1.5 Proofs That Are Not Valid

YOU WILL NEED
- grid paper
- ruler
- scissors

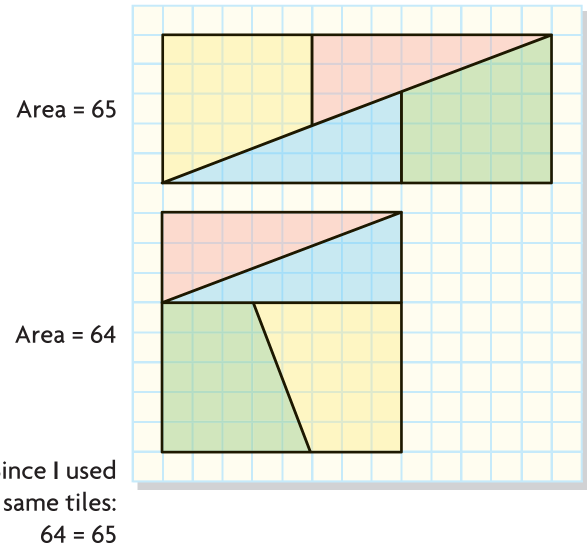
EXPLORE…
- Consider the following statement: There are tthree errorss in this sentence. Is the statement valid?

GOAL

Identify errors in proofs.

INVESTIGATE the Math

Moh was working with tiles on grid paper. He used right triangles and right trapezoids.



? How was it possible for 64 to equal 65?

A. Construct, as precisely as possible, the square figure on grid paper.

B. Separate the square into its right triangles and right trapezoids.

C. Cut out the shapes. Then reconfigure the shapes to make the rectangle.

D. Determine the accuracy of the positions of the shapes by looking for overlap or empty space.

E. Does Moh's rearrangement of tiles prove that 64 equals 65? Explain.

Reflecting

F. What does any overlap or empty space suggest about the areas of the figures?

G. How do the colours make the rectangle and the square appear to have the same area?

H. Explain how you can check for errors in your constructions.

APPLY the Math

EXAMPLE 1 Using reasoning to determine the validity of an argument

Athletes do not compete in both the Summer and Winter Olympics.
Hayley Wickenheiser has represented Canada four times at the Winter Olympics.
Therefore, Hayley Wickenheiser has not participated in the Summer Olympics.


Hayley Wickenheiser has represented Canada four times at the Winter Olympics.

Tia read these statements and knew that there was an error. Identify the error in the reasoning.

Tia's Solution

Athletes do not compete in both the Summer and Winter Olympics.

I did some research and found that 18 athletes have competed in both Games. This statement is not valid.

Hayley Wickenheiser has represented Canada four times at the Winter Olympics.

This statement is true. She has played on the national hockey team.

Therefore, Hayley Wickenheiser has not participated in the Summer Olympics.

The conclusion was false because the first statement was false. Hayley played for Canada in the softball competition in the 2000 Summer Olympics.

Your Turn

Zack is a high school student. All high school students dislike cooking. Therefore, Zack dislikes cooking. Where is the error in the reasoning?

Communication *Tip*

Stereotypes are generalizations based on culture, gender, religion, or race. There are always counterexamples to stereotypes, so conclusions based on stereotypes are not valid.

EXAMPLE 2 Using reasoning to determine the validity of a proof

Bev claims he can prove that $3 = 4$.

Bev's Proof

Suppose that: $a + b = c$

This statement can be written as: $4a - 3a + 4b - 3b = 4c - 3c$

After reorganizing, it becomes: $4a + 4b - 4c = 3a + 3b - 3c$

Using the distributive property, $4(a + b - c) = 3(a + b - c)$

Dividing both sides by $(a + b - c)$, $4 = 3$

Show that Bev has written an **invalid proof**.

invalid proof
A proof that contains an error in reasoning or that contains invalid assumptions.

Pru's Solution

premise
A statement assumed to be true.

Suppose that:

$$a + b = c \quad ✔$$

Bev's **premise** was made at the beginning of the proof. Since variables can be used to represent any numbers, this part of the proof is valid.

$$4a - 3a + 4b - 3b = 4c - 3c \quad ✔$$

Bev substituted $4a - 3a$ for a since $4a - 3a = a$.
Bev substituted $4b - 3b$ for b since $4b - 3b = b$.
Bev substituted $4c - 3c$ for c since $4c - 3c = c$.

$$4a + 4b - 4c = 3a + 3b - 3c \quad ✔$$

I reorganized the equation and I came up with the same result that Bev did when he reorganized. Simplifying would take me back to the premise. This part of the proof is valid.

$$4(a + b - c) = 3(a + b - c) \quad ✔$$

Since each side of the equation has the same coefficient for all the terms, factoring both sides is a valid step.

$$\frac{4(a + b - c)}{(a + b - c)} = \frac{3(a + b - c)}{(a + b - c)}$$

This step appears to be valid, but when I looked at the divisor, I identified the flaw.

$$a + b = c$$
$$a + b - c = c - c$$
$$a + b - c = 0$$

When I rearranged the premise, I determined that the divisor equalled zero.

Dividing both sides of the equation by $a + b - c$ is not valid. Division by zero is undefined.

Your Turn

How could this type of false proof be used to suggest that $65 = 64$?

EXAMPLE 3 Using reasoning to determine the validity of a proof

Liz claims she has proved that $-5 = 5$.

Liz's Proof

I assumed that $-5 = 5$.
Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$
This means that my assumption, $-5 = 5$, must be correct.

Where is the error in Liz's proof?

Simon's Solution

I assumed that $-5 = 5$.

Liz started off with the false assumption that the two numbers were equal.

Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$

Everything that comes after the false assumption doesn't matter because the reasoning is built on the false assumption.

Even though $25 = 25$, the underlying premise is not true.

$-5 \neq 5$

Liz's conclusion is built on a false assumption, and the conclusion she reaches is the same as her assumption.

If an assumption is not true, then any argument that was built on the assumption is not valid.

Circular reasoning has resulted from these steps. Starting with an error and then ending by saying that the error has been proved is arguing in a circle.

circular reasoning
An argument that is incorrect because it makes use of the conclusion to be proved.

Your Turn

How is an error in a premise like a counterexample?

EXAMPLE 4 Using reasoning to determine the validity of a proof

Hossai is trying to prove the following number trick:
Choose any number. Add 3. Double it. Add 4. Divide by 2. Take away the number you started with.

Each time Hossai tries the trick, she ends up with 5. Her proof, however, does not give the same result.

Hossai's Proof

n	Choose any number.
$n + 3$	Add 3.
$2n + 6$	Double it.
$2n + 10$	Add 4.
$2n + 5$	Divide by 2.
$n + 5$	Take away the number you started with.

Where is the error in Hossai's proof?

Sheri's Solution

$1 \longrightarrow 5$
$10 \longrightarrow 5$

I tried the number trick twice, for the number 1 and the number 10. Both times, I ended up with 5. The math trick worked for Hossai and for me, so the error must be in Hossai's proof.

n ✔

The variable n can represent any number. This step is valid.

$n + 3$ ✔

Adding 3 to n is correctly represented.

$2n + 6$ ✔

Doubling a quantity is multiplying by 2. This step is valid. Its simplification is correct as well.

$2n + 10$ ✔

Adding 4 to the expression is correctly represented, and the simplification is correct.

$2n + 5$ ✘

The entire expression should be divided by 2, not just the constant. This step is where the mistake occurred.

I corrected the mistake:

$$\frac{2n + 10}{2} = n + 5$$

$$n + 5 - n = 5$$

I completed Hossai's proof by subtracting n. I showed that the answer will be 5 for any number.

Your Turn

Is there a number that will not work in Hossai's number trick? Explain.

EXAMPLE 5 Using reasoning to determine the validity of a proof

Jean says she can prove that $\$1 = 1¢$.

Jean's Proof

$1 can be converted to 100¢.
100 can be expressed as $(10)^2$.
10 cents is one-tenth of a dollar.
$(0.1)^2 = 0.01$
One hundredth of a dollar is one cent, so $\$1 = 1¢$.

How can Jean's friend Grant show the error in her reasoning?

Grant's Solution

$1 can be converted to 100¢. ✔

It is true that 100 cents is the same as $1.

100 can be expressed as $(10)^2$. ✔

It is true that $(10)^2$ is $10 \cdot 10$, which is 100.

10 cents is one-tenth of a dollar. ✔

It is true that 10 dimes make up a dollar.

$(0.1)^2 = 0.01$ ✔

Arithmetically, I could see that this step was true. But Jean was ignoring the units. It doesn't make sense to square a dime. The units $¢^2$ and $\$^2$ have no meaning.

A dollar is equivalent to (10)($0.10) or 10(10¢), not to (10¢)(10¢) or ($0.10)($0.10).

$\$1 \neq 1¢$

Your Turn

Does Grant's explanation fully show the error in Jean's reasoning? Explain.

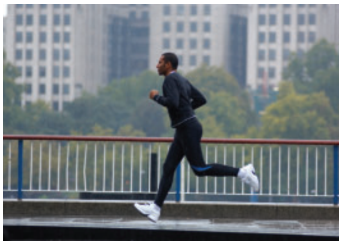
In Summary

Key Idea

- A single error in reasoning will break down the logical argument of a deductive proof. This will result in an invalid conclusion, or a conclusion that is not supported by the proof.

Need to Know

- Division by zero always creates an error in a proof, leading to an invalid conclusion.
- Circular reasoning must be avoided. Be careful not to assume a result that follows from what you are trying to prove.
- The reason you are writing a proof is so that others can read and understand it. After you write a proof, have someone else who has not seen your proof read it. If this person gets confused, your proof may need to be clarified.



Calories burned while running depend on mass, distance, and time. A runner of mass 70 kg who runs 15 km in 1 h will burn about 1000 Calories.

CHECK Your Understanding

1. Determine the error in each example of deductive reasoning.

a) All runners train on a daily basis. Gabriel is a runner. Therefore, Gabriel trains daily.

b) All squares have four right angles. Quadrilateral *PQRS* has four right angles. Therefore, *PQRS* is a square.

2. According to this proof, $5 = 7$. Identify the error.

Proof

$$\begin{aligned} 1 &= 1 + 1 \\ 2(1) &= 2(1 + 1) \\ 2(1) + 3 &= 2(1 + 1) + 3 \\ 2 + 3 &= 4 + 3 \\ 5 &= 7 \end{aligned}$$

PRACTISING

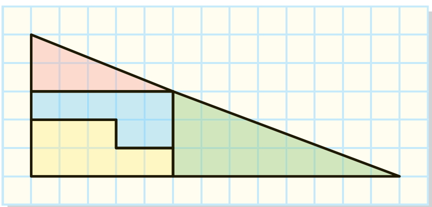
3. Mickey says he can prove that $2 = 0$. Here is his proof.

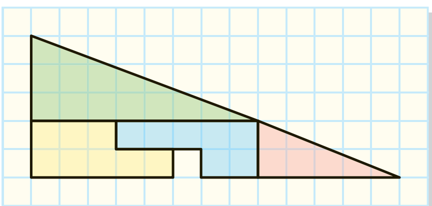
Let both a and b be equal to 1.

$a = b$	Transitive property
$a^2 = b^2$	Squaring both sides
$a^2 - b^2 = 0$	Subtracting b^2 from both sides
$(a - b)(a + b) = 0$	Factoring a difference of squares
$\dfrac{(a - b)(a + b)}{(a - b)} = \dfrac{0}{(a - b)}$	Dividing both sides by $a - b$
$1(a + b) = 0$	Simplifying
$a + b = 0$	
$1 + 1 = 0$	Substitution
$2 = 0$	

Explain whether each statement in Mickey's proof is valid.

4. Noreen claims she has proved that $32.5 = 31.5$.





Is Noreen's proof valid? Explain.

5. Ali created a math trick in which she always ended with 4. When Ali tried to prove her trick, however, it did not work.

Ali's Proof

n	I used n to represent any number.
$2n$	Multiply by 2.
$2n + 8$	Add 8.
$2n + 4$	Divide by 2.
$n + 4$	Subtract your starting number.

Identify the error in Ali's proof, and explain why her reasoning is incorrect.

6. Connie tried this number trick:
- Write down the number of your street address.
- Multiply by 2.
- Add the number of days in a week.
- Multiply by 50.
- Add your age.
- Subtract the number of days in a year.
- Add 15.

Connie's result was a number in which the tens and ones digits were her age and the rest of the digits were the number from her street address. She tried to prove why this works, but her final expression did not make sense.

Let n represent any house number.

$2n$	Multiply by 2.
$2n + 7$	Add the number of days in a week.
$100n + 350$	Multiply by 50.
Let a represent any age.	
$100n + 350 + a$	Add your age.
$100n + 350 + a - 360$	Subtract the number of days in a year.
$100n + a + 5$	Add 15.

a) Try this number trick to see if you get the same result as Connie.
b) Determine the errors in her proof, and then correct them.
c) Explain why your final algebraic expression describes the result of this number trick.

7. According to this proof, $2 = 1$. Determine the error in reasoning.
Let $a = b$.

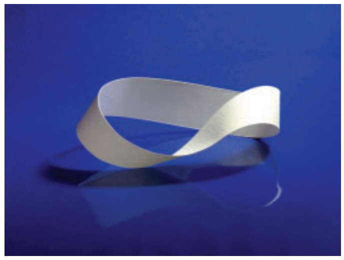
$$\begin{aligned} a^2 &= ab && \text{Multiply by } a. \\ a^2 + a^2 &= a^2 + ab && \text{Add } a^2. \\ 2a^2 &= a^2 + ab && \text{Simplify.} \\ 2a^2 - 2ab &= a^2 + ab - 2ab && \text{Subtract } 2ab. \\ 2a^2 - 2ab &= a^2 - ab && \text{Simplify.} \\ 2(a^2 - ab) &= 1(a^2 - ab) && \text{Factor.} \\ 2 &= 1 && \text{Divide by } (a^2 - ab). \end{aligned}$$

Closing

8. Discuss with a partner how false proofs can appear to be both reasonable and unreasonable at the same time. Summarize your discussion.

Extending

9. Brittney said she could prove that a strip of paper has only one side. She took a strip of paper, twisted it once, and taped the ends together. Then she handed her friend Amber a pencil, and asked Amber to start at any point and draw a line along the centre of the paper without lifting the pencil. Does a strip of paper have only one side? Why or why not?



10. Brenda was asked to solve this problem:

    Three people enjoyed a meal at a Thai restaurant. The waiter brought a bill for \$30. Each person at the table paid \$10.

    Later the manager realized that the bill should have been for only \$25, so she sent the waiter back to the table with \$5.

    The waiter could not figure out how to divide \$5 three ways, so he gave each person \$1 and kept \$2 for himself.

    Each of the three people paid \$9 for the meal.

    $9 \cdot 3 = 27$

    The waiter kept \$2.

    $27 + 2 = 29$

    What happened to the other dollar?

Does the question make sense? How should Brenda answer it?