UNIT

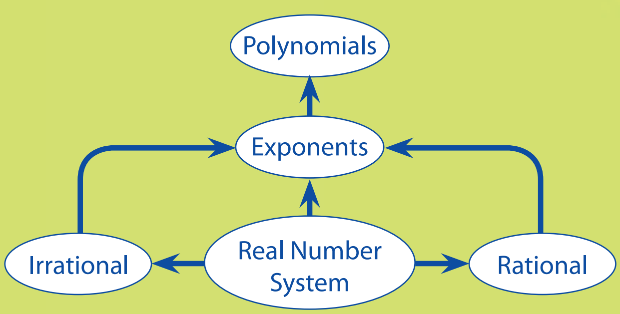
2 Algebra and Number

Consider how much of your daily life involves numbers! How do numbers relate to the careers you see here? Whatever you aspire to be, numbers will play an important role in what you do.

Number, and more specifically the real number system, forms the foundation for mathematics. One field of mathematics, algebra, includes the multiplication of polynomials and factoring of trinomials. In this unit, you will use square roots, cube roots, and irrational numbers along with exponents to solve problems in a variety of contexts. You will also develop and use algebraic skills to solve problems involving the multiplying and factoring of polynomials.

Your Algebra and Number Organizer

You can use this algebra and number organizer to see how the concepts in this unit are connected. You will see this organizer on the first page of each chapter. The concepts covered in the chapter will be highlighted.



Looking Ahead

In this unit, you will solve problems involving …

- square roots and cube roots
- integral and rational exponents
- irrational numbers, including radicals
- multiplying polynomials
- factoring polynomials

Unit 2 Project

Math in Art

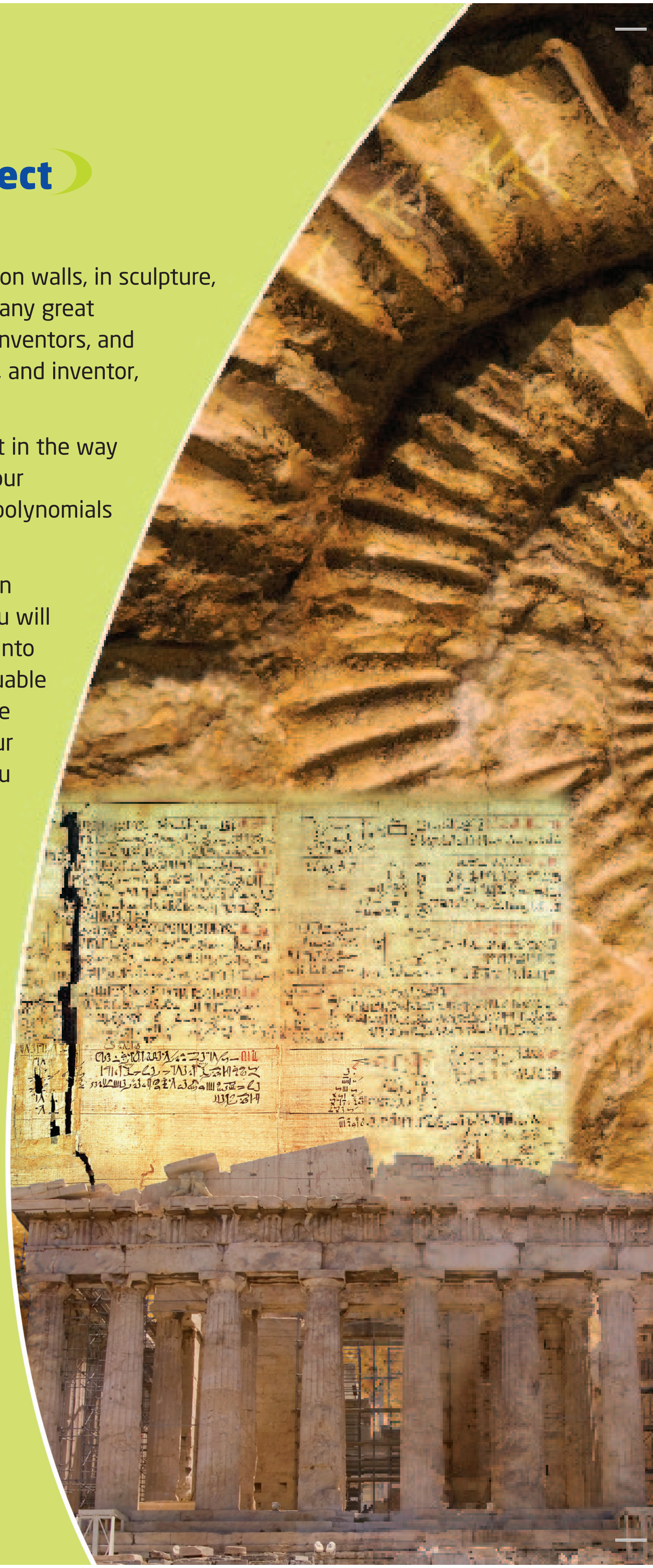
Art takes many different forms. It can be seen on walls, in sculpture, in architecture, on clothing, and even in nature. Many great mathematicians were also philosophers, historians, inventors, and artists. Leonardo da Vinci was a mathematician, artist, and inventor, who incorporated mathematics into his art.

In the Unit 2 project, you will discover that there is art in the way that numbers interact with each other. You will use your understanding of irrational numbers, exponents, and polynomials to incorporate math into your own work of art.

Unit Project questions and activities are included in Chapters 4 and 5. As you move through Chapter 4, you will discover how mathematics historically has been built into art. As you move through Chapter 5, you will gain valuable insights needed to create your own piece of art. At the end of Chapter 5, you will create your piece of art. Your final presentation will include a description of how you incorporated mathematics in your art.

While completing your project, you will ...

- discover relationships among irrational numbers, exponents, polynomials, and art (Chapters 4 and 5)
- apply your understanding of radicals to analyse the golden ratio (Chapter 4)
- create your own art and explain how you used mathematics in its development (Chapter 5)




BABYLONIAN NUMERALS

CHAPTER 4

Exponents and Radicals

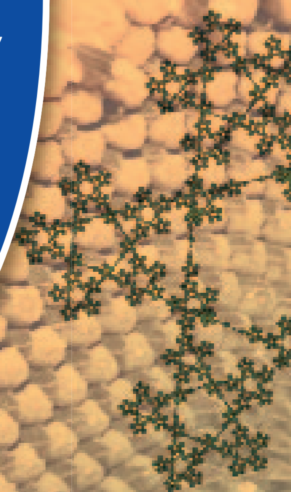
Exponents have been used to help model and solve problems since the time of the Babylonians, about 4000 years ago. For example, Plimpton 322 is a stone tablet written in Babylonian script (about 1900 to 1600 B.C.E.). The tablet includes sets of three positive numbers, such as 3, 4, and 5. These can be the measurements of the sides of a right triangle. These numbers are known as Pythagorean triples.

Today, we use exponents for solving problems that range from calculating interest earned on savings to estimating how fast bacteria can grow.

Big Ideas

When you have completed this chapter, you will be able to ...

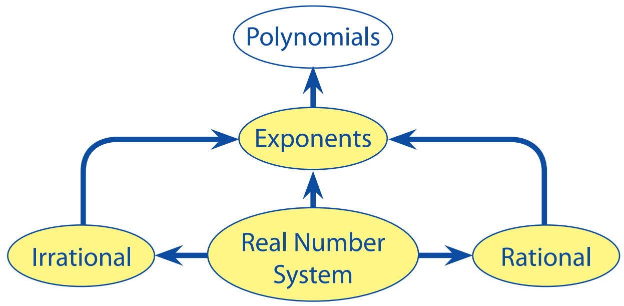
- solve problems that involve square roots and cube roots
- solve problems involving powers with integral and rational exponents
- represent, identify, and simplify irrational numbers

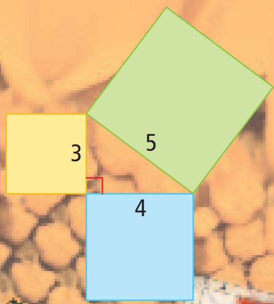


Key Terms

perfect square
square root
perfect cube
cube root
prime factorization
irrational number
radical
radicand
index
mixed radical
entire radical

Your Algebra and Number Organizer





Artist

Artists create art to communicate ideas. In addition to artistic and technical skills, artists are problem solvers who often use math concepts to represent reality. Artists use a variety of methods and materials to create their works. Some artists use concrete materials to create their designs. Multimedia artists and animators use computer design software to model objects in 3-D.

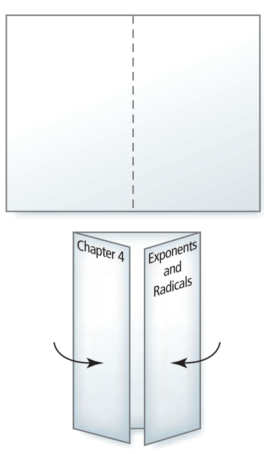
Web Link

If you are interested in learning more about artists, go to www.mhrmath10.ca and follow the links.

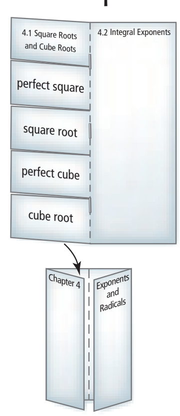
Foldables Study Tool

Make the following Foldable™ to take notes on what you will learn in Chapter 4.

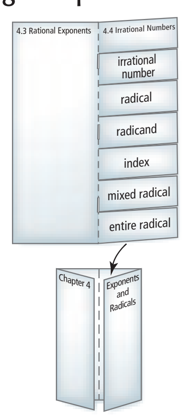
1. Fold a sheet of 11 × 17 paper as shown. On the outside front flaps, add the labels shown below.



2. Fold and label a sheet of 8.5 × 11 paper as shown. Cut tabs along the lines on the left half. Attach the tabbed page inside the left flap.



3. Fold and label another sheet of 8.5 × 11 paper as shown. Cut tabs along the lines on the right half. Attach the tabbed page inside the right flap.



4. On the centre panel of the Foldable™, write the title Exponent Laws. On the back, write the title What I Need to Work On.

4.1 Square Roots and Cube Roots

Focus on …

- determining the square root of a perfect square and explaining the process
- determining the cube root of a perfect cube and explaining the process
- solving problems involving square roots or cube roots

Workers apply what they know about surface area and volume when working with square shapes and cubes.

A house painter must calculate the surface area of the walls of a house when preparing a cost estimate. If you know the area of a square wall, how could you calculate the side lengths?

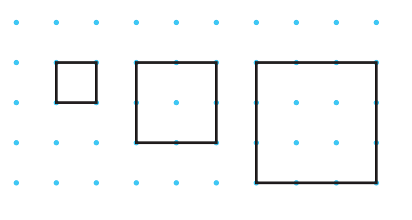
A designer must calculate the size of the case required to enclose a speaker for a sound system. If you know the volume of a cube-shaped box, how could you calculate the edge lengths?

Materials

- square dot paper
- isometric dot paper

Investigate Square Roots and Cube Roots

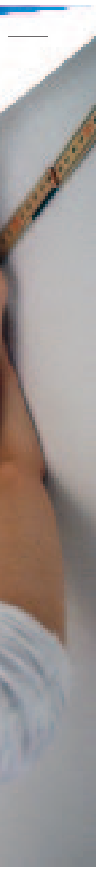
1. a) Determine the area of each square shown. Record the information in a table.



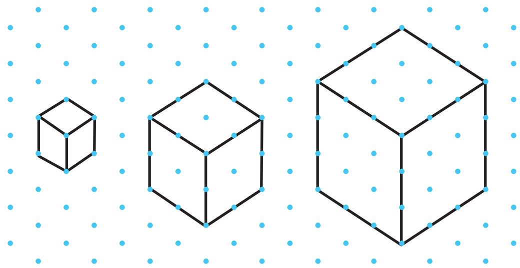
Side Length	Area in Exponential Form	Area

b) Extend the pattern for squares with dimensions of 4, 5, and 6 units.

c) What is the relationship between the side length of a square and the area of the square?



2. **a)** Determine the volume of each cube shown. Record the information in a table.



Edge Length	Volume in Exponential Form	Volume

b) Extend the pattern for cubes with dimensions of 4, 5, and 6 units.

c) What is the relationship between the edge length of a cube and the volume of the cube?

3. **Reflect and Respond**

Discuss with a partner.

a) What strategy could you use to find the side length of a square if you were given the area?

b) What strategy could you use to find the edge length of a cube if you were given the volume?

c) Explain, using a diagram, how you could predict
- the side length of a square with an area of 64 square units
- the edge length of a cube with a volume of 343 cubic units

Link the Ideas

Perfect squares and **square roots** are related to each other. The number 25 is a perfect square. It is formed by multiplying two factors of 5 together.

$(5)(5)$ or $5^2 = 25$

The symbol for square root is $\sqrt{\ }$.

The square root of 25 is 5, or $\sqrt{25} = \sqrt{(5)(5)}$
$= \sqrt{5^2}$
$= 5$

perfect square
- a number that can be expressed as the product of two equal factors
- for example, $16 = (4)(4)$ or 4^2

square root
- one of two equal factors of a number
- for example, $\sqrt{49} = \sqrt{(7)(7)}$
$= 7$

perfect cube

- a number that is the product of three equal factors
- for example, $64 = (4)(4)(4)$ or 4^3

cube root

- one of three equal factors of a number
- for example, $\sqrt[3]{512} = \sqrt[3]{(8)(8)(8)}$ $= 8$

Perfect cubes and **cube roots** are related to each other. The number 27 is a perfect cube. It is formed by multiplying three factors of 3 together.

$(3)(3)(3)$ or $3^3 = 27$

The symbol for cube root is $\sqrt[3]{\ }$.

The cube root of 27 is 3, or $\sqrt[3]{27} = \sqrt[3]{(3)(3)(3)}$

$= \sqrt[3]{3^3}$

$= 3$

Some numbers are both perfect squares and perfect cubes.

$64 = (8)(8)$ and $64 = (4)(4)(4)$

$= 8^2$ $\quad$ $= 4^3$

Therefore, 64 is a perfect square and a perfect cube.

Example 1 Identify Perfect Squares and Perfect Cubes

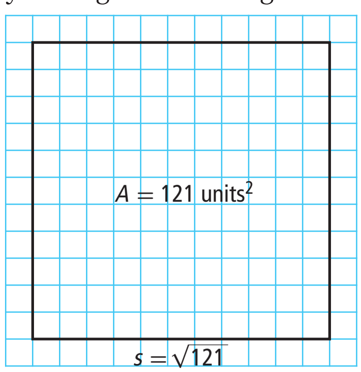
State whether each of the following numbers is a perfect square, a perfect cube, both, or neither.

a) 121 **b)** 729 **c)** 356

Solution

a) To decide whether 121 is a perfect square you might use a diagram.

$10^2 = 100$ Too low
$12^2 = 144$ Too high
$11^2 = 121$ Correct!



A square with side lengths of 11 units has an area of 121 units^2.

$(11)(11) = 121$.

Therefore, 121 is a perfect square.

To decide whether 121 is a perfect cube, you could use guess and check.

$4^3 = 64$ Too low
$5^3 = 125$ Too high

No whole number cubed results in a product of 121.

Therefore, 121 is not a perfect cube.

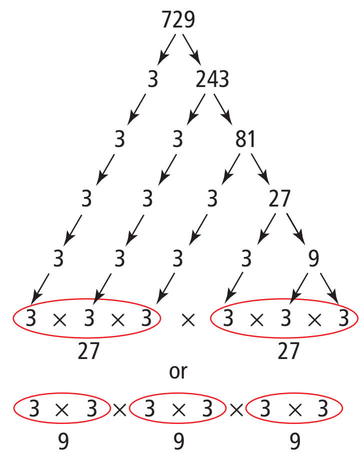
Web Link

To learn more about perfect squares and square roots, go to www.mhrmath10.ca and follow the links.

To learn more about perfect cubes and cube roots, go to www.mhrmath10.ca and follow the links.

b) For 729, you might use **prime factorization**. Prime factorization involves writing a number as the product of its prime factors. A factor tree helps organize the prime factors.

Record the prime factorization for 729. Then, identify the factors that can be squared or cubed to form the product 729.



These two groups indicate the square root of 729.

These three groups indicate the cube root of 729.

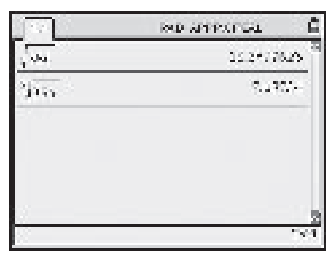
You can write 729 as the product of (27)(27) = 27^2. Therefore, 729 is a perfect square.

You can write 729 as the product of (9)(9)(9) = 9^3. Therefore, 729 is a perfect cube.

c) For 356, you might use a calculator.

C 356 $\sqrt{x}$ 18.867962

C 356 2nd $\sqrt[x]{y}$ 3 = 7.08734



Key sequences vary among calculators. Check the key sequence for determining square roots and cube roots of numbers on your calculator. Record the correct sequence for your calculator.

Since the square root is not a whole number, 356 is not a perfect square.
Since the cube root is not an integer, 356 is not a perfect cube.
The number 356 is neither a perfect square nor a perfect cube.

Your Turn

State whether each number is a perfect square, a perfect cube, both, or neither. Use a variety of methods.

a) 125 **b)** 196 **c)** 4096

prime factorization

- the process of writing a number written as a product of its prime factors.
- the prime factorization of 24 is 2 × 2 × 2 × 3.

Web Link

To learn more about prime factorization and to use a prime factorization tool, go to www.mhrmath10.ca and follow the links.

Did You Know?

Between 1850 and 1750 B.C.E., the Babylonians were applying the Pythagorean relationship. They recorded tables of square roots and cube roots on clay tablets. This was long before Pythagoras was born.

Did You Know?

Canada is the largest producer of uranium in the world. It provides about one third of the world's supply. Uranium is mined mainly in Northern Ontario and Saskatchewan. The mines in Saskatchewan provide the highest grade uranium.

Example 2 Solve Problems Involving Square Roots and Cube Roots

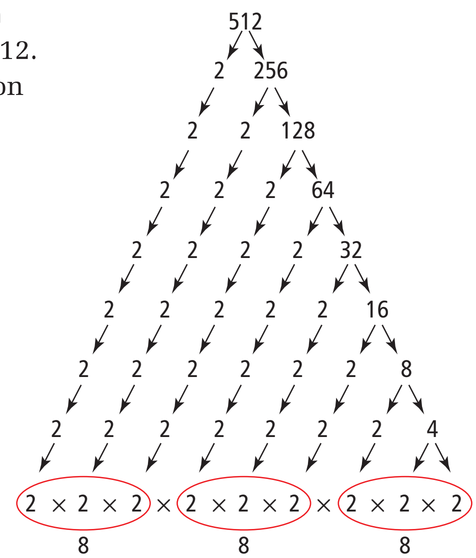
The uranium that Saskatchewan produces in a year has a volume of about 512 m^3. If this volume were made into a single cube, what would be the dimensions of the cube?

Solution

The volume of a cube of length x is given by $V = x^3$. Determine the dimensions of the cube, x, by calculating the cube root of the volume, or $x = \sqrt[3]{V}$.

Method 1: Use Prime Factorization

Determine the cube root of 512. Record the prime factorization for 512. Then, identify the factors that can be cubed to form 512.



Since there are three equal groups, you know that 512 is a perfect cube.

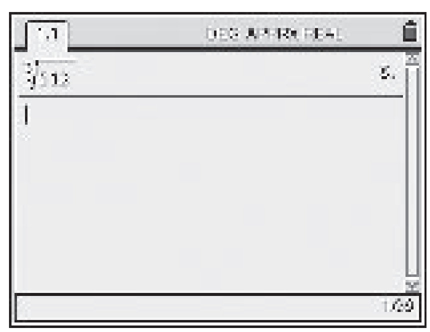
How do you know that 512 is not a perfect square?

The cube root of 512 is 8.
The cube would be 8 m in length, height, and width.

Method 2: Use a Calculator

C 512 2nd $\sqrt[x]{y}$ 3 = 8.

The cube would be 8 m in length, height, and width.



Your Turn

a) A floor mat for gymnastics is a square with an area of 196 m^2. What is its side length?

b) The volume of a cubic box is 27 000 $in.^3$ Use two methods to determine its dimensions.

Key Ideas

- A perfect square is the product of two equal factors. One of these factors is called the square root.

  36 is a perfect square: $\sqrt{36} = 6$ because $6^2 = 36$

- A perfect cube is the product of three equal factors, One of these factors is called the cube root.

  -125 is a perfect cube: $\sqrt[3]{-125} = -5$ because $(-5)^3 = -125$

- Numbers can be both perfect squares and perfect cubes.

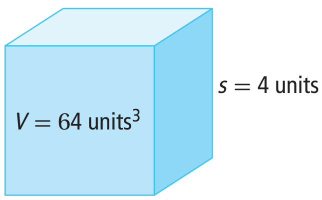
  15 625 is a perfect square: $125^2 = 15\ 625$

  15 625 is a perfect cube: $25^3 = 15\ 625$

- You can use diagrams or manipulatives, factor trees, or a calculator to solve problems involving square roots and cube roots.

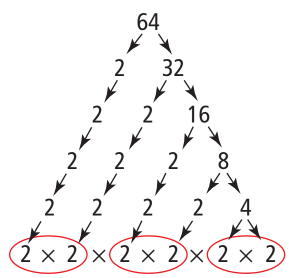
  Determine the cube root of 64.

  - Use a diagram.



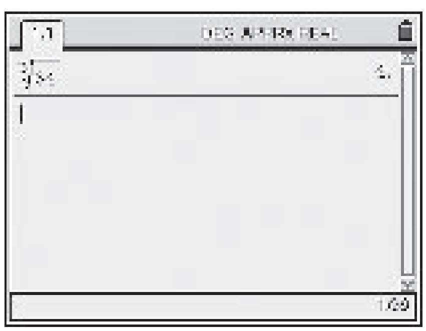
The edge lengths represent the cube root:
(4)(4)(4) = 64.

  - Use prime factorization.



There are three equal groups of 4. Therefore, the cube root of 64 is 4.

  - Use a calculator.



Check Your Understanding

Practise

1. What is the value of each expression? Express your answers as integers or fractions.

a) 7^2 **b)** -50^2 **c)** $(-3)^2$

d) $\frac{4^2}{5}$ **e)** $\frac{3}{2^2}$ **f)** $\left(\frac{3}{4}\right)^2$

2. Evaluate. Give your answers as integers or fractions.

a) 2^3 **b)** -4^3 **c)** $(-5)^3$

d) $\frac{2^3}{4}$ **e)** $\frac{3}{6^3}$ **f)** $\left(\frac{2}{3}\right)^3$

3. What is the value of each expression?

a) $\sqrt{49}$ **b)** $\sqrt{169}$ **c)** $\sqrt{(25)(4)}$

d) $\frac{16}{\sqrt{64}}$ **e)** $\frac{\sqrt{36}}{3}$ **f)** $\sqrt{9x^2}$

4. Evaluate.

a) $\sqrt[3]{1}$ **b)** $\sqrt[3]{(8)(27)}$ **c)** $\sqrt[3]{8000}$

d) $\frac{\sqrt[3]{64}}{2}$ **e)** $\sqrt[3]{\frac{27}{125}}$ **f)** $\sqrt[3]{64a^3}$

5. Identify each number as a perfect square, a perfect cube, or both. Support your answers using a diagram or a factor tree.

a) 1 **b)** 1000 **c)** 81

d) 169 **e)** 216 **f)** 1024

6. State whether each of the following numbers is a perfect square, a perfect cube, both, or neither.

a) 144 **b)** 2197 **c)** 16

d) 225 **e)** 15 625 **f)** 117 649

7. Evaluate using prime factorization. Explain the process.

a) $\sqrt{100}$ **b)** $\sqrt[3]{8}$ **c)** $\sqrt{81}$

d) $\sqrt[3]{27}$ **e)** $\sqrt{144}$ **f)** $\sqrt{576}$

8. Calculate.

a) $\sqrt{196}$ **b)** $\sqrt[3]{4096}$ **c)** $\sqrt[3]{9261}$

d) $\sqrt[3]{3375}$ **e)** $\sqrt{961}$ **f)** $\sqrt[3]{4913}$

9. Connor needs to replace the edging on a square rug. If the rug has an area of 25 m^2, what length of edging does he need?

10. Serena collected all the garbage she created in one year. The volume of the cube it formed was 343 ft^3. What was the edge length of the cube?

Apply

11. A square wrestling mat has an area of 1444 ft^2.
 a) Before calculating the side length of the mat, estimate two whole numbers between which the answer falls. Which number do you think the answer is closer to?
 b) Calculate the side length.
 c) How does your estimate compare to the calculated answer?

12. Star quilts are squares with a minimum area of 1 m^2 and a maximum area of 9 m^2. What are the possible whole number dimensions of such a quilt?

Did You Know?

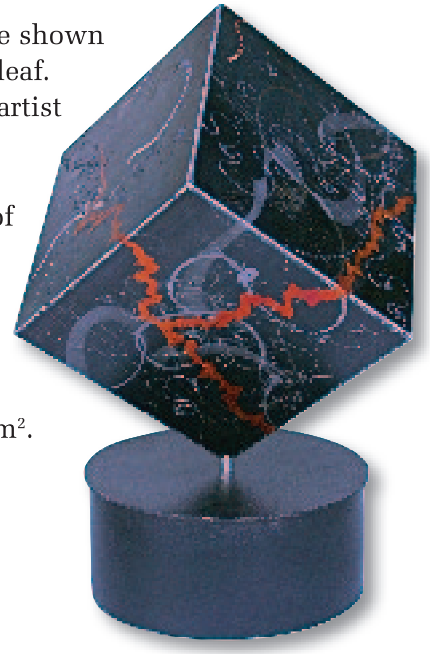
The star quilt is a pattern used by many cultures including the Lakota, Dakota, other Sioux nations, and Europeans. It was inspired from the design for buffalo robes. When buffalo were no longer available, the star quilt replaced the buffalo robe in Aboriginal traditions.

13. **Unit Project** The mural shown below was originally created to celebrate Alberta's Centennial in 2005. It was installed at the Centre d'arts visuels de l'Alberta in Edmonton, AB. The mural symbolizes the unity of the francophone communities throughout Alberta. Your art class decides to create a mural mosaic. Your mosaic will highlight the regions of the province or territory where you live.
 a) The class mosaic will be composed of 15-cm by 15-cm squares. How many squares will be needed to create a mural that covers an area of 2.7 m^2?
 b) Design a mural to show a geometric representation of square roots.
 c) How is the mural a geometric representation of square roots?

Les régions se racontent (The regions tell their story)

14. A recycling depot compresses cardboard into cubic bales. If each bale has a volume of 46 656 in.^{3}, what are its edge lengths?

15. **Unit Project** The cubic sculpture shown here is made of steel with copper leaf. It was created by Tony Bloom, an artist from Canmore, AB.



a) If it has a volume of 4913 in.^{3}, what is the length of one edge of the cube?

b) Explain how the sculpture is a geometric representation of a cube root.

16. The surface area of a die is 600 mm^2. What is the volume of the die?

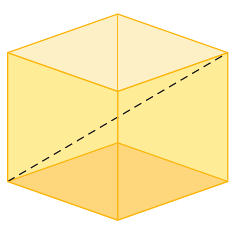
Extend

17. Meteorologists use the formula $D^3 = 684t^2$ to describe violent storms, such as tornadoes and hurricanes. D is the diameter of the storm, in kilometres, and t is the number of hours it will last.

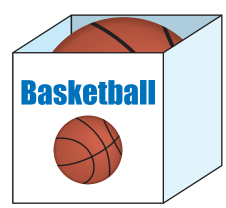
a) If a storm lasts for 4 h, what is its diameter?

b) If the diameter of a hurricane is 30 km, how long will it last?

18. A cube has a volume of 3375 cm^3. What is the diagonal distance through the cube from one corner to the opposite corner?



19. A manufacturer is designing an open, cube-shaped box to hold a basketball. The basketball has a volume of 2304π cm^3.

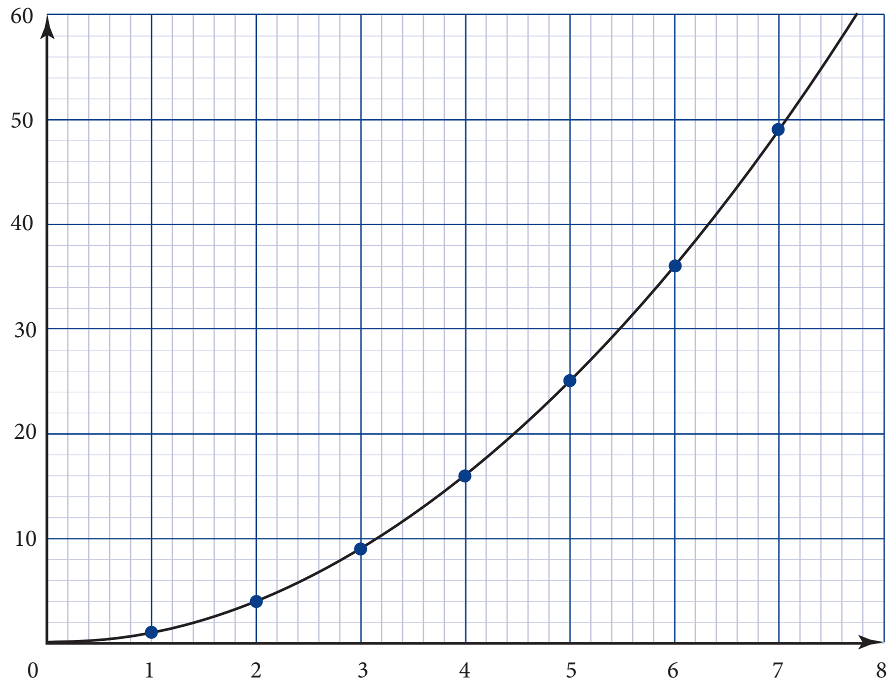


a) How much cardboard is needed to create the smallest box possible using the least amount of material? Do not include seam overlap in your calculations.

b) What is the volume of the box? What are its dimensions?

Create Connections

20. The following graph can be used to determine squares and square roots.



a) Use the graph to complete the following table of values.

Number	0	1	2	3			6	7	
Number Squared	0	1			16	25			64

b) Based on the table, how would you label the axes on the graph?

c) What does each small unit represent on the horizontal axis? vertical axis?

d) Explain how you could use the graph to find the value for 5^2.

e) How could you use the graph to evaluate $\sqrt{49}$?

f) Show how you could use the graph to determine the approximate value for $\sqrt{18}$. Multiply your answer by itself. How close is your product to 18?

g) What is an approximation for $(6.2)^2$?

21. a) Make an arithmetic question, involving a square root, that has a value of $\frac{2}{3}$.

b) Make an arithmetic question, involving a cube root, that has a value of $\frac{2}{3}$.

4.2 Integral Exponents

Focus on ...

- applying the exponent laws to expressions using rational numbers or variables as bases and integers as exponents
- converting a power with a negative exponent to an equivalent power with a positive exponent
- solving problems that involve powers with integral exponents

The Rhind mathematical papyrus (RMP) is a valuable source of information about ancient Egyptian mathematics. This practical handbook includes problems that illustrate how Egyptians solved problems related to surveying, building, and accounting. The RMP was written in approximately 1650 B.C.E. How do archaeologists know this?

One way to assess the age of organic matter is by using carbon-14 dating. While they are living, all living things absorb radioactive carbon-14. Papyrus is made from papyrus plants. As soon as the papyrus dies, it stops taking in new carbon. The carbon-14 decays at a constant, known rate and is not replaced. Scientists can measure the amount of carbon-14 remaining. They use a formula involving exponents to accurately assess the age of the papyrus.

Is the quantity of carbon-14 increasing or decreasing? Do you think the exponent in the formula would be positive or negative? Why?

Did You Know?

Carbon-14 dating is accurate for dating artifacts up to about 60 000 years old.

Investigate Negative Exponents

Materials
- ruler

1. On a sheet of paper, draw a line 16 cm long and mark it as shown.

2. Mark a point halfway between 0 and 16. Label the point with its value and its equivalent value in exponential form (2^x). Repeat this procedure until you reach a value of 1 cm.

a) How many times did you halve the line segment to reach 1 cm?

b) What do you notice about the exponents as you keep reducing the line segment by half?

3. a) Mark the halfway point between 0 and 1. What fraction does this represent?

b) Using the pattern established in step 2, what is the exponential form of the fraction?

c) Halve the remaining line segment two more times.

4. Use a table to summarize the line segment lengths and the equivalent exponential form in base 2.

5. Reflect and Respond

a) Describe the pattern you observe in the exponents as the distance is halved.

b) Is there a way to rewrite each fraction so that it is expressed as a power with a positive exponent? Try it. Compare this form to the equivalent power with a negative exponent. What is the pattern?

c) Create a general form for writing any power with a negative exponent as an equivalent power with a positive exponent.

6. a) Carbon-14 has a half-life of 5700 years. This means the rate of decay is $\frac{1}{2}$ or 2^{-1} every 5700 years. What fraction of carbon-14 would be present in organic material that is 11 400 years old? 17 100 years old? Express each answer as a power with a negative exponent. Explain how you arrived at your answers.

b) Suggest types of situations when a negative exponent might be used.

Did You Know?

The *half-life* of a radioactive element is the amount of time it takes for half of the atoms in a sample to decay. The half-life of a radioactive element is constant. It does not depend upon the quantity or the amount of time that has gone by.

Link the Ideas

You can use the exponent laws to help simplify expressions with integral exponents.

Exponent Law	
Note that a and b are rational or variable bases and m and n are integral exponents.	
Product of Powers	$(a^m)(a^n) = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Power of a Power	$(a^m)^n = a^{mn}$
Power of a Product	$(ab)^m = (a^m)(b^m)$
Power of a Quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
Zero Exponent	$a^0 = 1, a \neq 0$

To simplify expressions with integral exponents, you can use the following principle as well as the exponent laws.

A power with a negative exponent can be written as a power with a positive exponent.

- $a^{-n} = \frac{1}{a^n}, a \neq 0$ $\qquad 2^{-3} = \frac{1}{2^3}$
- $\frac{1}{a^{-n}} = a^n, a \neq 0$ $\qquad \frac{1}{2^{-3}} = 2^3$

Example 1 Multiply or Divide Powers With the Same Base

Write each product or quotient as a power with a single exponent.

a) $(5^8)(5^{-3})$

b) $(0.8^{-2})(0.8^{-4})$

c) $\frac{x^5}{x^{-3}}$

d) $\frac{(2x)^3}{(2x)^{-2}}$

Solution

Use the exponent laws for multiplying or dividing powers with the same base and integral exponents.

a) Method 1: Add the Exponents

$$(5^8)(5^{-3}) = 5^{8+(-3)}$$
$$= 5^5$$

How do you know that you can add the exponents?

Method 2: Use Positive Exponents

Convert the power with a negative exponent to one with a positive exponent. Rewrite as a division statement. Then, since the bases are the same, you can subtract the exponents.

$$\begin{aligned}(5^8)(5^{-3}) &= (5^8)\left(\frac{1}{5^3}\right)\\ &= \frac{5^8}{5^3}\\ &= 5^{8-3}\\ &= 5^5\end{aligned}$$

How do you know that you can subtract the exponents?

b) Method 1: Add the Exponents

$$\begin{aligned}(0.8^{-2})(0.8^{-4}) &= 0.8^{-2+(-4)}\\ &= 0.8^{-6}\end{aligned}$$

Method 2: Use Positive Exponents

$$\begin{aligned}(0.8^{-2})(0.8^{-4}) &= \left(\frac{1}{0.8^2}\right)\left(\frac{1}{0.8^4}\right)\\ &= \frac{1}{(0.8^2)(0.8^4)}\\ &= \frac{1}{0.8^{(2+4)}}\\ &= \frac{1}{0.8^6}\end{aligned}$$

Which method do you prefer? Why?

c) $$\begin{aligned}\frac{x^5}{x^{-3}} &= x^{5-(-3)}\\ &= x^{5+3}\\ &= x^8\end{aligned}$$

What strategy was used?

d) Method 1: Subtract the Exponents

$$\begin{aligned}\frac{(2x)^3}{(2x)^{-2}} &= (2x)^{3-(-2)}\\ &= (2x)^5\end{aligned}$$

Method 2: Use Positive Exponents

$$\begin{aligned}\frac{(2x)^3}{(2x)^{-2}} &= (2x)^3\,(2x)^2\\ &= (2x)^{3+2}\\ &= (2x)^5\end{aligned}$$

Your Turn

Simplify each product or quotient.

a) $(2^{-3})(2^5)$

b) $\frac{7^{-5}}{7^3}$

c) $\frac{(-3.5)^4}{(-3.5)^{-3}}$

d) $\frac{(3y)^2}{(3y)^{-6}}$

Example 2 Powers of Powers

Write each expression as a power with a single, positive exponent. Then, evaluate where possible.

a) $(4^3)^{-2}$ **b)** $[(a^{-2})(a^0)]^{-1}$ **c)** $\left(\frac{2^4}{2^6}\right)^{-3}$ **d)** $\left[\left(\frac{3}{4}\right)^{-2}\left(\frac{3}{4}\right)^{-2}\right]^{-2}$

Solution

a) Multiply the exponents. Then, rewrite as a positive exponent.

$$\begin{aligned}(4^3)^{-2} &= 4^{(3)(-2)}\\ &= 4^{-6}\\ &= \frac{1}{4^6}\\ &= \frac{1}{4096}\end{aligned}$$

b) Since the bases are the same, you can multiply the powers by adding the exponents. Raise the result to the exponent −1. Then, multiply.

$$\begin{aligned}[(a^{-2})(a^0)]^{-1} &= (a^{-2+0})^{-1}\\ &= (a^{-2})^{-1}\\ &= a^{(-2)(-1)}\\ &= a^2\end{aligned}$$

How could you use your knowledge of the exponent laws for zero exponents to help simplify the original expression?

c) Method 1: Simplify Within the Brackets

Since the bases are the same, you can subtract the exponents. Raise the result to the exponent −3. Then, multiply.

$$\begin{aligned}\left(\frac{2^4}{2^6}\right)^{-3} &= [2^{(4-6)}]^{-3}\\ &= (2^{-2})^{-3}\\ &= 2^{(-2)(-3)}\\ &= 2^6\\ &= 64\end{aligned}$$

Method 2: Raise Each Power to an Exponent

Raise each power to the exponent −3. Then, divide the resulting powers by subtracting the exponents, since they have the same base.

$$\begin{aligned}\left(\frac{2^4}{2^6}\right)^{-3} &= \frac{(2^4)^{-3}}{(2^6)^{-3}}\\ &= \frac{2^{(4)(-3)}}{2^{(6)(-3)}}\\ &= \frac{2^{-12}}{2^{-18}}\\ &= 2^{-12-(-18)}\\ &= 2^6\\ &= 64\end{aligned}$$

Which method do you prefer? Why?

Did You Know?

John Wallis was a professor of geometry at Oxford University in England in 1655. He was the first to explain the significance of zero and negative exponents. He also introduced the current symbol for infinity, ∞.

d) Add the exponents. Raise the resulting power to the exponent −2.

$$\left[\left(\frac{3}{4}\right)^{-2}\left(\frac{3}{4}\right)^{4}\right]^{-2} = \left[\left(\frac{3}{4}\right)^{-2+4}\right]^{-2}$$
$$= \left[\left(\frac{3}{4}\right)^{2}\right]^{-2}$$
$$= \left(\frac{3}{4}\right)^{(2)(-2)}$$
$$= \left(\frac{3}{4}\right)^{-4}$$
$$= \frac{1}{\left(\frac{3}{4}\right)^{4}}$$
$$= \left(\frac{4}{3}\right)^{4}$$
$$= \frac{256}{81}$$

How do you know that you can add the exponents?

Why is base now $\frac{4}{3}$ instead of the original base of $\frac{3}{4}$?

Is it true in all cases that you can express a rational number with a negative exponent as its reciprocal with a positive exponent? Try it out.

Your Turn

Simplify and evaluate where possible.

a) $[(0.6^3)(0.6^{-3})]^{-5}$ **b)** $[(t^{-4})(t^3)]^{-3}$ **c)** $\left(\frac{x^6}{x^4}\right)^{-2}$ **d)** $\left[\frac{(y^2)^0}{(y)^3}\right]^{-3}$

Example 3 Apply Powers With Integral Exponents

It is estimated that there are 117 billion grasshoppers in an area of 39 000 km^2 of Saskatchewan. Approximately how many grasshoppers are there per square kilometre?

Solution

Method 1: Use Arithmetic

Divide the number of grasshoppers by the total area.

$$\text{grasshoppers per square kilometre} = \frac{117\,000\,000\,000}{39\,000}$$
$$= 3\,000\,000$$

There are approximately 3 000 000 grasshoppers per square kilometre.

Method 2: Use Exponent Rules

Since you cannot enter numbers as large as 117 billion directly into most calculators, rewrite them using exponential form. Then, use the exponent rules to calculate the power of 10.

$$\text{grasshoppers per square kilometre} = \frac{(117)(10^9)}{(39)(10^3)}$$
$$= (3)(10^{(9-3)})$$
$$= (3)(10^6)$$

There are approximately 3 000 000 grasshoppers per square kilometre.

Is it possible to enter numbers expressed using exponential form directly into your calculator? How would doing this help calculate the answer?

Did You Know?

The clear-winged grasshopper is a pest of grasses and cereal grain crops. These insects can completely destroy barley and wheat fields early in the season. Agricultural field workers conduct grasshopper surveys and produce forecasts to help assess the need for control measures to protect crops.

Your Turn

Manitoba Agriculture, Food and Rural Initiatives staff conducted a grasshopper count. In one 25-km^2 area, there were 401 000 000 grasshoppers. Use the following table to assess the degree of grasshopper infestation in this area.

Grasshopper Density
0–4 per square metre = very light
4–8 per square metre = light
8–12 per square metre = moderate
12–24 per square metre = severe
24 per square metre = very severe

Key Ideas

- A power with a negative exponent can be written as a power with a positive exponent.

$$3^{-4} = \frac{1}{3^4} \qquad \left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \left(\frac{3}{2}\right)^2 \qquad \frac{1}{2^{-5}} = 2^5$$

- You can apply the above principle to the exponent laws.

Exponent Law	Example
Note that *a* and *b* are rational or variable bases and *m* and *n* are integral exponents.	
Product of Powers $(a^m)(a^n) = a^{m+n}$	$(3^{-2})(3^4) = 3^{-2+4}$ $= 3^2$
Quotient of Powers $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{x^3}{x^{-5}} = x^{3-(-5)}$ $= x^8$
Power of a Power $(a^m)^n = a^{mn}$	$(0.75^4)^{-2} = 0.75^{(4)(-2)}$ $= 0.75^{-8}$ or $\frac{1}{0.75^8}$
Power of a Product $(ab)^m = (a^m)(b^m)$	$(4z)^{-3} = \frac{1}{(4z)^3}$ or $\frac{1}{64z^3}$
Power of a Quotient $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\left(\frac{t}{3}\right)^{-2} = \left(\frac{3}{t}\right)^2$ $= \frac{3^2}{t^2}$ $= \frac{9}{t^2}$
Zero Exponent $a^0 = 1, a \neq 0$	$(4y^2)^0 = 1$ $-(4y^2)^0 = -1$

Check Your Understanding

Practise

1. For each situation, identify when a positive and a negative exponent would be used.

a) calculating the population growth of a city since 2005 using the expression $150\ 000(1.005)^n$

b) calculating the amount of a radioactive substance remaining from a known sample amount using the expression $25\left(\frac{1}{2}\right)^n$

c) determining how many bacteria are present in a culture after h hours using the expression $500(2)^h$

2. Write each expression with positive exponents.

a) b^{-3} **b)** xy^{-4} **c)** $2x^{-2}$

d) $2x^2y^{-1}$ **e)** $-4x^{-5}$ **f)** $-2x^{-3}y^{-4}$

3. Daniel was rewriting the expression $\frac{2x^{-3}}{y^5}$ with positive exponents. He quickly recorded $\frac{2}{x^3y^5}$. Is his answer correct? Justify your answer.

4. Simplify each expression. State the answer using positive exponents.

a) $(4^3)(4^{-5})$ **b)** $\frac{3^{-4}}{3^{-2}}$ **c)** $\frac{12^3}{12^7}$

d) $\left(\frac{8^{-1}}{8^0}\right)^3$ **e)** $(5^4)^{-2}$ **f)** $[(3^2)(2^{-5})]^3$

g) $\left(\frac{5^2}{4^2}\right)^{-1}$ **h)** $(3.2^{-2})^{-3}$ **i)** $4[(2)^{-1}(2)^{-2}]^{-1}$

5. Simplify each expression by restating it using positive exponents only.

a) $\frac{1}{s^2t^{-6}}$ **b)** $[(h)^7(h)^{-2}]^{-2}$ **c)** $\frac{8t}{t^{-3}}$

d) $(2x^{-4})^3$ **e)** $\left(\frac{n^4}{n^{-4}}\right)^{-3}$ **f)** $[(xy^4)^{-3}]^{-2}$

6. Simplify, then evaluate. Express your answers to four decimal places, where necessary.

a) $(0.5^2)^{-3}$ **b)** $\left[\left(\frac{2}{3}\right)^3\right]^{-3}$ **c)** $[(5)(5^3)]^{-1}$

d) $\left(\frac{6^4}{6^4}\right)^{-3}$ **e)** $\left(\frac{8}{8^3}\right)^{-4}$ **f)** $\left[\left(\frac{3}{4}\right)^{-4} \div \left(\frac{3}{4}\right)^2\right]^{-1}$

7. A mountain pine beetle population can double every year if conditions are ideal. Assume the forest in Jasper National Park, AB, has a population of 20 000 beetles. The formula $P = 20\ 000(2)^n$ can model the population, P, after n years.
   a) How many beetles were there in the forest four years ago? eight years ago?
   b) If the conditions remain ideal, how many beetles will there be two years from now?

8. French-language publishing sales in Canada increased by a growth rate of 1.05 per year from 1996 to 2000. There were sales of $300 000 in 1996. The formula $S = 300\ 000(1.05)^n$ models the sales, S, after n years. Assume that the growth rate stays constant. What would be the projected sales for 2010?

Apply

9. The bacterium *Escherichia coli* is commonly found in the human intestine. A single bacterium has a width of 10^{-3} mm. The head of a pin has a diameter of 1 mm. How many *Escherichia coli* bacteria can fit across the diameter of a pin?

10. A culture of bacteria in a lab contains 2000 bacterium cells. The number of cells doubles every day. This relationship can be modelled by the equation $N = 2000(2)^t$, where N is the estimated number of bacteria cells and t is the time in days.
   a) How many cells were present for each amount of time?
      i) after two days
      ii) after one week
      iii) two days ago
   b) What does $t = 0$ indicate?

11. The Great Galaxy in Andromeda is about 2 200 000 light years from Earth. Light travels 5 900 000 000 000 miles in a year. How many miles is the Great Galaxy in Andromeda from Earth?

12. A red blood cell is about 0.0025 mm in diameter. How large would it appear if it were magnified 10^8 times?

13. Wildlife biologists are tracking the whooping crane population growth at Wood Buffalo National Park, AB. The crane population increased by a growth rate of 7.3% per year from 2002 to 2008. There were 174 whooping cranes in 2002. The rate of growth can be modelled using the formula $P = 174(1.073)^n$, where P is the estimated population and n is the number of years. If conditions remain constant, what is the projected crane population

a) in 2014?

b) in 2011?

Did You Know?

The whooping crane is an endangered species. One of the three wild populations of whooping cranes in North America summers at Wood Buffalo National Park in Alberta. It winters in Texas. Projects are ongoing to help the wild population recover.

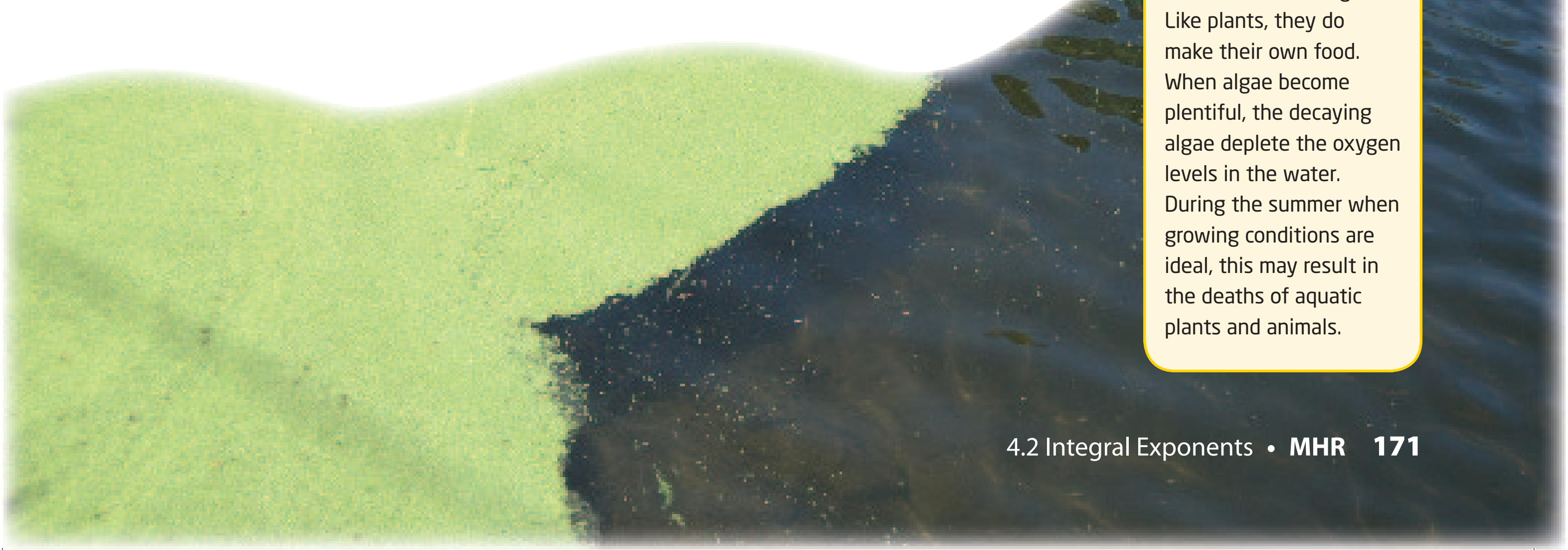
14. There are approximately $(3.2)(10^{18})$ atoms in 1 mg of lead. How many atoms are there in a kilogram of lead? Hint: 1 kg = 10^6 mg.

15. Over time, all rechargeable batteries lose their charge, even when not in use. A 12-volt nickel-metal hydride (NiMH) battery, commonly used in power tools, will lose approximately 30% of its charge every month if not recharged. This situation can be modelled by the formula $V = 12(0.70^m)$, where V is the estimated voltage of the battery in volts, and m is the number of months the battery is not used. What is the estimated voltage of an unused battery after 3 months? Assume the battery was initially fully charged.

16. The fraction of the surface area of a pond covered by algae cells doubles every week. Today, the pond surface is fully covered with algae. This situation can be modelled by the formula $C = \left(\frac{1}{2}\right)^t$, where C is the fraction of the surface area covered by algae t weeks ago. When was 25% of the pond covered?

Did You Know?

Algae grow in water and are neither plant, nor animal, nor fungus. Like plants, they do make their own food. When algae become plentiful, the decaying algae deplete the oxygen levels in the water. During the summer when growing conditions are ideal, this may result in the deaths of aquatic plants and animals.



17. Abby, Kevin, and Caleigh are organizing a 12-hour Famine to raise money to help children in developing countries. Participants are to collect pledges in one of two ways. They can ask for a flat rate of \$30 per pledge or they can ask for a pledge of \$0.01 for the first hour and then, every hour after that, double the pledge from the previous hour. Each hourly pledge using the doubling approach can be modelled by the formula $P = 0.01(2)^h$, where P is the hourly pledge amount and h is the number of hours of participation in the famine.

a) Which pledge approach do you think would raise more money?

b) If the maximum number of hours students can participate in the famine is 12, what is the pledge amount for the last hour?

Did You Know?

A team of Canadian scientists learned previously that crude oil spilled on beaches in Nova Scotia did not break down, due to poor soil conditions. The low nutrient concentrations in the soil limited the growth of natural bacteria. Adding fertilizer to the soil increased the rate of bacterial growth. The scientists applied what they learned to the oil spill in the Arctic, and it worked.

18. Following the 1989 Exxon Valdez oil spill, 100 km of Arctic shoreline was contaminated. Crude oil is made up of thousands of compounds. It takes many different kinds of naturally occurring bacteria to break the oil down. Lab technicians identified and counted the bacteria. They monitored how well the oil was degrading. More bacteria and less oil were signs that the shoreline was recovering. The number of bacteria needed to effectively break down an oil spill is 1 000 000 per millilitre of oil. The bacteria double in number every two days. The starting concentration of bacteria is 1000 bacteria per millimetre. This situation can be modelled by the equation $C = 1000(2^d)$, where C is the estimated concentration of bacteria and d is the number of 2-day periods the bacteria grow. Approximately how long would it take for the bacteria to reach the required concentration?

19. From 2001 to 2006 the population of Lloydminster increased at an average annual rate of 2.52%. This can be modelled using the formula $P = 13\ 145(1.0252)^n$, where P is the estimated population and n is the number of years since 2001.

a) What was the population of Lloydminster in 2004?

b) If this rate of increase stays the same, what will the population be in 2012?

Extend

20. Calculate the value of x that makes each statement true.

a) $x^{-4} = \frac{81}{16}$

b) $\left(\frac{1}{3}\right)^x = 81$

c) $\left(\frac{3}{4}\right)^x = \frac{64}{27}$

d) $(-5)^x = \frac{1}{25}$

21. Use your knowledge of equivalent powers to evaluate $\left(\frac{2^5}{8^2}\right)^{-2}$.

22. A fundraising committee plans to donate $64 000 to six community agencies as follows. The first agency will receive $\frac{1}{2}$. The second agency will receive $\frac{1}{2}$ of what is left. The third agency will receive $\frac{1}{2}$ of what is left, and so on down the line.

a) What fraction of the money will each agency get?

b) How much money will each agency get?

c) Will there be any remaining money after the committee makes these donations? If so, how much?

d) According to the pattern, how many agencies could be supported if no agency is to receive less than $125?

23. The intensity of light from a stage light decreases exponentially with the thickness of the coloured gels covering it. The intensity, I, in watts per square centimetre, can be calculated using the formula $I = 1200\left(\frac{4}{5}\right)^n$, where n is the number of coloured gels used. What is the intensity of light with

a) no gels? b) 2 gels? c) 4 gels?

24. The number, N, of radium atoms remaining in a sample that started at 400 atoms can be represented by the equation $N = 400(2)^{\frac{-t}{1600}}$, where t is time, in years.

a) How many atoms are left after 3200 years?

b) What does $t = 0$ represent?

c) What do negative values of t represent?

Create Connections

25. Use a pattern of your choice. Describe the relationship between a negative exponent and its equivalent form with a positive exponent.

26. Describe a problem where negative exponents are used to model a real-life situation. What does the negative exponent represent in your problem?

27. a) Which power is larger: 3^5 or 3^4? Explain how you know. What can you conclude about comparing powers with the same base?

b) Develop an example that shows how to compare powers with the same exponents and different bases. What can you conclude about comparing powers with the same exponents?

c) Which of the following powers is the greatest? How do you know? Arrange the powers from least to greatest.

2^{666} 3^{555} 4^{444} 5^{333} 6^{222}

4.3 Rational Exponents

Focus on ...

- applying the exponent laws to expressions using rational numbers or variables as bases and rational exponents
- solving problems that involve powers with rational exponents

On a piano keyboard, the pitches of any two adjacent keys are related by a ratio equal to $\sqrt[12]{2}$. This is defined as the number that, when multiplied by itself 12 times, results in 2. You may have noticed that there is no $\sqrt[12]{}$ key on your calculator. How can a piano technician evaluate this number?

Roots other than the square root often occur in science, technology, music, art, and other disciplines. How can you represent such roots in a way that makes them easy to work with?

Investigate Rational Exponents

1. According to the product rule for powers

$$\left(9^{\frac{1}{2}}\right)\left(9^{\frac{1}{2}}\right) = 9^{\frac{1}{2}+\frac{1}{2}}$$
$$= 9^1$$
$$= 9$$

You can reverse these statements to get

$$9^1 = 9^{\frac{1}{2}+\frac{1}{2}}$$
$$= \left(9^{\frac{1}{2}}\right)\left(9^{\frac{1}{2}}\right)$$

What is the value of $9^{\frac{1}{2}}$? Check your answer with a calculator.

2. Predict values for $4^{\frac{1}{2}}$, $16^{\frac{1}{2}}$, $36^{\frac{1}{2}}$, and $49^{\frac{1}{2}}$. Use a calculator to check your predictions. Were you correct?

3. Predict the value of $8^{\frac{1}{3}}$. Explain your thinking. Check your prediction.

4. Reflect and Respond

a) Explain how determining $49^{\frac{1}{2}}$ and your definition for square root are related.

b) Express the 12th root of 2 as a power. Evaluate using your calculator. Express the answer to six decimal places.

c) Use your calculator to determine the 12th power of your answer to part b). Explain why the answer is not 2.

Link the Ideas

You can use the exponent laws to help simplify expressions with rational exponents.

Exponent Law	
Note that a and b are rational or variable bases and m and n are rational exponents.	
Product of Powers	$(a^m)(a^n) = a^{m+n}$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Power of a Power	$(a^m)^n = a^{mn}$
Power of a Product	$(ab)^m = (a^m)(b^m)$
Power of a Quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
Zero Exponent	$a^0 = 1, a \neq 0$

To simplify expressions with rational exponents, you can use the following principle as well as the exponent laws.

- $a^{-n} = \frac{1}{a^n}, a \neq 0$ $\qquad 3^{-0.2} = \frac{1}{3^{0.2}}$
- $\frac{1}{a^{-n}} = a^n, a \neq 0$ $\qquad \frac{1}{3^{-0.2}} = 3^{0.2}$

Example 1 Multiply or Divide Powers With the Same Base

Write each product or quotient as a power with a single exponent.

a) $\left(5^{\frac{1}{3}}\right)\left(5^{\frac{5}{3}}\right)$ **b)** $(x^5)\left(x^{-\frac{1}{2}}\right)$ **c)** $\frac{3^{-\frac{3}{4}}}{3^{0.25}}$ **d)** $\frac{8^{1.8}}{16^{0.3}}$

Solution

Use the exponent laws for multiplying or dividing powers with the same base and rational exponents.

a) Since the bases are the same, you can add the exponents.

$$\begin{aligned}\left(5^{\frac{1}{3}}\right)\left(5^{\frac{5}{3}}\right) &= 5^{\left(\frac{1}{3}+\frac{5}{3}\right)}\\ &= 5^{\frac{6}{3}}\\ &= 5^2\end{aligned}$$

b) Since the bases are the same, you can add the rational exponents.

$$(x^5)\left(x^{-\frac{1}{2}}\right) = x^{\left[5 + \left(-\frac{1}{2}\right)\right]}$$
$$= x^{\frac{9}{2}}$$

Convert to $x^{\left[\frac{10}{2} + \left(-\frac{1}{2}\right)\right]}$.

c) Convert the rational exponents so both are fractions or decimal numbers. Then, since the bases are the same, you can subtract the exponents.

$$\frac{3^{-\frac{3}{4}}}{3^{0.25}} = \frac{3^{-0.75}}{3^{0.25}}$$
$$= 3^{-0.75 - 0.25}$$
$$= 3^{-1} \text{ or } \frac{1}{3}$$

d) Convert to the same base. Then, subtract the exponents.

$$\frac{8^{1.8}}{16^{0.3}} = \frac{(2^3)^{1.8}}{(2^4)^{0.3}}$$
$$= \frac{2^{5.4}}{2^{1.2}}$$
$$= 2^{4.2}$$

How can you use powers of 2 to convert to the same base?

Your Turn

Write each expression as a power with a single exponent.

a) $(x^{1.5})(x^{3.5})$ **b)** $\left(p^{\frac{-5}{4}}\right)\left(p^{\frac{1}{2}}\right)$

c) $\frac{4^{\frac{1}{2}}}{4^{0.5}}$ **d)** $\frac{1.5^{\frac{4}{3}}}{1.5^{\frac{1}{6}}}$

Example 2 Simplify Powers With Rational Exponents

Write each expression as a power with a single, positive exponent. Then, evaluate where possible.

a) $(4x^3)^{0.5}$ **b)** $\left[(x^3)\left(x^{\frac{3}{2}}\right)\right]^{\frac{1}{2}}$ **c)** $\left(\frac{3^4}{16}\right)^{-0.75}$

Solution

a) Raise each term to the exponent, then multiply the exponents.

$$(4x^3)^{0.5} = (4^{0.5})(x^3)^{0.5}$$
$$= 2x^{(3)(0.5)}$$
$$= 2x^{1.5} \text{ or } 2x^{\frac{3}{2}}$$

What is the value of $4^{0.5}$?

b) Method 1: Add the Exponents

Since the bases are the same, you can add the exponents. Raise the result to the exponent $\frac{1}{2}$. Then, multiply.

$$\begin{aligned}\left[(x^3)\left(x^{\frac{3}{2}}\right)\right]^{\frac{1}{2}} &= \left(x^{\left(3 + \frac{3}{2}\right)}\right)^{\frac{1}{2}} \\ &= \left(x^{\frac{9}{2}}\right)^{\frac{1}{2}} \\ &= x^{\left(\frac{9}{2}\right)\left(\frac{1}{2}\right)} \\ &= x^{\frac{9}{4}}\end{aligned}$$

$$\begin{aligned}3 + \frac{3}{2} &= \frac{6}{2} + \frac{3}{2} \\ &= \frac{9}{2}\end{aligned}$$

Method 2: Apply Power of a Power

Raise each power to the exponent $\frac{1}{2}$. Then, add the exponents of the resulting powers.

$$\begin{aligned}\left[(x^3)\left(x^{\frac{3}{2}}\right)\right]^{\frac{1}{2}} &= \left[(x^3)^{\frac{1}{2}}\right]\left[\left(x^{\frac{3}{2}}\right)^{\frac{1}{2}}\right] \\ &= \left[x^{(3)\left(\frac{1}{2}\right)}\right]\left[x^{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}\right] \\ &= \left(x^{\frac{3}{2}}\right)\left(x^{\frac{3}{4}}\right) \\ &= x^{\left(\frac{6}{4} + \frac{3}{4}\right)} \\ &= x^{\frac{9}{4}}\end{aligned}$$

c) Convert the base to a single fraction with the same exponent. Then, raise the result to the exponent $-\frac{3}{4}$.

$$\begin{aligned}\left(\frac{3^4}{16}\right)^{-0.75} &= \left(\frac{3^4}{2^4}\right)^{-\frac{3}{4}} \\ &= \left[\left(\frac{3}{2}\right)^4\right]^{-\frac{3}{4}} \\ &= \left(\frac{3}{2}\right)^{(4)\left(-\frac{3}{4}\right)} \\ &= \left(\frac{3}{2}\right)^{-3} \\ &= \left(\frac{2}{3}\right)^3 \\ &= \frac{8}{27}\end{aligned}$$

Why is $\left(\frac{3}{2}\right)^{-3}$ the same as $\left(\frac{2}{3}\right)^3$?

Your Turn

Simplify and evaluate where possible.

a) $(27x^6)^{\frac{2}{3}}$ **b)** $\left[\left(t^{\frac{4}{3}}\right)\left(t^{\frac{1}{3}}\right)\right]^9$ **c)** $\left(\frac{x^3}{64}\right)^{-\frac{2}{3}}$

Example 3 Apply Powers With Rational Exponents

Food manufacturers use a beneficial bacterium called *Lactobacillus bulgaricus* to make yoghurt and cheese. The growth of 10 000 bacteria can be modelled using the formula $N = 10\ 000(2)^{\frac{h}{42}}$, where N is the number of bacteria after h hours.

a) What does the value 2 in the formula tell you?

b) How many bacteria are present after 42 h?

c) How many more bacteria are present after 2 h?

d) How many bacteria are present after 105 h?

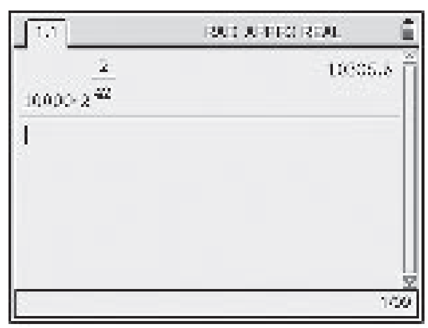
Solution

a) The value 2 indicates that the number of bacteria doubles every 42 h.

b) Substitute the value $h = 42$ into the formula and evaluate.

$N = 10\ 000(2)^{\frac{42}{42}}$
$N = 10\ 000(2)^1$
$N = 20\ 000$
There are 20 000 bacteria after 42 h.

c) Substitute the value $h = 2$ into the formula and evaluate.



$N = 10\ 000(2)^{\frac{2}{42}}$
$N = 10\ 000(1.033\ 558\ldots)$
$N = 10335.58\ldots$
$10\ 335.58\ldots - 10\ 000 = 335.58\ldots$

Why do you subtract 10 000?

There are approximately 336 more bacteria after 2 h.

d) Substitute the value $h = 105$ into the formula and evaluate.

$N = 10\ 000(2)^{\frac{105}{42}}$
$N = 10\ 000(5.656\ 854\ldots)$
$N = 56\ 568.54\ldots$
There are approximately 56 569 bacteria after 105 h.

Your Turn

Cody invests \$5000 in a fund that increases in value at the rate of 12.6% per year. The bank provides a quarterly update on the value of the investment using the formula $A = 5000(1.126)^{\frac{q}{4}}$, where q represents the number of quarterly periods and A represents the final amount of the investment.

a) What is the relationship between the interest rate of 12.6% and the value 1.126 in the formula?

b) What is the value of the investment after the 3rd quarter?

c) What is the value of the investment after 3 years?

Key Ideas

- You can write a power with a negative exponent as a power with a positive exponent.

  $(-9)^{-1.3} = \frac{1}{(-9)^{1.3}}$ $\qquad \frac{1}{2^{-3.2}} = 2^{3.2}$

- You can apply the above principle to the exponent laws for rational exponents.

Exponent Law	Example
Note that a and b are rational or variable bases and m and n are rational exponents	
Product of Powers $(a^m)(a^n) = a^{m+n}$	$\left(x^{\frac{3}{5}}\right)\left(x^{\frac{6}{5}}\right) = x^{\frac{3}{5}+\frac{6}{5}}$ $= x^{\frac{9}{5}}$
Quotient of Powers $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{4s^{2.5}}{12s^{0.5}} = \frac{1}{3}s^{(2.5-0.5)}$ $= \frac{1}{3}s^2$ or $\frac{s^2}{3}$
Power of a Power $(a^m)^n = a^{mn}$	$(t^{3.3})^{\frac{1}{3}} = t^{(3.3)\left(\frac{1}{3}\right)}$ $= t^{1.1}$
Power of a Product $(ab)^m = (a^m)(b^m)$	$\left(8x^{\frac{1}{2}}\right)^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}\left(x^{\frac{1}{2}}\right)^{\frac{2}{3}}$ $= 4x^{\frac{2}{6}}$ or $4x^{\frac{1}{3}}$ How does expressing 8 as 2^3 help simplify?
Power of a Quotient $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\left(\frac{x^3}{y^6}\right)^{\frac{1}{3}} = \frac{(x^3)^{\frac{1}{3}}}{(y^6)^{\frac{1}{3}}}$ $= \frac{x}{y^2}$
Zero Exponent $a^0 = 1, a \neq 0$	$(-2)^0 = 1$ $-2^0 = -1$

- A power with a rational exponent can be written with the exponent in decimal or fractional form.

  $x^{\frac{3}{5}} = x^{0.6}$

Check Your Understanding

Practise

1. Use the exponent laws to simplify each expression. Where possible, compute numerical values.

a) $(x^3)\left(x^{\frac{7}{3}}\right)$ **b)** $\left(b^{\frac{1}{5}}\right)\left(b^{\frac{9}{5}}\right)$ **c)** $(a^2)^{\frac{3}{2}}$

d) $(k^{4.8})(k^3)$ **e)** $(16)^{0.25}$ **f)** $\left(\frac{-8a^6}{27}\right)^{\frac{1}{3}}$

g) $\left(2x^{\frac{1}{3}}\right)\left(-4x^{\frac{5}{3}}\right)$ **h)** $(9x^2)^{\frac{3}{2}}$ **i)** $(25x^2)^{0.5}$

2. Use the exponent laws to simplify each expression. Leave your answers with positive exponents.

a) $(x^3)\left(x^{\frac{-2}{3}}\right)$ **b)** $(81^{-0.25})^3$ **c)** $\dfrac{(m^{-2})^{\frac{2}{3}}}{\left(m^{\frac{1}{2}}\right)^4}$

d) $(9p^2)^{-\frac{1}{2}}\left(p^{-\frac{3}{2}}\right)$ **e)** $\left[\dfrac{x^{-2}}{(xy)^4}\right]^{1.5}$ **f)** $\left[\dfrac{4x^{-2}}{9y^{-4}}\right]^{-\frac{5}{2}}$

3. For each of the following, use the exponent laws to help identify a value for p that satisfies the equation.

a) $(x^p)^{\frac{1}{3}} = x^{\frac{2}{3}}$ **b)** $(x^p)\left(x^{\frac{3}{4}}\right) = x^2$

c) $\dfrac{x^p}{x^{-2}} = x^{\frac{5}{2}}$ **d)** $\left(-3x^{\frac{5}{2}}\right)\left(px^{-\frac{1}{2}}\right) = \dfrac{-3}{4}x^2$

e) $\left(\dfrac{9a^{-4}}{25}\right)^p = \dfrac{3}{5a^2}$ **f)** $(2^{-p})(3^p) = \dfrac{27}{8}$

4. Evaluate without using a calculator. Leave your answers as rational numbers.

a) $8^{\frac{2}{3}}$ **b)** $16^{\frac{1}{4}}$ **c)** $-27^{\frac{4}{3}}$

d) $\left(3^{\frac{1}{6}}\right)\left(3^{\frac{5}{6}}\right)$ **e)** $\left(\dfrac{36x^0}{25}\right)^{1.5}$ **f)** $\dfrac{6^{-2}}{36^{-\frac{1}{2}}}$

5. Evaluate using a calculator. Express your answers to four decimal places, if necessary.

a) $(81^{-0.25})^3$ **b)** $(8^3)(8^{1.2})$ **c)** $\left(\dfrac{2^5}{5^2}\right)^{-\frac{3}{2}}$

d) $\left(\dfrac{2^3}{8^2}\right)^{\frac{2}{3}}$ **e)** $\left(\dfrac{-64}{6^{\frac{1}{2}}}\right)^{\frac{4}{3}}$ **f)** $\dfrac{\left(2^{\frac{1}{2}}\right)^3}{16}$

6. Whonnock Lake, BC is stocked with rainbow trout. The population grows at a rate of 10% per month. The number of trout stocked is given by the expression $250(1.1)^n$, where n is the number of months since the start of the trout season.

Calculate the number of trout

a) 5 months after the season opens

b) $4\frac{1}{2}$ months after the season opens

c) 2 months before the season opens

d) $3\frac{1}{2}$ months before the season opens

Did You Know?

In BC, nearly 900 lakes and streams are stocked annually with trout, char, and kokanee.

Apply

7. For each solution, find the step where an error was made. What is the correct answer? Compare your corrections with those of a classmate.

a) $\frac{t^{1.2}}{t^{-0.5}} = t^{(1.2 - 0.5)}$
$= t^{0.7}$

b) $(16x^2)^{0.5} = (16^{0.5})(x^2)^{0.5}$
$= 8x^{(2)(0.5)}$
$= 8x$

8. Kelly has been saving money she earned from her paper route for the past two years. She has saved $1000 to put towards the purchase of a car when she graduates high school. Kelly has two options for investing the money. If she deposits the money into a 3-year term deposit, it earns 1.5% interest per year, but if she deposits the money into a 2-year term deposit, it will earn 2% interest per year. The formula for calculating the value of her investment is $A = 1000(1 + i)^n$, where A is the amount of money at the end of the term, i is the interest rate as a decimal number, and n is the number of years the money is in the term deposit.

a) Which term deposit will give her the most interest?

b) How much more interest does this option pay?

9. From the beginning of 2003 to the beginning of 2007, the population of Manitoba increased at an average annual rate of 0.5%. This situation can be modelled with the equation $P = 1.1619(1.005)^n$, where P is the population, in millions, and n is the number of years since the beginning of 2003.

a) What do you think the number 1.1619 represents?

b) Assuming that the growth rate continues, what will be the population of Manitoba after 15.5 years?

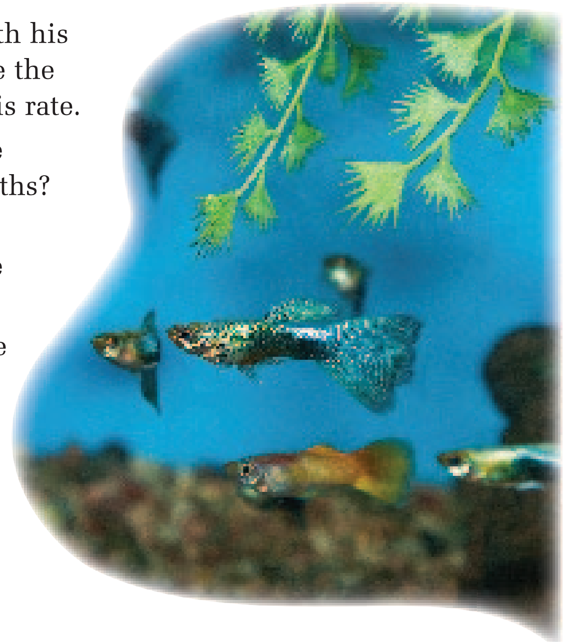
c) Assuming that the growth rate was the same prior to 2003, what was the population of Manitoba at the beginning of 1999?

10. Chris buys six guppies. Every month his guppy population doubles. Assume the population continues to grow at this rate.

    a) How many guppies will there be after 1 month? 2 months? 3 months? n months?

    b) How many guppies will there be after 6.5 months?

    c) Can the fish population continue to grow at this rate? Explain.



11. A mutual fund with an initial value of \$10 000 is decreasing in value at a rate of 12% per year. This situation can be represented by the equation $V = 10\ 000(0.88)^n$, where V represents the value of the fund and n the number of years.

    a) At this rate, what will be the value of the mutual fund in 5 years and 3 months?

    b) If the rate of loss was the same for previous years, what was the value of the fund 3.5 years ago?

12. Martine uses a photographic enlarger that can enlarge a picture to 150% of its previous size. This situation can be modelled by the formula $S = 1.5^t$, where S is the percent increase in the picture size as a decimal number and t is the number of times the enlarger is used.

    150% means 1.5 times.

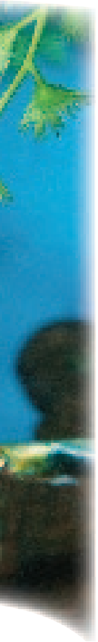
    a) By how many times is a picture enlarged if the enlarger is used 5 times?

    b) How many times would the enlarger need to be used to make a picture at least 25 times as large as the original?

13. Water blocks out sunlight in proportion to its depth. In Qamani'tuaq Lake, NU, $\frac{9}{10}$ of the sunlight reaching the surface of the water can still be seen at a depth of 1 m. This situation can be modelled by the formula $S = 0.9^d$, where S is the fraction of sunlight seen at a depth of d metres. How much sunlight can be seen at a depth of

    a) 7.8 m?

    b) 2.75 m?



14. Under certain conditions, the temperature, T, in degrees Celsius, of a cooling object can be modelled using the formula $T = 40(10^{-0.1t})$. In this formula, t is time, in minutes. What is the temperature

a) after 10 min? **b)** after 0.25 h?

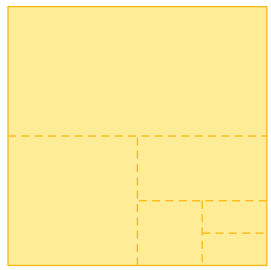
15. For any planetary system, the orbital radius of a planet, R, in metres, can be predicted using the formula $R = (KT^2)^{\frac{1}{3}}$. In this formula, K is a constant for the system, and T is the orbital period of the planet. The value of K for objects orbiting the sun is $(3.37)(10^{18})$. If it takes Mercury $(7.60)(10^6)$ s to orbit the sun, what is the orbital radius of Mercury, in metres?

Did You Know?

Johannes Kepler (1571–1630) was a German mathematician and astronomer. He was the first to correctly explain planetary motion.

Extend

16. When a sheet of paper is folded in half, the area of the paper is reduced by half. This situation can be represented by the equation $A = A_0\left(\frac{1}{2}\right)^f$. In this equation, A_0 represents the starting area of the piece of paper and f the number of consecutive folds. How many folds are needed before the area of the folded paper is less than 1% of the original area? Is this possible? Try it.



17. Julia is a veterinarian who needs to determine the remaining concentration of a particular drug in a horse's bloodstream. She can model the concentration using the formula $C = C_0\left(\frac{1}{2}\right)^{\frac{t}{4}}$, where C is an estimate of the remaining concentration of drug in the bloodstream in milligrams per millilitre of blood, C_0 is the initial concentration, and t is the time in hours that the drug is in the bloodstream. At 10:15 a.m. the concentration of drug in the horse's bloodstream was 40 mg/mL.

a) If only a single dose of the drug is given, what will the approximate concentration of the drug be 6 h later?

b) Julia needs to administer a second dose of the drug when the concentration in the horse's bloodstream is down to 30 mg/mL. Estimate after how many hours this would occur.

Create Connections

18. Describe a problem where rational exponents are used to model a real-life situation. Discuss with a classmate what the rational exponent represents in your problem.

19. Describe at least one common error made when simplifying expressions that include powers with rational exponents. Think of at least one strategy you can use to avoid making the error.

4.4 Irrational Numbers

Focus on ...

- representing, identifying, and simplifying irrational numbers
- converting between powers with rational exponents and radicals
- converting between mixed radicals and entire radicals
- solving problems involving radicals

A golden rectangle has sides that are in a ratio that is pleasing to the eye. The ratio of the length to the width in a golden rectangle is called the golden ratio. Many artists use the golden ratio when composing their paintings. This painting titled *Coming Rain* by Ayla Bouvette appears to have several golden rectangles.

The golden ratio is an **irrational number**. Irrational numbers are often called artistic numbers, since they appear in art, architecture, nature, and geometry.

irrational number

- a number that cannot be expressed in the form $\frac{a}{b}$, where a and b are integers, and $b \neq 0$
- cannot be expressed as a terminating or repeating decimal
- $\pi = 3.1415...$ $\sqrt{5} = 2.236...$

Did You Know?

Ayla Bouvette is an artist with Okanagan, Anishnaabe, and Red River Métis heritage. Her work uses the themes and designs of her heritage. She sells prints of her works to support local powwow fundraising.

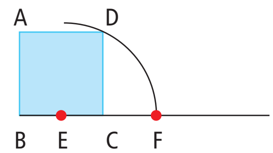
Investigate the Golden Rectangle

Materials
- ruler
- compasses

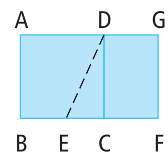
Unit Project

1. Draw a square on a blank sheet of paper. Label the vertices of the square ABCD. Measure and record the side length of the square.

2. Complete the following steps.
- Mark the midpoint of side BC as E.
- Extend line BC so that it is about double in length.
- Use compasses to draw an arc with radius DE so that it intersects line BC at point F.



3. Complete the golden rectangle by drawing DG and FG.



4. a) Calculate the length of DE to four decimal places.

b) Measure the length of DE. How does the actual measurement compare to your calculated value?

c) Calculate the length of BF to four decimal places. Hint: DE is the same length as EF.

d) Measure the length of BF. How does the actual measurement compare to your calculated value?

5. Reflect and Respond

a) The ratio of the length to the width in a golden rectangle is called the golden ratio. Write an exact expression for the golden ratio.

b) What is the approximate value of the golden ratio, to two decimal places?

c) In the painting on page 184 describe the golden rectangles you see. Discuss your ideas with a classmate.

6. a) Look for three rectangular shapes in the classroom that you think may be in the golden ratio. Use a table to organize your findings.
- Measure the length and width of each shape.
- Calculate the ratios of the sides as you did for the rectangle you drew.

b) How do the ratios compare? How close were the rectangles you chose to golden rectangles?

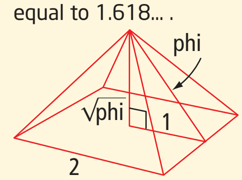
c) Compare your results with those of a classmate.

WWW Web Link

For more information about the golden ratio and the golden rectangle, go to www.mhrmath10.ca and follow the links.

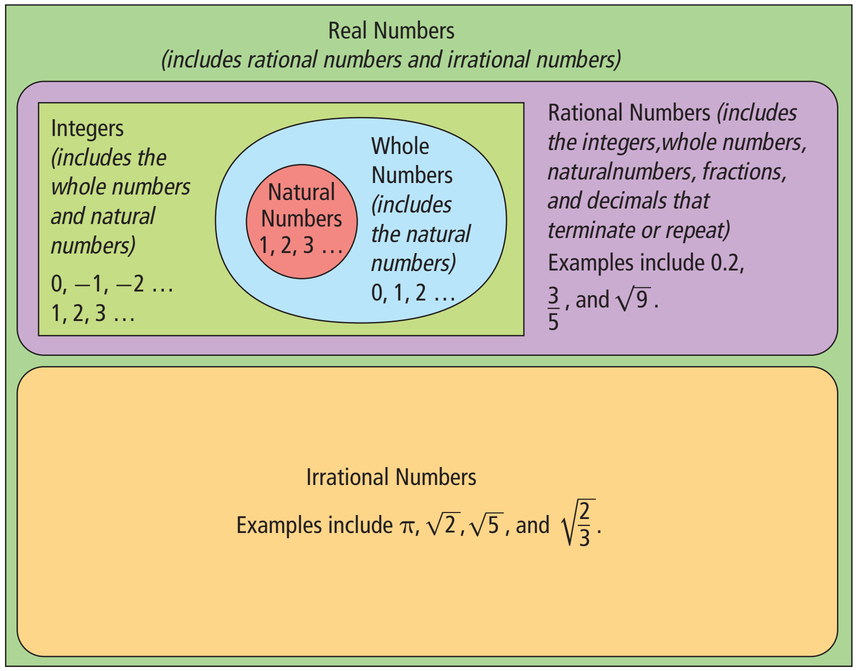
Did You Know?

The Rhind papyrus of Egypt records the building of the Great Pyramid of Giza in 4700 B.C.E. The pyramid has proportions according to the golden ratio. The golden ratio is often represented as *phi,* or φ. This is approximately equal to 1.618... .



Link the Ideas

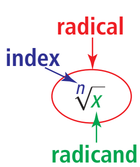
The rational numbers include the natural numbers, whole numbers, and integers. These numbers and the irrational numbers form a set called the real numbers.



What subsets do integers belong to? whole numbers? natural numbers?

Powers with fractional exponents can be written as **radicals** in the form $x^{\frac{1}{n}} = \sqrt[n]{x}$, where $n \neq 0$. When n is even, x cannot be negative, since the product of an even number of equal factors is always positive.

radical
- consists of a root symbol, an index, and a radicand



- can be rational, $\sqrt{4}$, or irrational, $\sqrt{2}$

radicand
- the quantity under the radical sign

index
- indicates what root to take

For example, $\left(3^{\frac{1}{2}}\right)\left(3^{\frac{1}{2}}\right) = 3^{\left(\frac{1}{2} + \frac{1}{2}\right)}$
$= 3$

You know that $(\sqrt{3})(\sqrt{3}) = 3$.

Therefore, $3^{\frac{1}{2}} = \sqrt{3}$.

A power can be expressed as a radical in the form

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = (\sqrt[n]{x})^m$$

or

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

where m and n are integers.

For example, $2^{\frac{3}{4}}$ can be written as $\sqrt[4]{2^3}$ or $\left(\sqrt[4]{2}\right)^3$.

A fractional exponent can be written in decimal form.

For example, $\sqrt[5]{6^3} = 6^{\frac{3}{5}}$
$= 6^{0.6}$

If the radicand is a number, you can evaluate a power with a fractional or decimal exponent.

For example, $\sqrt{5^3} = 5^{\frac{3}{2}}$
$= 11.1803...$

When the index is 2, it is commonly not written.

Did You Know?

The most famous irrational number is likely pi. The Babylonians (about 2000 to 1600 B.C.E.) were the first to approximate pi to a value of 3. Since that time, pi has been calculated to 1 241 100 000 000 decimal places.

The value of pi is 3.14159265368979323846264338327 95...

Example 1 Convert From a Power to a Radical

Express each power as an equivalent radical.

a) $64^{\frac{1}{2}}$

b) $16^{\frac{3}{4}}$

c) $(8x^2)^{\frac{1}{3}}$

Solution

Write each power as a radical.

Use the denominator of the exponent as the index of the radical.

a) $64^{\frac{1}{2}} = \sqrt{64}$

b) $16^{\frac{3}{4}} = \left(\sqrt[4]{16}\right)^3$

c) $(8x^2)^{\frac{1}{3}} = \sqrt[3]{8x^2}$

Your Turn

Express each power as a radical.

a) $10^{\frac{1}{4}}$

b) $1024^{\frac{1}{3}}$

c) $(x^4)^{\frac{3}{8}}$

Example 2 Convert From a Radical to a Power

Express each radical as a power with a rational exponent.

a) $\sqrt[4]{4^3}$ **b)** $\sqrt[5]{3^4}$ **c)** $\sqrt{s^3}$

Solution

Write each radical as a power.
Use the index as the denominator of the exponent.

a) $\sqrt[4]{4^3} = (4^3)^{\frac{1}{4}}$
$= 4^{\frac{3}{4}}$
$= 4^{0.75}$

b) $\sqrt[5]{3^4} = 3^{\frac{4}{5}}$

c) $\sqrt{s^3} = (s^3)^{\frac{1}{2}}$
$= s^{\frac{3}{2}}$

Your Turn

Express each radical as a power.

a) $\sqrt{125}$ **b)** $\sqrt[3]{y^5}$ **c)** $\sqrt[n]{27^2}$

Example 3 Convert Mixed Radicals to Entire Radicals

mixed radical

- the product of a rational number and a radical
- for example, $3\sqrt{2}$, and $\frac{1}{2}\sqrt[3]{6}$

entire radical

- the product of 1 and a radical
- for example, $\sqrt{32}$, and $\sqrt[3]{2^5}$

Express each **mixed radical** as an equivalent **entire radical**.

a) $5\sqrt{11}$

b) $2\sqrt[3]{5}$

c) $1.5\sqrt[3]{6}$

Solution

a) $5\sqrt{11} = \sqrt{(5^2)}\sqrt{(11)}$
$= \sqrt{(5^2)(11)}$
$= \sqrt{(25)(11)}$
$= \sqrt{275}$

What is the index?
How does it help you convert to an entire radical?

b) $2\sqrt[3]{5} = \sqrt[3]{(2^3)}\sqrt[3]{(5)}$
$= \sqrt[3]{(2^3)(5)}$
$= \sqrt[3]{(8)(5)}$
$= \sqrt[3]{40}$

c) $1.5\sqrt[3]{6} = \sqrt[3]{(1.5^3)}\,\sqrt[3]{(6)}$ or $\frac{3}{2}\sqrt[3]{6} = \sqrt[3]{\left(\frac{3}{2}\right)^3}\,\sqrt[3]{(6)}$

$= \sqrt[3]{(1.5^3)(6)}$

$= \sqrt[3]{(3.375)(6)}$

$= \sqrt[3]{20.25}$

$= \sqrt[3]{\left(\frac{3}{2}\right)^3(6)}$

$= \sqrt[3]{\left(\frac{27}{8}\right)(6)}$

$= \sqrt[3]{\frac{81}{4}}$

What could you do to express the radical in lowest terms?

Your Turn

Convert each mixed radical to an equivalent entire radical.

a) $9\sqrt[3]{4}$ **b)** $4.2\sqrt{18}$ **c)** $\frac{1}{2}\sqrt{10}$

Example 4 Convert Entire Radicals to Mixed Radicals

Express each entire radical as an equivalent mixed radical.

a) $\sqrt{27}$ **b)** $\sqrt{50}$ **c)** $\sqrt{48}$ **d)** $\sqrt[4]{80}$

Solution

a) $\sqrt{27} = \sqrt{(9)(3)}$

$= \sqrt{9}\,\sqrt{3}$

$= 3\sqrt{3}$

What value is a perfect square? How does this help you?

b) $\sqrt{50} = \sqrt{(25)(2)}$

$= \sqrt{25}\,\sqrt{2}$

$= 5\sqrt{2}$

c) $\sqrt{48} = \sqrt{(16)(3)}$ or $\sqrt{48} = \sqrt{(4)(12)}$

$= \sqrt{16}\,\sqrt{3}$

$= 4\sqrt{3}$

$= \sqrt{4}\,\sqrt{12}$

$= 2\sqrt{(4)(3)}$

$= 2\sqrt{4}\,\sqrt{3}$

$= 2(2)\sqrt{3}$

$= 4\sqrt{3}$

d) $\sqrt[4]{80} = \sqrt[4]{(2)(2)(2)(2)(5)}$

$= \sqrt[4]{(2^4)(5)}$

$= 2\sqrt[4]{5}$

How does prime factorization help you?

Your Turn

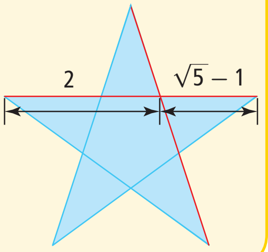
Convert each entire radical to an equivalent mixed radical.

a) $\sqrt{40}$ **b)** $\sqrt{108}$ **c)** $\sqrt[3]{32}$

Did You Know?

The pentagram is also called the star polygon. Inside a pentagram is a pentagon.

Each length of a pentagram intersects two other lengths of the pentagram. The intersection points divide each length according to the golden ratio.



Web Link

For more information about pentagrams and how to draw one, go to www.mhrmath10.ca and follow the links.

Example 5 Order Irrational Numbers

Order these irrational numbers from least to greatest.

$2\sqrt{18}$ $\sqrt{8}$ $3\sqrt{2}$ $\sqrt{32}$

Solution

Method 1: Express Each Irrational Number as an Entire Radical

$$\begin{aligned} 2\sqrt{18} &= \sqrt{(2^2)(18)} \\ &= \sqrt{72} \end{aligned}$$

$$\sqrt{8} = \sqrt{8}$$

$$\begin{aligned} 3\sqrt{2} &= \sqrt{(3^2)(2)} \\ &= \sqrt{18} \end{aligned}$$

$$\sqrt{32} = \sqrt{32}$$

The radicals in order from least to greatest are $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$, and $\sqrt{72}$ or $\sqrt{8}$, $3\sqrt{2}$, $\sqrt{32}$, and $2\sqrt{18}$.

In this case, how could you have used mixed radicals to order the irrational numbers?

Method 2: Estimate the Approximate Values and Plot on a Number Line

Estimate the value of each radical.

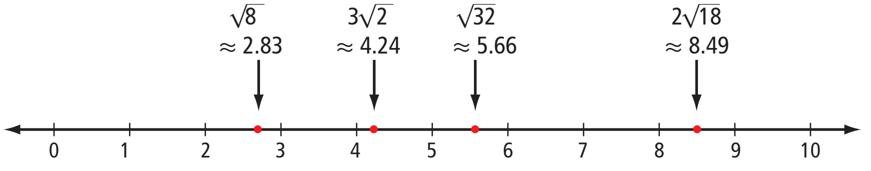
$$\begin{aligned} 2\sqrt{18} &= (2)(4.243...) \\ &= 8.485... \\ &\approx 8.49 \end{aligned}$$

$$\begin{aligned} \sqrt{8} &= 2.828... \\ &\approx 2.83 \end{aligned}$$

$$\begin{aligned} 3\sqrt{2} &= (3)(1.414...) \\ &= 4.243... \\ &\approx 4.24 \end{aligned}$$

$$\begin{aligned} \sqrt{32} &= 5.657... \\ &\approx 5.66 \end{aligned}$$

Plot the approximations on a number line.



Using the approximations on the number line, the radicals in order from least to greatest are $\sqrt{8}$, $3\sqrt{2}$, $\sqrt{32}$, and $2\sqrt{18}$.

Your Turn

Use two different methods to order the following irrational numbers from greatest to least: $2\sqrt{54}$, $\sqrt{192}$, $5\sqrt{10}$.

Example 6 Solve Problems Involving Irrational Numbers

The Seabee Mine is located at Laonil Lake, SK. In 2007, the mine produced a daily average of gold great enough to fill a cube with a volume of 180 cm^3. If five days of gold production is cast into a cube, what is its edge length?

Solution

The volume of gold produced in five days is (5)(180) = 900 cm^3. The formula for the volume, V, of a cube is $V = s^3$, where s is the length of one side. Substitute 900 for the volume. Solve for s by taking the cube root of both sides of the equation.

Alternatively, you can raise both sides to the exponent $\frac{1}{3}$.

$$900 = s^3 \qquad \text{or} \qquad 900 = s^3$$
$$\sqrt[3]{900} = \sqrt[3]{s^3} \qquad\qquad 900^{\frac{1}{3}} = (s^3)^{\frac{1}{3}}$$
$$9.654\ 89... = s \qquad\qquad 9.654\ 89... = s$$

The edge of the cube would be approximately 9.7 cm long.

Your Turn

Assume the Seabee Mine doubles its daily gold production to 360 cm^3. What is the edge length of a cube of gold produced in a five-day period?

Did You Know?

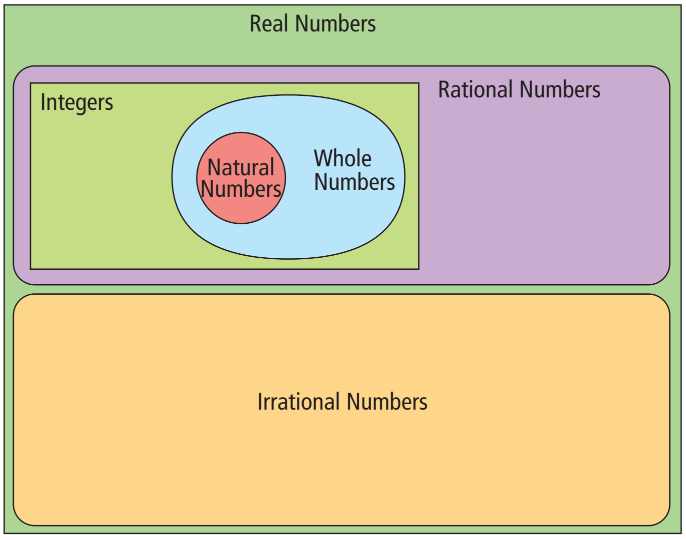
To date, the total amount of gold that has been extracted from Earth would fill only two Olympic sized swimming pools.

Did You Know?

It can be profitable to extract gold from ore grades as low as 0.5 g per 1000 kg of ore. This grade of ore is so low in gold that the gold is not visible. Ore grades of about 30 g per 1000 kg are needed before you can see gold.

Key Ideas

- Rational numbers and irrational numbers form the set of real numbers.



- Radicals can be expressed as powers with fractional exponents.

  $$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

  The index of the radical has the same value as the denominator of the fractional exponent.

  $\sqrt[3]{10} = 10^{\frac{1}{3}}$ $\qquad$ $\sqrt[5]{7^3} = 7^{\frac{3}{5}}$

- Radicals can be entire radicals such as $\sqrt{72}$, $\sqrt[5]{96}$, and $\sqrt[3]{\frac{54}{8}}$. They can also be mixed radicals such as $6\sqrt{2}$, $2\sqrt[5]{3}$, and $\frac{3\sqrt{2}}{2}$. You can convert between entire radicals and mixed radicals.

- You can order radicals that are irrational numbers using different methods:
  - Use a calculator to produce approximate values.
  - Express each irrational number as an entire radical.

Check Your Understanding

Practise

1. Express each power as an equivalent radical.

a) $4^{\frac{3}{2}}$ **b)** $32^{\frac{1}{5}}$ **c)** $64^{0.5}$

d) $\left(\frac{1}{100}\right)^{\frac{1}{4}}$ **e)** $\left(\frac{y^4}{x^3}\right)^{\frac{1}{3}}$ **f)** $(m^n)^{\frac{3}{2}}$

2. Express each radical as a power.

a) $\sqrt{(12p)^3}$ **b)** $\sqrt[5]{5^3}$ **c)** $\sqrt[4]{x^3}$

d) $\sqrt[3]{\frac{s^3}{t^5}}$ **e)** $\sqrt{y^{\frac{5}{3}}}$ **f)** $\sqrt[n]{8}$

3. Evaluate each expression. State the result to four decimal places, if necessary.

a) $\sqrt{0.36}$ **b)** $(27)^{\frac{1}{3}}$ **c)** $4\sqrt{17}$

d) $(65)^{\frac{2}{3}}$ **e)** $0.3(22)^{\frac{1}{2}}$ **f)** $\frac{\sqrt{36}}{\sqrt{7}}$

4. Express each mixed radical as an equivalent entire radical.

a) $3\sqrt{11}$ **b)** $7\sqrt{2}$ **c)** $3\sqrt{5}$

d) $2\sqrt{7}$ **e)** $3\sqrt{3}$ **f)** $10\sqrt{6}$

5. Express each mixed radical as an equivalent entire radical.

a) $2\sqrt[3]{7}$ **b)** $3\sqrt[3]{3}$ **c)** $10\sqrt[3]{5}$

d) $4\sqrt[3]{2}$ **e)** $3\sqrt[4]{2}$ **f)** $2\sqrt[4]{5}$

6. Express each entire radical as an equivalent mixed radical.

a) $\sqrt{12}$ **b)** $\sqrt{50}$ **c)** $\sqrt{48}$

d) $\sqrt{72}$ **e)** $\sqrt{45}$ **f)** $\sqrt{500}$

7. Express each entire radical as an equivalent mixed radical.

a) $\sqrt[3]{24}$ **b)** $\sqrt[3]{54}$ **c)** $\sqrt[3]{243}$

d) $\sqrt[3]{40}$ **e)** $\sqrt[4]{32}$ **f)** $\sqrt[4]{243}$

8. Order each set of numbers from least to greatest. Then, identify the irrational numbers.

a) $\frac{5}{8}$ $0.\overline{6}$ $\sqrt{0.25}$ $\sqrt[3]{0.84}$

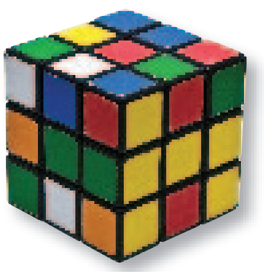
b) $3\sqrt{28}$ $\sqrt{225}$ $15\frac{4}{5}$ $\sqrt[4]{625}$

9. Plot each set of numbers on a number line. Which of the numbers in each set is irrational?

a) $3\sqrt{4}$ $6.\overline{6}$ $\sqrt{39}$ $\sqrt[3]{515}$

b) $4\frac{3}{11}$ $\sqrt[3]{125}$ $\frac{4\sqrt{125}}{5}$ $3\sqrt{8}$

10. The Rubik's Cube is a mechanical puzzle. Calculate the edge length of a Rubik's Cube with a volume of 38.44 cm^3, to three decimal places.



11. Pacific halibut are the largest of all flatfish. The relationship between the length and mass of Pacific halibut can be approximated using the equation $l = 0.46\sqrt[3]{m}$. In this equation, l is the length, in metres, and m is the mass, in kilograms. Use the equation to predict the length of a 25-kg Pacific halibut.

Did You Know?

The range of the Pacific halibut extends along the Pacific coast to the Bering Sea. These fish are important to coastal First Nations who harvest them for food and ceremonial purposes.

Apply

12. Police can estimate the speed of a car by the length of the skid marks made when the driver braked. The formula is $v = \sqrt{30df}$. In this formula, v is the speed, in miles per hour, d is the length of the skid marks, in feet, and f is the coefficient of friction. What was the speed of a vehicle if the skid marks were 75 ft long and the coefficient of friction was 0.7?

13. **Unit Project** Christina is a weaver in Pangnirtung, NU. The dimensions of the tapestry that she is working on represent the golden ratio.

a) If the longer dimension of the tapestry is 60 cm, what is the shorter dimension? Express the answer to the nearest hundredth of a centimetre.

b) What is the total area of the tapestry?

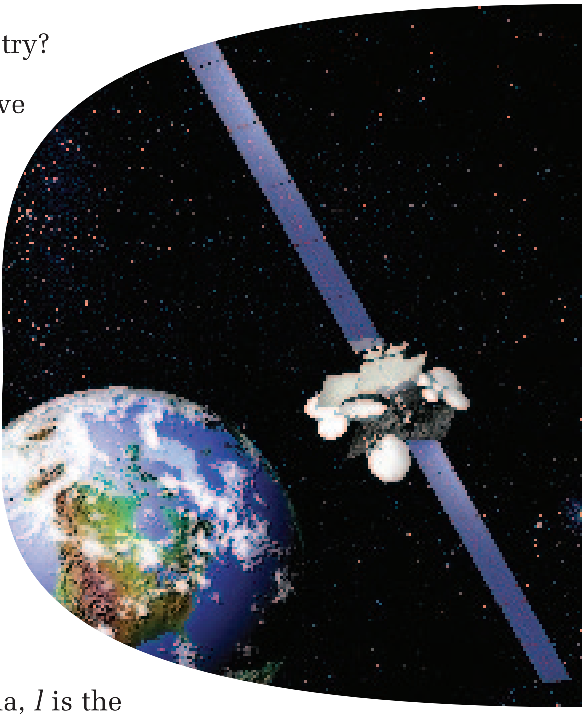
Did You Know?

The velocity of a satellite in *geosynchronous orbit* around Earth matches the rotation of Earth. Since the orbital velocity matches Earth's rotation, the satellite appears to stay in one spot over Earth. However, the satellite is actually travelling at more than 11 000 km/h.

14. When a satellite is h kilometres above Earth, the time, t, in minutes, to complete one orbit is given by the formula $t = \frac{\sqrt{(6370 + h)^3}}{6024}$.

a) A telecommunications satellite is placed 30 km above Earth. How long does it take the satellite to make one orbit?

b) A satellite is placed in geosynchronous orbit about Earth. What must its altitude be?



Web Link

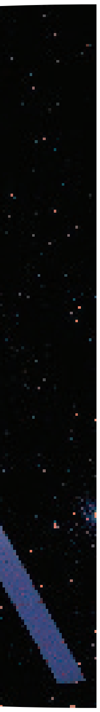
For information about Canada's satellites, go to www.mhrmath10.ca and follow the links.

15. The formula $l = \frac{8t^2}{\pi^2}$ represents the swing of a pendulum. In this formula, l is the length of the pendulum, in feet, and t is the time, in seconds, it takes to swing back and forth once. What is the length of a pendulum that makes one swing in 2 s?

16. An electronics store owner researched the number of customers who would attend a limited-time sale. She modelled the relationship between the sales discount and the length of the sale using the formula $N = 580\sqrt[3]{Pt}$. In this formula, N is the number of customers expected, P is the percent of the sales discount, and t is the number of hours of the sale. What sales discount should the store offer in order to attract 500 shoppers in 8 h?

Kira made a start to the solution. Complete her work.

$$N = 580\sqrt[3]{Pt}$$
$$500 = 580\sqrt[3]{P(8)}$$
$$\frac{500}{580} = \frac{580}{580}\sqrt[3]{8P}$$



17. The amount of current, I, in amperes, that an appliance uses can be calculated by the formula $I = \left(\frac{P}{R}\right)^{\frac{1}{2}}$, where P is the power, in watts, and R is the resistance, in ohms. How much current does an appliance use if $P = 120$ W and $R = 3\ \Omega$? Express your answer to one decimal place.

18. **Unit Project** Many aspects of nature, such as the spiral patterns of leaves and seeds, can be described using the Fibonacci sequence. The sequence is 1, 1, 2, 3, 5, 8, 13, … . The expression for the nth term of the Fibonacci sequence is called Binet's formula. The formula is $F_n = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n$. Use Binet's formula to find F_3.

Web Link

Pine cones, which grow in spirals, show the Fibonacci spirals. Try to locate spiral patterns in the pine cone shown. To learn more about Fibonnaci spirals, go to www.mhrmath10.ca and follow the links.

Extend

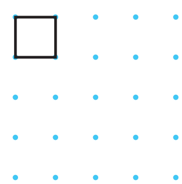
19. Express as a power with a rational exponent.

a) $\sqrt{\sqrt{2^{\frac{4}{5}}}}$ **b)** $\sqrt[4]{\sqrt{256}}$

20. Is the statement $\sqrt[4]{(-x)^4} = x$ sometimes, always, or never true? Explain your reasoning.

Create Connections

21. A 1-by-1 square can be drawn on 5-by-5 dot paper as shown.



a) Draw as many different sized squares as possible on a piece of 5-by-5 dot paper. How many did you find?

b) What is the side length of each square? Which side lengths are rational and which are irrational?

c) What is the area of each square?

22. Describe the relationship between a radical and its equivalent power with a rational exponent.

23. Research the history of up to three algebraic or mathematical symbols of your choice. You might consider the radical sign, pi, phi, or zero. For each symbol, explain who developed it and why they created it.

24. **Unit Project** Use what you have learned about radicals to analyse the golden ratio. Use the following methods as a guide.
- Make a timeline about the history of the golden ratio.
- Explain the exact relationship between the dimensions of the golden rectangle and the golden ratio.
- Use a visual to help describe one other example of the golden ratio.

4 Review

4.1 Square Roots and Cube Roots, pages 152-161

1. Which of the following numbers are perfect squares, perfect cubes, or both?

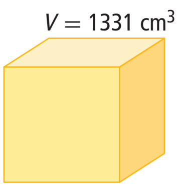
a) 81 **b)** 64 **c)** 125
d) 121 **e)** 729 **f)** 196

2. Use prime factorization to evaluate $\sqrt{144}$.

3. Calculate.

a) $\sqrt{121}$ **b)** $\sqrt[3]{216}$ **c)** $\sqrt[3]{8000}$

4. What are the dimensions of the cube?



5. Jan wants to build a fence around her garden to prevent deer from eating the vegetables. The deer fencing costs \$15 per metre. How much will it cost to enclose a square garden with an area of 81 m^2?

4.2 Integral Exponents, pages 162-173

6. Write as a power with a positive exponent.

a) $(x^{-4})^2$ **b)** $\dfrac{s^3}{s^{-3}}$

c) $\dfrac{(-2.6)^4}{(-2.6)^{-2}}$ **d)** $\dfrac{(4k)^2}{(4k)^{-3}}$

7. Evaluate each expression. Express your answers to four decimal places, if necessary.

a) $(3^{-2})^{-2}$ **b)** $\left[\dfrac{4.5}{(3^2)(1.5)}\right]^3$

c) $[(1.2^3)(1.2^{-2})]^{-4}$ **d)** $\left[\dfrac{(4x^{-2})^{-2}}{(4x)^3}\right]^2$

8. A ball is dropped from a height of 3 m and allowed to bounce freely. The height, h, in metres, that it rebounds can be modelled using the formula $h = 3(0.7)^n$. In this formula, n is the number of bounces.

a) How high does the ball reach on the third bounce? Express the answer to two decimal places.

b) After how many bounces does the ball reach a maximum height of 0.5 m?

9. A radioactive element has a half-life of one week. The formula for the amount of the element remaining is $A = 500\left(\frac{1}{2}\right)^n$, where n is the number of weeks. How much of a 500-g sample of the element

a) remains after five weeks?

b) was there four weeks ago?

10. In a national park, the caribou population increased by a growth factor of 1.05 per year over a 15-year period from 1993 to 2008. There were 1400 caribou at the beginning of 1993 This situation can be modelled by the formula $P = 1400(1.05)^n$, where P is the estimated caribou population and n is the number of years since 1993.

a) If the growth rate remained constant, how many caribou were there after 1 year? 2 years? 3 years? n years?

b) How many caribou will there be at the beginning of 2011?

c) Assume that the growth rate was the same before 1993. How many caribou were there at the beginning of 1990?

4.3 Rational Exponents, pages 174–183

11. Simplify each expression. Express the answers with positive exponents.

a) $\left(x^{\frac{-4}{3}}\right)^{\frac{1}{4}}$ **b)** $\frac{4^{\frac{2}{5}}}{4^{-0.6}}$ **c)** $(16g^8)^{\frac{-3}{4}}$ **d)** $\left(\frac{t^2}{0.5t^{\frac{-1}{3}}}\right)^3$

12. **a)** Use mental math to simplify $\left(\frac{1}{3}\right)^{-1}(27)^{\frac{1}{3}}$. Show your thinking.

b) What shortcuts have you developed for working with negative exponents? Discuss your strategies with a classmate.

13. Evaluate each expression. Express your answers to four decimal places, if necessary.

a) $\left(12^{\frac{2}{5}}\right)(12^{0.4})$ **b)** $\left(3^{\frac{-2}{3}}\right)^3$ **c)** $\left(\frac{0.5^{0.5}}{0.5^{-2}}\right)^4$ **d)** $\left(\frac{8^4}{2}\right)^{\frac{1}{2}}$

14. Barb incorrectly simplified $(27x^4)^{\frac{2}{3}}$ as $18x^{\frac{8}{3}}$. What error did she make? What is the correct answer?

15. Jason invested \$6500 into a fund that increases in value at a rate of 6.2% per year. Jason receives a statement on the value of his investment based on the formula $A = 6500(1.062)^t$, where A is the total value of the investment and t is the number of annual periods. What is the value of Jason's fund after three annual periods?

16. The speed that a tsunami can travel is modelled by the equation $s = 356(d)^{\frac{1}{2}}$. In the equation, s is the speed, in kilometres per hour, and d is the average depth of the water, in kilometres. What is the depth of a tsunami travelling at a speed of 145 km/h?

17. The cost, C, to make computer chips can be calculated using the formula $C = 1000n^{\frac{2}{3}} + 1500$, where n is the number of chips. What is the cost of producing 10 000 computer chips?

Did You Know?

A tsunami can travel at speeds up to 800 km/h. In 1964, the strongest earthquake of the century in North America resulted in a tsunami travelling at more than 700 km/h toward the BC coast. The 4.3-m wave caused extensive flooding and property damage in Port Alberni, BC.

4.4 Irrational Numbers, pages 184–195

18. Write each power as an equivalent radical.

a) $x^{\frac{3}{5}}$ **b)** $(27t^2)^{\frac{2}{3}}$ **c)** $\left(\frac{g^3}{18}\right)^{0.5}$

19. Express each radical as a power.

a) $\sqrt{(xp)^5}$ **b)** $\sqrt[3]{2^5}$ **c)** $3\sqrt[5]{x^4}$

20. Convert each mixed radical to an equivalent entire radical.

a) $3\sqrt{12}$ **b)** $2\sqrt{10}$ **c)** $4\sqrt[3]{5}$ **d)** $-2\sqrt[3]{2}$

21. Express each entire radical as an equivalent mixed radical.

a) $\sqrt{180}$ **b)** $\sqrt{192}$ **c)** $\sqrt[3]{128}$ **d)** $\sqrt[4]{48}$

22. Identify the irrational numbers in each set. Then, order all the numbers from greatest to least.

a) $0.\overline{24}$ $\frac{\pi}{3}$ $\sqrt{0.9}$ $\sqrt[5]{96}$ **b)** $18^{\frac{1}{2}}$ $\sqrt{36}$ $6.\overline{2}$ $2\sqrt[3]{27}$

23. Use a graphic organizer of your choice to represent the similarities and differences between rational and irrational numbers. Explain the key ideas on your organizer.

Traditional Cree hand drum, from Winnipeg, MB showing feathers, the moon and sun, and animal footprints. Traditional Cree hand drums are used in ceremony, cultural events such as round dances, and as artwork. Traditional drums should always be handled with respect following appropriate protocol.

24. a) A circular drum head has an area of 300 cm². What is the approximate radius of the drum? Express your answer to one decimal place.

b) What is the surface area of a drum head with a radius of 12 cm?

4 Practice Test

Multiple Choice

For #1 to #7, choose the best answer.

1. Which of the following numbers is both a perfect square and a perfect cube?

A 8 **B** 16 **C** 32 **D** 64

2. Which expression is not equivalent to $(2x^3)^{-4}$?

A $\frac{16}{x^{12}}$ **B** $\frac{1}{16x^{12}}$

C $\frac{2^{-4}}{x^{12}}$ **D** $\frac{x^{-12}}{16}$

3. Oxalic acid is $(3.0)(10^3)$ times as acidic as acetic acid. Acetic acid is $(2.8)(10^4)$ as strong as boric acid. How many times as acidic is oxalic acid as boric acid?

A $(9.3)(10^1)$ **B** $(3.1)(10^4)$

C $(5.8)(10^7)$ **D** $(8.4)(10^7)$

Did You Know?

Oxalic acid is found in rhubarb. Acetic acid is found in vinegar. Boric acid is an antiseptic used to treat minor burns.

4. Certain bacteria in soil can double in number every 4 h. If the initial population is 2000, what will the approximate population of the bacteria be after 16 h?

A 131 070 000 **B** 128 000

C 32 000 **D** 2064

5. Which of the following is equivalent to $\sqrt[4]{405}$?

A $3\sqrt[4]{5}$ **B** $5\sqrt[4]{3}$

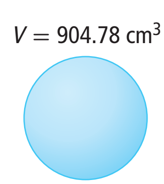
C $5\sqrt[4]{81}$ **D** $81\sqrt[4]{3}$

6. Which power is equivalent to $\sqrt{2^3(3)^{-2}}$?

A $\frac{2^{\frac{3}{2}}}{3}$ **B** $2^{\frac{3}{2}}(3)$

C $\frac{3}{2^{\frac{3}{2}}}$ **D** $2^{-\frac{3}{2}}(3)$

7. The radius, r, of a sphere is given by the equation $r = \sqrt[3]{\frac{3V}{4\pi}}$, where V is the volume of the sphere. What is the approximate radius of this sphere?



A 2.3 cm **B** 6.0 cm

C 12.87 cm **D** 72.0 cm

Short Answer

8. A cube-shaped storage container has a volume of 10.648 m^3.

a) Before calculating, estimate two whole numbers between which the edge length of the container lies. Which number do you think the actual value is closer to?

b) What are the dimensions of the container?

9. Explain why any irrational number cannot be a rational number.

10. a) The whole grid of a Sudoku puzzle has an area of 169 cm^2. What are the dimensions of the grid?

b) What is the side length of each of the nine sections? Express your answer to the nearest tenth of a centimetre.

Did You Know?

A Sudoku puzzle is a Japanese logic puzzle on a grid that has nine 3-by-3 sections.

5	8				7			
		6	4	5		8		
		7		3				1
1					9		2	
8								6
	6		2					4
7				9		2		
		1		2	3	9		
			8				1	3

11. Amaiya in Kugaaruk, NU is video conferencing with Ed in Muenster, SK. She sends the following homework solution to him. What error did she make? Find the step and correct the answer.

$$\frac{2^{-\frac{1}{2}}}{2^{0.5}} = \frac{2^{-0.5}}{2^{0.5}}$$
$$= 2^{-0.5 + 0.5}$$
$$= 2^{0}$$
$$= 1$$

12. The remaining mass, M, of radioactive iodine-131 is given by the equation $M = M_0\left(2^{-\frac{t}{8}}\right)$, where M_0 is the original mass, in grams, and t is time, in days. If the original mass is 250 g, how much iodine-131 remains after 24 days? Express the answer to two decimal places.

13. As you go higher above the surface of Earth, the distance to the horizon becomes greater. The distance, d, in kilometres, to the horizon is given by the formula $d = 3.572\sqrt{h}$, where h is the height of the object, in metres. What is the distance to the horizon from the top of Mount Robson, which has a height of 3959 m? State the result to one decimal place.

Did You Know?

Mount Robson, BC, is the highest point in the Canadian Rockies.

14. A town's rabbit population is growing. The expected population can be modelled using the formula $P = 1580(1.018)^n$, where P is the estimated population and n is the number of years.

a) What is the current rabbit population?

b) What is the expected population in 6 years?

Extended Response

15. Ming and Brian were ordering the irrational numbers $3\sqrt{27}$, $3\sqrt{48}$, $4\sqrt{12}$, and $6\sqrt{3}$ from least to greatest.

a) Ming plans to draw a number line. Describe the rest of her method.

b) Brian converted $4\sqrt{12}$ to $\sqrt{192}$. Describe the rest of his method.

c) Use the method of your choice to order the irrational numbers.