# 1.3 Using Reasoning to Find a Counterexample to a Conjecture

**YOU WILL NEED**

- calculator
- ruler
- compass

***EXPLORE...***

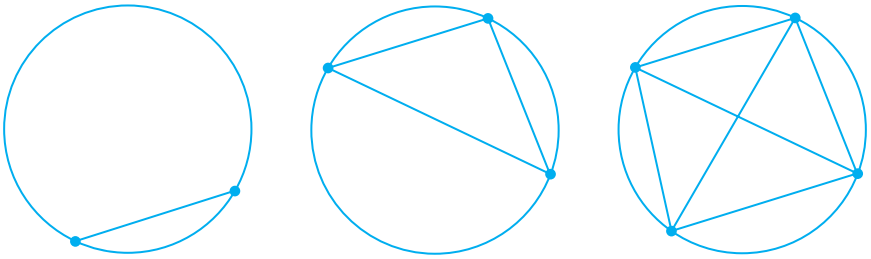
- Six, twelve, ten, one, fifty … Conjecture: All but one of the vowels (a, e, i, o, u, and y) are used to spell numbers. Gather evidence to support or deny this conjecture.

**GOAL**

Invalidate a conjecture by finding a contradiction.

## *LEARN ABOUT the Math*

Kerry created a series of circles. Each circle had points marked on its circumference and joined by chords.



As the number of points on the circumference increased, Kerry noticed a pattern for the number of regions created by the chords.

| Number of Points | 2 | 3 | 4 |
|---|---|---|---|
| Number of Regions | 2 | 4 | 8 |

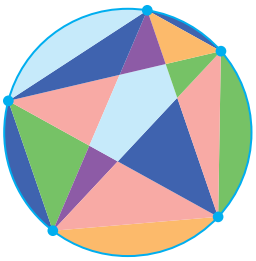
She made the following conjecture: As the number of connected points on the circumference of a circle increases by 1, the number of regions created within the circle increases by a factor of 2.

? How can Kerry test the validity of her conjecture?

**EXAMPLE 1** Testing a conjecture

Gather more evidence to test Kerry's conjecture.

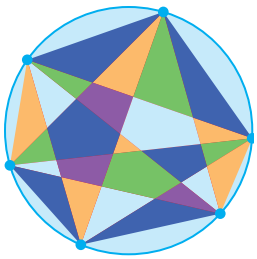
**Zohal's Solution**



| Number of Points | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Number of Regions | 2 | 4 | 8 | 16 | |

I drew another circle and identified five points on its circumference. Then I joined the pairs of points with chords. I coloured the resulting regions to make them easier to count.

My diagram had 16 regions. This supported Kerry's conjecture because the pattern for the resulting regions was $2^1$, $2^2$, $2^3$, $2^4$.



I drew another circle and identified six points on its circumference. Then I joined the pairs of points with chords and coloured the regions.

When I counted, I got only 31 regions, not $2^5$ or 32 as Kerry's conjecture predicts.

| Number of Points | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Number of Regions | 2 | 4 | 8 | 16 | 31 |

The number of regions did not increase by a factor of 2. This **counterexample** disproves Kerry's conjecture.

**counterexample**
An example that invalidates a conjecture.

## Reflecting

**A.** Why do you think Zohal started her development of further evidence by using five points on the circumference of a circle?

**B.** Why is only one counterexample enough to disprove a conjecture?

# *APPLY the Math*

### EXAMPLE 2 Connecting to previous conjectures

In Lesson 1.1, page 9, Francesca and Steffan made conjectures about the difference between consecutive squares.

Steffan's conjecture: The difference between consecutive perfect squares is always an odd number.

Francesca's conjecture: The difference between consecutive perfect squares is always a prime number.

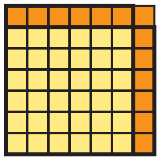
How can these conjectures be tested?

### Luke's Solution: Communicating about Steffan's conjecture and more trials

Steffan's conjecture was true for the pairs of consecutive squares he chose: $2 \times 2$ and $3 \times 3$, $3 \times 3$ and $4 \times 4$, and $5 \times 5$ and $6 \times 6$.

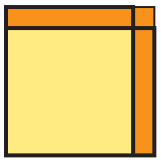
First, I tried $1 \times 1$ and $2 \times 2$. I made the same tile squares as Steffan. When I took away the yellow square, I was left with a pair of tiles that shared an edge with the yellow square and a single tile in the top right corner.

Next, I chose $4 \times 4$ and $5 \times 5$, since Steffan had skipped over these values. I was left with two groups of tiles, each with the same value as a side of the yellow square, plus one extra tile in the top right corner.



I tried consecutive squares of $6 \times 6$ and $7 \times 7$. The difference again showed the same pattern: two groups of tiles, each with the same value as a side of the yellow square, plus a single tile in the top right corner.

These three examples support Steffan's conjecture.



I visualized what the difference would look like for any pair of consecutive squares. There would always be two groups of orange tiles, each with the same value as a side of the smaller yellow square, plus one unpaired orange square in the corner. The total value of the two equal groups would always be an even number, since 2 times any number is even. The unpaired tile would make the difference odd.

All this evidence strengthens the validity of Steffan's conjecture. However, it doesn't prove the conjecture since I haven't tried all the possible cases.

### Pierre's Solution: Connecting more evidence to Francesca's conjecture

Francesca used the consecutive squares of 1 and 2, 3 and 4, and 8 and 9.

I chose values so I could start to fill the gaps between the values that Francesca chose.

$3^2 - 2^2 = 5$

Five is a prime number.

$5^2 - 4^2 = 9$

The next gap was 4 and 5. Nine is not a prime number.

Francesca's conjecture, that the difference between consecutive squares is always a prime number, was disproved since a counterexample was found.

### *Your Turn*

**a)** Find another counterexample to Francesca's conjecture.

**b)** Can you find a counterexample to Steffan's conjecture? Explain.

## EXAMPLE 3 Using reasoning to find a counterexample to a conjecture

Matt found an interesting numeric pattern:

$1 \cdot 8 + 1 = \mathbf{9}$
$12 \cdot 8 + 2 = \mathbf{98}$
$123 \cdot 8 + 3 = \mathbf{987}$
$1234 \cdot 8 + 4 = \mathbf{9876}$

Matt thinks that this pattern will continue.
Search for a counterexample to Matt's conjecture.

### Kublu's Solution

$\mathbf{1} \cdot 8 + \mathbf{1} = 9$
$1\mathbf{2} \cdot 8 + \mathbf{2} = 98$
$12\mathbf{3} \cdot 8 + \mathbf{3} = 987$
$123\mathbf{4} \cdot 8 + \mathbf{4} = 9876$

The pattern seemed to be related to the first factor (the factor that wasn't 8), the number that was added, and the product.

| | A | B |
|---|---|---|
| 1 | 1 · 8 + 1 | 9 |
| 2 | 12 · 8 + 2 | 98 |
| 3 | 123 · 8 + 3 | 987 |
| 4 | 1234 · 8 + 4 | 9876 |
| 5 | 12345 · 8 + 5 | 98765 |
| 6 | 123456 · 8 + 6 | 987654 |
| 7 | 1234567 · 8 + 7 | 9876543 |
| 8 | 12345678 · 8 + 8 | 98765432 |
| 9 | 123456789 · 8 + 9 | 987654321 |

I used a spreadsheet to see if the pattern continued.

The spreadsheet showed that it did.

$12345678\mathbf{10} \cdot 8 + \mathbf{10} = 98\,765\,431\,290$
$1234567\mathbf{80} \cdot 8 + \mathbf{10} = 9\,876\,543\,130$
$12345678\mathbf{10} \cdot 8 + \mathbf{0} = 98\,765\,431\,280$
$1234567\mathbf{80} \cdot 8 + \mathbf{0} = 9\,876\,543\,120$

When I came to the tenth step in the sequence, I had to decide whether to use 10 or 0 in the first factor and as the number to add. I decided to check each way that 10 and 0 could be represented.

The pattern holds true until 9 of the 10 digits are included. At the tenth step in the sequence, a counterexample is found.

Since the pattern did not continue, Matt's conjecture is invalid.

Revised conjecture: When the value of the addend is 1 to 9, the pattern will continue.

I decided to revise Matt's conjecture by limiting it.

### *Your Turn*

If Kublu had not found a counterexample at the tenth step, should she have continued looking? When would it be reasonable to stop gathering evidence if all the examples supported the conjecture? Justify your decision.

## In Summary

### Key Ideas

- Once you have found a counterexample to a conjecture, you have disproved the conjecture. This means that the conjecture is invalid.
- You may be able to use a counterexample to help you revise a conjecture.

### Need to Know

- A single counterexample is enough to disprove a conjecture.
- Even if you cannot find a counterexample, you cannot be certain that there is not one. Any supporting evidence you develop while searching for a counterexample, however, does increase the likelihood that the conjecture is true.

## *CHECK Your Understanding*

**1.** Show that each statement is false by finding a counterexample.

**a)** A number that is not negative is positive.

**b)** All prime numbers are odd.

**c)** All basketball players are tall.

**d)** The height of a triangle lies inside the triangle.

**e)** On maps, the north arrow always points up.

**f)** The square root of a number is always less than the number.

**g)** The sum of two numbers is always greater than the greater of the two numbers.

**h)** As you travel north, the climate gets colder.

**2.** Seth claims that all quadrilaterals with four equal sides are squares. Do you agree or disagree? Justify your decision.

## *PRACTISING*

**3.** Jim claims that whenever you multiply two whole numbers, the product is greater than either of the two factors. Do you agree or disagree? Justify your decision.

**4.** Rachelle claims that the sum of a multiple of 3 and a multiple of 6 must be a multiple of 6. Do you agree or disagree? Justify your decision.

**5.** Hannah examined these multiples of 9: 18, 45, 63, 27, 81, 108, 216. She claimed that the sum of the digits in any multiple of 9 will add to 9. Do you agree or disagree? Justify your decision.

**6.** Colin made the following conjecture: If a quadrilateral has two opposite angles that are right angles, the quadrilateral is a rectangle. Do you agree or disagree? Justify your decision.

**7.** Claire noticed that the digits 4, 5, 6, and 7 could be used to express each value from 1 to 5 as shown to the right. She conjectured that these digits could be used to express each value from 1 to 20. Explain, with examples, whether Claire's conjecture is reasonable.

| Number | Expression |
|---|---|
| 1 | $\frac{7-5}{6-4}$ |
| 2 | $7-6+5-4$ |
| 3 | $\frac{6(7-5)}{4}$ |
| 4 | $7+6-5-4$ |
| 5 | $5(\sqrt{64}-7)$ |

**8.** George noted a pattern that was similar to Matt's pattern in Example 3. George conjectured that the products would follow the pattern of ending with the digit 5 or 0. Gather evidence about George's conjecture. Does your evidence strengthen or disprove George's conjecture? Explain.

$$1 \cdot 4 + 1 = 5$$
$$12 \cdot 4 + 2 = 50$$
$$123 \cdot 4 + 3 = 495$$
$$1234 \cdot 4 + 4 = 4940$$

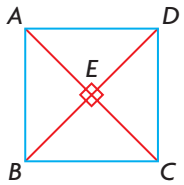
**9.** From questions 2 to 8, choose a conjecture that you have disproved. Based on your counterexample, revise the conjecture to make it valid.

**10.** Patrice studied the following table and made this conjecture: The sums of the squares of integers separated by a value of 2 will always be even.

| | | | | |
|---|---|---|---|---|
| $(-1)^2 + 1^2 = 2$ | $2^2 + 4^2 = 20$ | $(-3)^2 + (-5)^2 = 34$ | $4^2 + 6^2 = 52$ | $0^2 + 2^2 = 4$ |

Is Patrice's conjecture reasonable? Explain.

**11.** Geoff made the following conjecture: If the diagonals of a quadrilateral are perpendicular, then the quadrilateral is a square. Determine the validity of his conjecture. Explain your results.




**12.** Amy made the following conjecture: When any number is multiplied by itself, the product will be greater than this starting number. For example, in $2 \cdot 2 = 4$, the product 4 is greater than the starting number 2. Meagan disagreed with Amy's conjecture, however, because $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and $\frac{1}{4}$ is less than $\frac{1}{2}$. How could Amy's conjecture be improved? Explain the change(s) you would make.

**13.** Create a general statement that is true in some cases but not in every case. Provide examples that support your statement. Provide a counterexample.

**14.** Tim conjectured that all natural numbers can be written as the sum of consecutive natural numbers, based on these examples:

$10 = 1 + 2 + 3 + 4$ $\qquad$ $12 = 3 + 4 + 5$

$9 = 4 + 5$ $\qquad$ $94 = 22 + 23 + 24 + 25$

Do you agree or disagree with Tim's conjecture? Justify your decision.

**15.** Blake claimed that all odd numbers can be expressed as the sum of three prime numbers. Explain, with evidence, the reasonableness of his claim.

**16.** German mathematician Christian Goldbach made the conjecture that every even number greater than 2 is the sum of two prime numbers. For example:

$14 = 3 + 11$
$30 = 7 + 23$

This conjecture has become known as Goldbach's conjecture. No one has ever been able to prove that it is true for all even numbers, but no one has ever found a counterexample.

**a)** Find three other examples that support Goldbach's conjecture.
**b)** If a counterexample exists, describe what it would look like.

**17.** Jarrod discovered a number trick in a book he was reading: Choose a number. Double it. Add 6. Double again. Subtract 4. Divide by 4. Subtract 2.

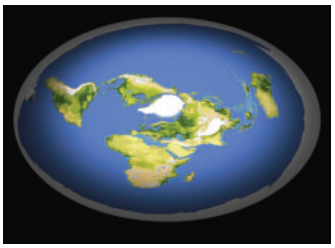
**a)** Try the trick several times. Make a conjecture about the relation between the number picked and the final result.
**b)** Can you find a counterexample to your conjecture? What does this imply?

## History | Connection

### Reasoning in Science

Scientific discoveries are often based on inductive reasoning. Scientists make conjectures after examining all the evidence they have. They test their conjectures by conducting experiments in which they compare how the universe actually behaves with how they predict it should behave. If the experiments have the predicted results, then the scientists' conjectures are strengthened. If the results contradict the conjectures, then the scientists use the results to revise their conjectures or to make new conjectures.

Many scientific conjectures have been changed over time as new information has come to light. One such conjecture relates to Earth itself. In ancient times, the world was believed to be flat. The flat world conjecture was held to be true until counterexamples required it to be changed. Aristotle, in about 330 BCE, was one of the first people to conjecture that the world was not a flat disc but a sphere. Pliny the Elder, in the 1st century CE, was able to suggest that the flat world conjecture was no longer valid because of the evidence that had been developed to contradict it. Pliny considered the possibility of an imperfect sphere. Modern evidence, from satellite images and spaceships, has provided no counterexamples to the spheroid theory, so this theory is generally accepted as fact today.



If Earth were a flat disc, it might look like this from space.

However, it actually looks like this.

**A.** What other conjectures about our universe have been revised after new evidence was gathered?
**B.** How does inductive reasoning play a part in our beliefs and understanding about our universe?

## Closing

**18.** What relationship exists among inductive reasoning, evidence, and counterexamples?

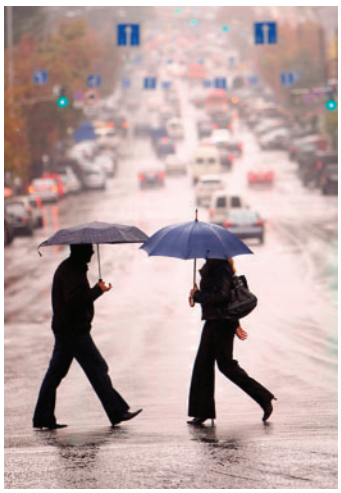
## Extending

**19.** Serge made the following conjecture: When 3 is subtracted from a perfect square that is greater than 4, the result is always a composite number. For example:

$15^2 - 3 = 222$

222 is a composite number because it is divisible by factors other than 1 and itself. Do you agree with Serge's conjecture? Justify your decision.

**20.** Environment Canada explains probability of precipitation forecasts as subjective estimates. These forecasts or estimates are actually conjectures based on numerical evidence and regional topography. They are important for people, such as building contractors, farmers, and surveyors, who work outside and for anyone else who is planning outdoor activities. As the time for precipitation comes closer, a forecaster's conjecture is revised to reflect newer data and increased accuracy.



**a)** Environment Canada's chart below seems to be written for adults who live in a city or its suburbs. Revise the chart so that it is written for you, by including activities that you participate in.

**b)** Explain your revisions. How did you decide which probabilities of precipitation could affect your activities?

**A User's Guide to Probability of Precipitation**

| | |
|---|---|
| 0% | No precipitation even though it may be cloudy. |
| 10% | Little likelihood of rain or snow: only 1 chance in 10. |
| 20% | No precipitation is expected. |
| 30% | If you go ahead with your outdoor plans, keep an eye on the weather. |
| 40% | An umbrella is recommended. Make alternate plans for outdoor activities that are susceptible to rain. Not a good day to pave the driveway. Keep your fingers crossed! |
| 50% | It's 50–50 on whether you get precipitation or not. |
| 60% | Want to water your lawn? The odds are favourable that Mother Nature might give you some help. |
| 70% | Consider the effect of precipitation on your plans for outdoor activities. The chance for no precipitation is only 3 in 10! |
| 80% | Rain or snow is likely. |
| 90% | The occurrence of precipitation is a near certainty. |
| 100% | Precipitation is a certainty. |

Environment Canada, Probability of Precipitation brochure

**21.** Mohammed claims that the expression $n^2 + n + 2$ will never generate an odd number for a positive integer value of $n$. Do you agree or disagree? Justify your decision.

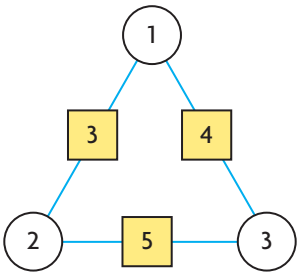
## Applying Problem-Solving Strategies

### Analyzing a Number Puzzle

Arithmagons are number puzzles that have addition as the central operation. They are based on polygons, with a circle at each vertex and a box on each side.

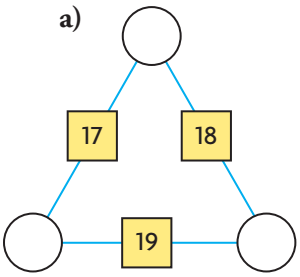
### The Puzzle

The number in each box of an arithmagon is the sum of the two numbers in the circles adjacent to the box.

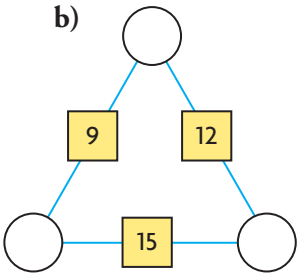



**A.** Solve the three triangular arithmagons below.

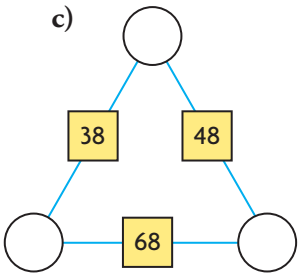
**a)**




**b)**




**c)**




**B.** Create your own arithmagon. Exchange arithmagons with a partner, and solve your partner's arithmagon.

### The Strategy

**C.** What patterns did you notice?

**D.** What relationship exists between the numbers in the circles and the numbers in the opposite boxes?

**E.** Describe the strategy you used to solve your partner's arithmagon.

**F.** Of the arithmagons you've encountered, which was easiest to solve? Explain.