$ **c)** $h(x) = 2(x + 1)(x - 7)$

b) $g(x) = -2x(x - 3)$

3. A quadratic function has an equation that can be written in the form $f(x) = a(x - r)(x - s)$. The graph of the function has x-intercepts $x = -2$ and $x = 4$ and passes through point $(5, 7)$. Write the equation of the quadratic function.

PRACTISING

4. For each quadratic function, determine the x-intercepts, the y-intercept, the equation of the axis of symmetry, and the coordinates of the vertex of the graph.

a) $f(x) = (x - 1)(x + 1)$ **d)** $f(x) = -2(x - 2)(x + 1)$

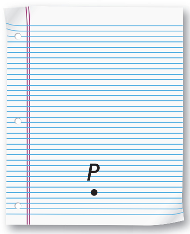
b) $f(x) = (x + 2)(x + 2)$ **e)** $f(x) = 3(x - 2)^2$

c) $f(x) = (x - 3)(x - 3)$ **f)** $f(x) = 4(x - 1)^2$

Math in Action

Paper Parabolas

On an 8.5 in. by 11 in. piece of lined paper, mark a point that is close to the centre, near the bottom edge. Label this point *P*.



- Fold the paper, at any angle, so that any point on the bottom edge of the paper touches point *P*. Crease the paper along the fold line, and then open the paper.
- Fold the paper again, but at a different angle, so that another point on the bottom edge of the paper touches point *P*. Crease the paper.
- Continue this process until you have many different creases in the paper on both sides of *P*.
  - What shape emerges?
  - Compare your shape with the shapes made by other students. How are the shapes the same? How are they different?
  - How does changing the location of point *P* affect the shape that is formed?
  - The creases intersect at several points. How could you determine whether a set of these points is on a parabola?

5. Sketch the graph of each function in question 4, and state the domain and range of the function.

6. Sketch the graph of

$$y = a(x - 3)(x + 1)$$

for $a = 3$. Describe how the graph would be different from your sketch if the value of a were 2, 1, 0, −1, −2, and −3.

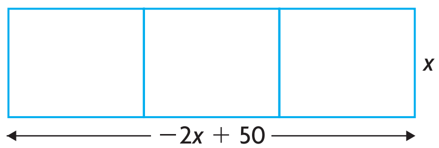
7. Sketch the graph of

$$y = (x - 3)(x + s)$$

for $s = 3$. Describe how the graph would be different from your sketch if the value of s were 2, 1, 0, −1, −2, and −3.8.

8. Byron is planning to build three attached rectangular enclosures for some of the animals on his farm. He bought 100 m of fencing. He wants to maximize the total area of the enclosures. He determined a function, $A(x)$, that models the total area in square metres, where x is the width of each rectangle:

$$A(x) = -2x^2 + 50x$$



a) Determine the maximum total area.
b) State the domain and range of the variables in the function.

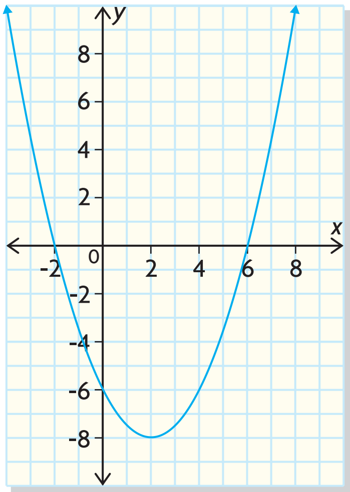
9. Paulette owns a store that sells used video games in Red Deer, Alberta. She charges $10 for each used game. At this price, she sells 70 games a week. Experience has taught her that a $1 increase in the price results in five fewer games being sold per week. At what price should Paulette sell her games to maximize her sales? What will be her maximum revenue?

10. For each quadratic function below
i) use partial factoring to determine two points that are the same distance from the axis of symmetry
ii) determine the coordinates of the vertex
iii) sketch the graph

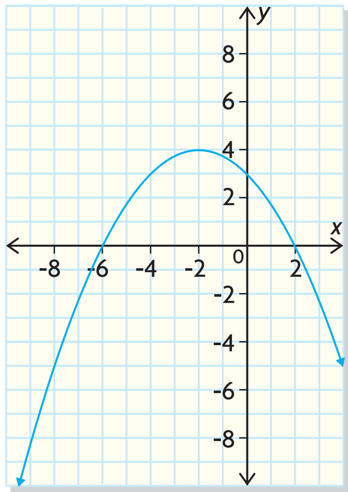
a) $f(x) = x^2 + 4x - 6$
b) $f(x) = x^2 - 8x + 13$
c) $f(x) = 2x^2 + 10x + 7$
d) $f(x) = -x^2 - 8x - 5$
e) $f(x) = -\frac{1}{2}x^2 + 2x - 3$
f) $f(x) = -2x^2 + 10x - 9$

11. Determine the equation of the quadratic function that defines each parabola.

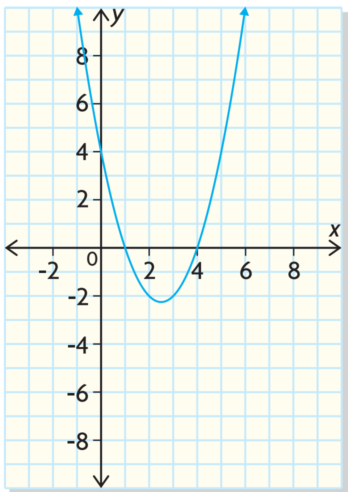
a)



c)



b)



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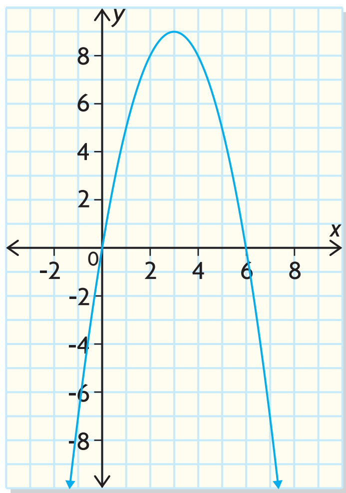


12. a) Use two different algebraic strategies to determine the equation of the axis of symmetry and the vertex of the parabola defined by the following function:

$$f(x) = -2x^2 + 16x - 24$$

b) Which strategy do you prefer? Explain.

13. Determine the quadratic function that defines a parabola with x-intercepts $x = -1$ and $x = 3$ and y-intercept $y = -6$. Provide a sketch to support your work.

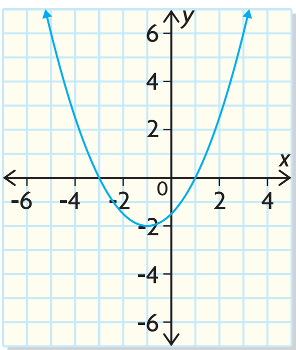
14. How many zeros can a quadratic function have? Provide sketches to support your reasoning.

15. On the north side of Sir Winston Churchill Provincial Park, located near Lac La Biche, Alberta, people gather to witness the migration of American white pelicans. The pelicans dive underwater to catch fish. Someone observed that a pelican's depth underwater over time could be modelled by a parabola. One pelican was underwater for 4 s, and its maximum depth was 1 m.
   a) State the domain and range of the variables in this situation.
   b) Determine the quadratic function that defines the parabola.

16. Elizabeth wants to enclose the backyard of her house on three sides to form a rectangular play area for her children. She has decided to use one wall of the house and three sections of fence to create the enclosure. Elizabeth has budgeted $800 for the fence. The fencing material she has chosen costs $16/ft. Determine the dimensions that will provide Elizabeth with the largest play area.

17. A water rocket was launched from the ground, with an initial velocity of 32 m/s. The rocket achieved a height of 44 m after 2 s of flight. The rocket was in the air for 6 s.
   a) Determine the quadratic function that models the height of the rocket over time.
   b) State the domain and range of the variables.

Closing



18. Identify the key characteristics you could use to determine the quadratic function that defines this graph. Explain how you would use these characteristics.

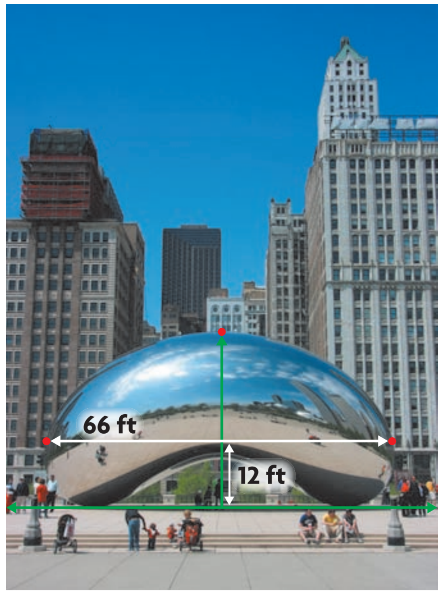
Extending

19. The Chicago Bean is a unique sculpture that was inspired by liquid mercury. It is 66 ft long and 33 ft high, with a 12 ft arch underneath. The top curved section that connects the three red dots in the photograph forms a parabola.

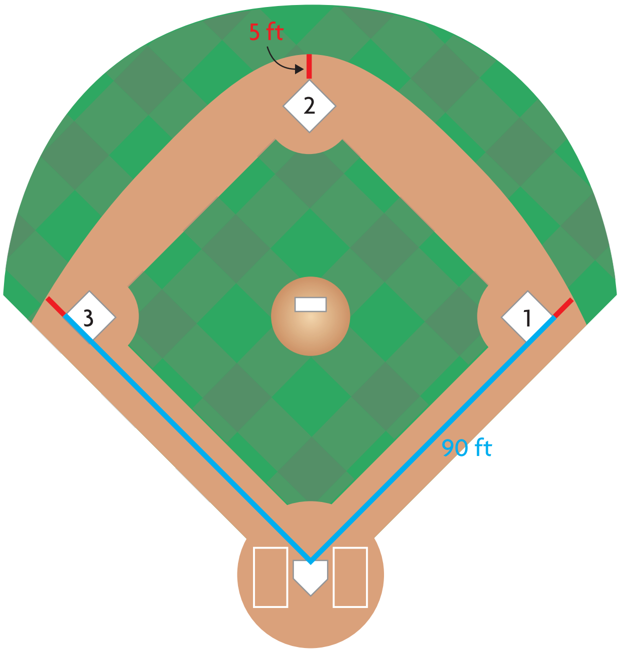
a) Determine the quadratic function that connects the three red dots. Assume that the ground (the green line) represents the x-axis of the graph. Write the function in standard form.

b) What are the domain and range of the variables?

c) If the parabola extended to the ground, what would the x-intercepts be, rounded to the nearest tenth?



20. A local baseball team has raised money to put new grass on the field. The curve where the infield ends can be modelled by a parabola. The foreman has marked out the key locations on the field, as shown in the diagram.



a) Determine a quadratic function that models the curve where the infield ends.

b) State the domain and range of the variables. Justify your decision.

c) Graph the quadratic function.

21. The National Basketball Association (NBA) mandates that every court must have the same dimensions. The length of the court must be 6 ft less than twice the width. The area of the court must be 4700 ft^2. Use this information to determine the dimensions of a basketball court used by the NBA.