4.3 The Ambiguous Case of the Sine Law

YOU WILL NEED
- calculator
- ruler
- protractor

GOAL

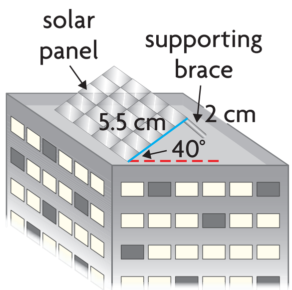
Analyze the ambiguous case of the sine law, and solve problems that involve the ambiguous case.

EXPLORE...
- Two sides in an obtuse triangle are 3 m and 4 m in length. The angle that is opposite the 3 m side measures 40°. Determine the measure of the angle that is opposite the 4 m side.

INVESTIGATE the Math

Naomi works for a company that makes supporting braces for solar panels. She is drawing a scale diagram to show solar panels that are going to be installed on the flat roof of a downtown high-rise. Each panel is 5.5 m long and must be tilted at 40° to the horizontal in order to maximize the strength of the Sun's rays. Naomi needs to choose the length of the supporting brace for each panel. Supporting braces are available in 1 m increments, starting at 2 m and going up to 6 m.

Naomi started with a 2 m brace and discovered that she could not complete the triangle.



I drew a scale diagram of the situation, using 1 cm to represent 1 m. I drew an angle of 40° first. Then I measured 5.5 cm along one of the arms to represent the solar panel.

I don't think a triangle with these measurements exists.

? How many different scale diagrams are possible, with the supporting braces that are available?

A. Work with a partner. Use a ruler and a protractor to construct a 40° angle connected to a 5.5 cm side, as shown in Naomi's diagram.

B. Calculate the height of any triangle formed using the 5.5 cm side and 40° angle.

C. The 2 cm side is too short. Try side lengths from 3 cm to 6 cm. The side that is opposite the 40° angle can be at any angle to the 5.5 cm side.

D. What length of supporting brace is necessary in order to have two possible triangles? Explain.

Reflecting

E. What range of supporting brace lengths result in two possible triangles?

F. What information was Naomi originally given? Will this type of information always lead to the **ambiguous case of the sine law**?

G. When dealing with a *SSA* situation, how does the height of the triangle help you determine the number of possible triangles?

ambiguous case of the sine law

A situation in which two triangles can be drawn, given the available information; the ambiguous case may occur when the given measurements are the lengths of two sides and the measure of an angle that is not contained by the two sides *(SSA)*.

APPLY the Math

EXAMPLE 1 Connecting the *SSA* situation to the number of possible triangles

Given each *SSA* situation for $\triangle ABC$, determine how many triangles are possible.

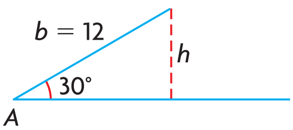
a) $\angle A = 30°$, $a = 4$ m, and $b = 12$ m

b) $\angle A = 30°$, $a = 6$ m, and $b = 12$ m

c) $\angle A = 30°$, $a = 8$ m, and $b = 12$ m

d) $\angle A = 30°$, $a = 15$ m, and $b = 12$ m

Saskia's Solution



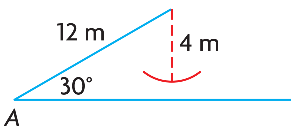
I drew the beginning of a triangle with a 30° angle and a 12 m side.

$$\sin 30° = \frac{h}{12}$$
$$12 \sin 30° = h$$
$$6 \text{ m} = h$$

I used the sine ratio to calculate the height of the triangle.

I can use this height as a benchmark to decide on side lengths opposite the 30° angle that will result in zero, one, or two triangles.

a) $\angle A = 30°$, $a = 4$ m, and $b = 12$ m

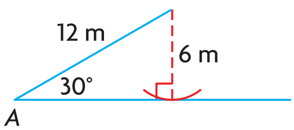


No triangles are possible.

Since $a < b$ and $a < h$, I knew that no triangles are possible.

I used a compass to be certain. I set the compass tips to represent 4 m. I placed one tip of the compass at the open end of the 12 m side and swung the pencil tip toward the other side. The pencil couldn't reach the base, so a 4 m side could not close the triangle.

b) $\angle A = 30°$, $a = 6$ m, and $b = 12$ m

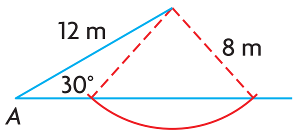


One triangle is possible.

Since $a < b$ and $a = h$, there is only one possible triangle, a right triangle.

A compass arc intersects the base at only one point.

c) $\angle A = 30°$, $a = 8$ m, and $b = 12$ m

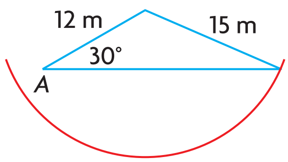


Two triangles are possible.

Since $a < b$ and $a > h$, there are two possible triangles.

A compass arc intersects the base at two points.

d) $\angle A = 30°$, $a = 15$ m, and $b = 12$ m



One triangle is possible.

Since $a > b$, only one triangle is possible.

A compass arc intersects the base at only one point.

Your Turn

Determine how many triangles are possible, given $\angle A = 120°$, $a = 15$ m, and $b = 12$ m.

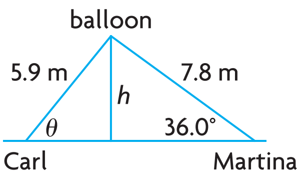
EXAMPLE 2 Solving a problem using the sine law

Martina and Carl are part of a team that is studying weather patterns. The team is about to launch a weather balloon to collect data. Martina's rope is 7.8 m long and makes an angle of 36.0° with the ground. Carl's rope is 5.9 m long. Assuming that Martina and Carl form a triangle in a vertical plane with the weather balloon, what is the distance between Martina and Carl, to the nearest tenth of a metre?

Sandra's Solution: Using the sine law and then the cosine law

Let h represent the height of the weather balloon.
Let θ represent the angle for Carl's rope.

Situation 1:

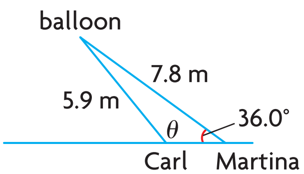


$$\sin 36.0 = \frac{h}{7.8}$$
$$7.8(\sin 36.0) = 7.8\left(\frac{h}{7.8}\right)$$
$$4.5847\ldots = h$$

I drew the triangle.

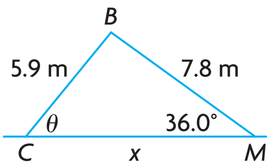
I noticed that this is a *SSA* situation. I had to determine the height of the triangle to determine if this is an ambiguous case.

Situation 2:



Carl's rope is longer than the height and shorter than Martina's rope, so there are two possible triangles. I drew the second triangle.

Situation 1:



$$\frac{\sin \theta}{7.8} = \frac{\sin 36°}{5.9}$$
$$\sin \theta = \frac{7.8 \sin 36°}{5.9}$$
$$\sin \theta = 0.7770...$$
$$\theta = \sin^{-1}(0.7770...)$$
$$\theta = 50.9932...°$$

I substituted the side lengths and angles (including θ) into the formula for the sine law and isolated θ.

$$\angle B = 180° - 36.0° - 50.9932...°$$
$$\angle B = 93.0067...°$$

The measures of the angles in a triangle sum to 180°.

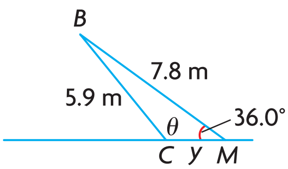
$$x^2 = 5.9^2 + 7.8^2 - 2(5.9)(7.8) \cos 93.0067...°$$
$$x^2 = 100.4777...$$
$$x = 10.0238...$$

I used the cosine law to determine the distance, x, between Martina and Carl. I substituted the known measurements into the cosine law.

In Situation 1, Martina and Carl are 10.0 m apart.

Situation 2:

I also considered the situation in which Carl is closer to Martina.



$$\frac{\sin \theta}{7.8} = \frac{\sin 36°}{5.9}$$
$$\sin \theta = \frac{7.8 \sin 36°}{5.9}$$
$$\sin \theta = 0.7770...$$
$$\theta = \sin^{-1}(0.7770...)$$
$$\theta = 50.9932...°$$

I used the sine law to determine θ.

$$\theta = 180° - 50.9932...°$$
$$\theta = 129.0067...°$$

I determined the measure of the supplementary angle, which is suitable for this situation.

$$\angle B = 180° - 36.0° - 129.0067...°$$
$$\angle B = 14.9932...°$$

The measures of the angles in a triangle sum to 180°.

$$y^2 = 5.9^2 + 7.8^2 - 2(5.9)(7.8) \cos 14.9932...°$$
$$y^2 = 6.7433...$$
$$y = 2.5968...$$

I can use $\angle B$ in the cosine law to determine the distance, y, between Martina and Carl.

I substituted the measure of $\angle B$ and the given side lengths into the cosine law.

In the second situation, Martina and Carl are 2.6 m apart.
Martina and Carl are either 10.0 m apart or 2.6 m apart.

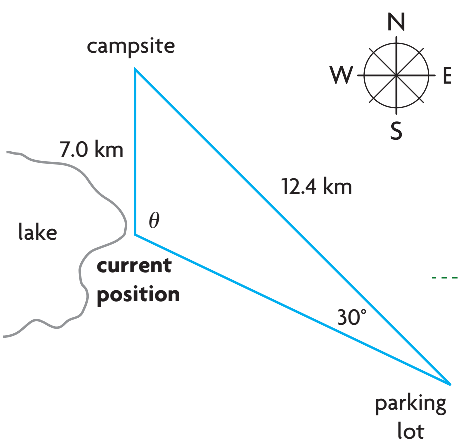
Your Turn

What length would Carl's rope need to be in order for there to be only one possible triangle that could model this situation?

EXAMPLE 3 Reasoning about ambiguity

Leanne and Kerry are hiking in the mountains. They left Leanne's car in the parking lot and walked northwest for 12.4 km to a campsite. Then they turned due south and walked another 7.0 km to a glacier lake. The weather was taking a turn for the worse, so they decided to plot a course directly back to the parking lot. Kerry remembered, from the map in the parking lot, that the angle between the path to the campsite and the path to the glacier lake measures about 30°. What compass direction should they follow to return directly to the parking lot?

Austin's Solution



Since I am given specific directions, I know exactly how to draw a sketch of the situation. There is only one way to draw the sketch, so this is not ambiguous.

Leanne and Kerry left the parking lot and walked northwest and then south.

Because the campsite is due north of the lake, I knew that the angle at the lake vertex of the triangle, θ, would help me determine the compass direction that Leanne and Kerry need to travel.

My diagram shows that Leanne and Kerry need to travel approximately southeast.

$$\frac{\sin \theta}{12.4} = \frac{\sin 30°}{7.0}$$
$$12.4\left(\frac{\sin \theta}{12.4}\right) = 12.4\left(\frac{\sin 30°}{7.0}\right)$$
$$\sin \theta = 0.8857...$$
$$\theta = \sin^{-1}(0.8857)$$
$$\theta = 62.3395...°$$

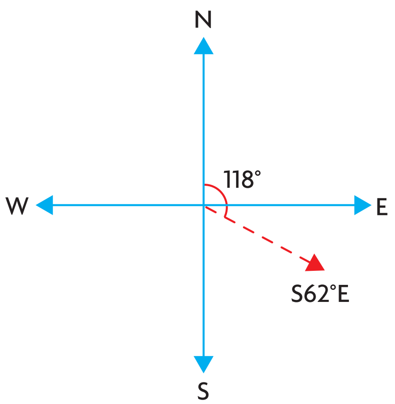
I noticed two side-angle pairs, so I substituted the values into the sine law and solved for θ.

The angle seemed too small, according to my diagram. To correct the angle measure, I needed the supplementary angle.

Correction:

$\theta = 180° - 62.3395...°$

$\theta = 117.6604...°$



$180° - 117.6604...° = 62.3395...°$

Leanne and Kerry would need to travel in the direction S62°E to reach the parking lot.

I subtracted the measure of the angle in my triangle from 180° to determine the direction of travel.

Your Turn

How far would Leanne and Kerry need to travel to reach the parking lot?

In Summary

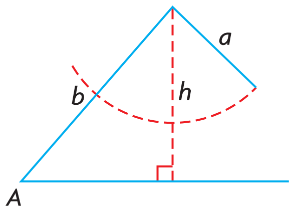
Key Idea

- The ambiguous case of the sine law may occur when you are given two side lengths and the measure of an angle that is opposite one of these sides. Depending on the measure of the given angle and the lengths of the given sides, you may need to construct and solve zero, one, or two triangles.

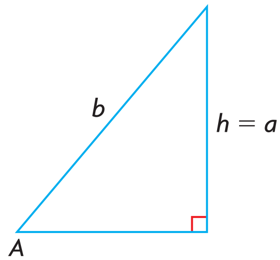
Need to Know

- In $\triangle ABC$ below, where h is the height of the triangle, $\angle A$ and the lengths of sides a and b are given, and $\angle A$ is acute, there are four possibilities to consider:

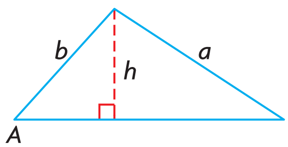
If $\angle A$ is acute and $a < h$, there is **no triangle.**



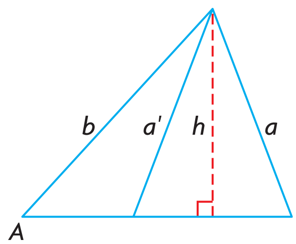
If $\angle A$ is acute and $a = h$, there is **one right triangle.**



If $\angle A$ is acute and $a > b$ or $a = b$, there is **one triangle.**

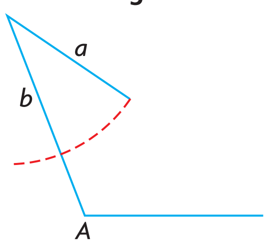


If $\angle A$ is acute and $h < a < b$, there are **two possible triangles.**

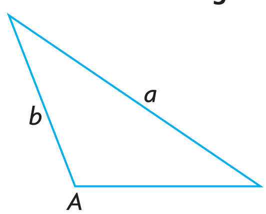


- If $\angle A$, a, and b are given and $\angle A$ is obtuse, there are two possibilities to consider:

If $\angle A$ is obtuse and $a < b$ or $a = b$, there is **no triangle.**



If $\angle A$ is obtuse and $a > b$, there is **one triangle.**



CHECK Your Understanding

1. Given each set of measurements for $\triangle ABC$, determine if there are zero, one, or two possibilities. Draw the triangle(s) to support your answer.
a) $\angle A = 75°$, $a = 4$ m, and $b = 12$ m
b) $\angle A = 50°$, $a = 10$ m, and $b = 6$ m
c) $\angle A = 115°$, $a = 3.0$ m, and $b = 9.0$ m
d) $\angle A = 62°$, $a = 2.8$ m, and $b = 3.0$ m

2. Decide whether each description of a triangle involves the *SSA* situation.
a) In $\triangle ABC$, $\angle B = 100°$, $a = 8$ cm, and $b = 10$ cm.
b) In $\triangle DEF$, $\angle D = 81°$, $e = 9$ cm, and $f = 8$ cm.
c) In $\triangle GHI$, $\angle G = 40°$, $i = 5$ cm, and $g = 4$ cm.
d) In $\triangle JKL$, $\angle L = 15°$, $j = 71$ cm, and $k = 36$ cm.
e) In $\triangle MNO$, $\angle O = 28°$, $m = 8.4$ cm, and $o = 4.0$ cm.
f) In $\triangle PQR$, $\angle Q = 95°$, $q = 1.0$ cm, and $r = 0.5$ cm.

3. Calculate the height of each triangle in question 2. Determine the number of triangles that are possible (zero, one, or two). Justify your answers.

PRACTISING

4. Decide whether each description of a triangle involves the *SSA* situation. If it does, determine the number of triangles (zero, one, or two) that are possible with the given measurements. Draw the triangle(s), and justify your answer.
a) In $\triangle ABC$, $\angle A = 51°$, $a = 5$ m, and $b = 14$ m.
b) In $\triangle ABC$, $\angle C = 30°$, $a = 6$ mm, and $c = 12$ mm.
c) In $\triangle ABC$, $\angle B = 40°$, $a = 12$ cm, and $b = 10$ cm.
d) In $\triangle ABC$, $\angle A = 155°$, $b = 15$ m, and $c = 12$ m.

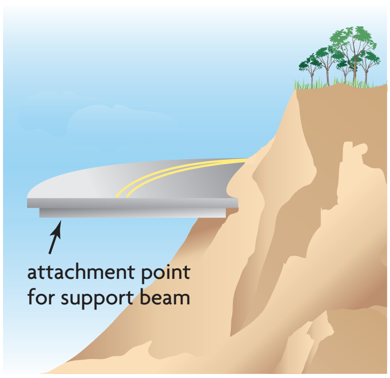
5. In $\triangle DEF$, $EF = 15.0$ cm and $\angle E = 37°$.
a) Calculate the height of the triangle from base ED.
b) Determine the possible lengths of side FD, so that there are zero, one, or two triangles that satisfy these conditions. Draw each triangle to support your answer.

6. A landowner claims that his property is triangular, with one side that is 430 m long and another side that is 110 m long. The angle that is opposite one of these sides measures 35°.
a) Determine the length of the third side of the property, to the nearest metre.
b) Improve the description of the property to avoid confusion.

7. The *Raven's Song,* a traditional Tsimshian cedar canoe, is paddled away from a dock, directly toward a navigational buoy that is 5 km away. After reaching the buoy, the direction of the canoe is altered and it is paddled another 3 km. From the dock, the angle between the buoy and the canoe's current position measures 12°.
   a) How far is the *Raven's Song* from the dock?
   b) Is this the only possible solution? Explain.

Bill Helin carved the *Raven's Song* from a 600-year-old cedar taken from the Nimpkish Valley. The canoe was created to carry a message of goodwill from the First Nations Peoples of the West Coast of British Columbia to the 1994 Commonwealth Games in Victoria.

8. An obtuse triangle has two known side lengths: 4.0 m and 4.2 m. The angle that is opposite the shorter side measures 64.0°.
   a) Calculate the obtuse angle in the triangle, to the nearest tenth of a degree.
   b) Is there only one possible answer? Explain.



9. Part of a highway is to be cantilevered out from a mountainside, as shown. The width of the highway is 22 m, and the angle of the mountain slope at the road measures 51°. An 18 m beam needs to be installed to support the highway. Calculate possible distances, downhill from the highway, where the support post could be fastened. What distance would you recommend? Explain.

10. A farmer finishes repairing a fence post and then walks 250 yd through his corn field. He turns and walks another 300 yd east, until he can see the fence post southwest of him. He realizes that he left some of his tools at the fence post and heads directly back to it. How far does he need to walk, to the nearest metre?

11. In an extreme adventure triathalon, participants swim 1.7 km from a dock to one end of an island, run 1.5 km due north along the length of the island, and then kayak back to the dock. From the dock, the angle between the lines of sight to the ends of the island measures 15°. How long is the kayak leg of the race?

12. Carol is flying a kite on level ground. The string of the kite forms an angle of 50° with the ground. Two other girls, standing different distances from Carol, see the kite at angles of elevation of 66° and 35°. One girl is 11 m from Carol. All three girls are standing in a line. For each question below, state all possible answers to the nearest metre.

a) How high is the kite above the ground?

b) How long is the string?

c) How far is the second girl from Carol?

13. The Huqiu Tower in China was built in 961 CE. When the tower was first built, its height was 47 m. Since then, it has tilted 2.8°. It is now called China's Leaning Tower. There is a point on the ground where you can be equidistant from both the top and the bottom of the tower. How far is this point from the base of the tower? Round your answer to the nearest metre.

14. Create a *SSA* problem with zero, one, or two possible triangles. Exchange problems with a classmate. Sketch the situation described in your classmate's problem, and determine the number of possible triangles.

15. Draw a *SSA* situation in which there is no possible triangle.

a) Label the sides and angle, and use trigonometry to confirm that there is no possible triangle.

b) Determine the angle that would be necessary for there to be one possible triangle.

c) What angle would be necessary for there to be two possible triangles?

Closing

16. In $\triangle LMN$, $\angle L$ is acute. Using a sketch, explain the relationship among $\angle L$, sides l and m, and the height of $\triangle LMN$ for each situation below.

a) Only one triangle is possible.

b) Two triangles are possible.

c) No triangle is possible.

Extending

17. In $\triangle DEF$, $d = 13.0$ cm, $f = 15.0$ cm, and $\angle D = 26°$. The two possible locations for vertex F are F_1 and F_2.

a) Calculate the area of $\triangle DEF_1$.

b) Calculate the area of $\triangle DEF_2$.

c) Calculate the area of $\triangle F_1EF_2$.

d) Discuss with a classmate an alternative solution for determining the area of $\triangle F_1EF_2$.

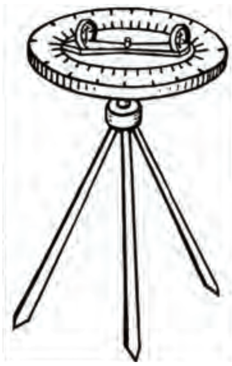
History | Connection

Dioptras and Theodolites

YOU WILL NEED

- protractor
- string
- straw
- metre stick or tape measure

Surveying has played an important role in most cultures, including ancient cultures. For example, surveying tools and techniques were used to design the pyramids in ancient Egypt, map North America, and determine the boundaries of many nations. Tape measures, plumb lines, and levels were some of the original surveying tools. With the development of trigonometry, tools for measuring angles became important. One of these tools was the dioptra. It consisted of a sighting tube attached to a protractor, and it was used in ancient Greece and Rome to measure angles in the vertical and horizontal planes. Historians speculate that the Romans used dioptras as early as 2600 years ago, when building tunnels and aqueducts.

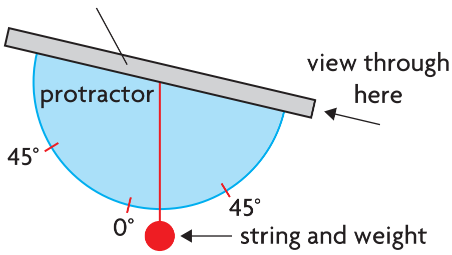


This instrument was modified to include a sighting telescope, which can measure angles to the nearest 2″ or 0.06% of a degree.

Theodolites are still used in mapping and building. They can cost upward of $10 000.

In the 16th century, more accurate tools were developed. A polimetrum, later called a theodolite, was used to measure horizontal angles. In the 18th century, the theodolite was combined with the altazimuth, an instrument for measuring vertical angles. The new, combined instrument became known as the modern theodolite or the transit theodolite.

A. Use a straw, a protractor, and a plumb line to construct your own dioptra.



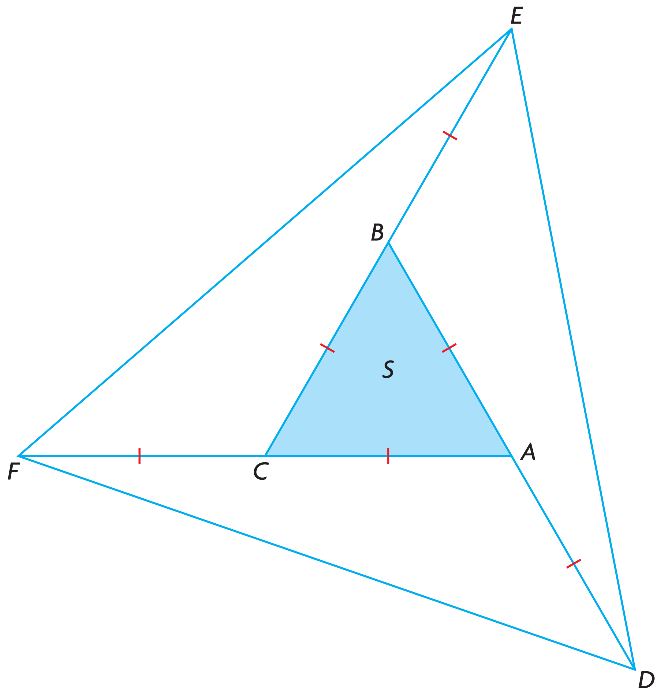
B. Use your dioptra and concepts of trigonometry to determine the height of a building or a tree.

Applying Problem-Solving Strategies

Analyzing an Area Puzzle

YOU WILL NEED

- scissors
- calculator



$\triangle ABC$ is an equilateral triangle with side lengths of 5 cm. Each side has been extended to the vertices of $\triangle DEF$. All the extended segments (CF, AD, and BE) are also 5 cm.

The Puzzle

A. Estimate how many $\triangle ABC$s could fit into the area of $\triangle DEF$.

B. Using scissors and extra cutouts of $\triangle ABC$, determine exactly how many $\triangle ABC$s fit into $\triangle DEF$.

The Strategy

C. Describe the strategy you used to solve this puzzle.

Variation

D. Try using trigonometry to solve this puzzle.

E. Create a similar puzzle using a different regular polygon.