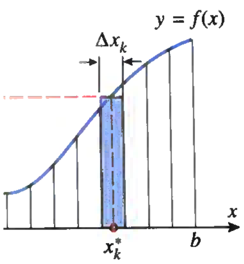
8.1 AREA BETWEEN TWO CURVES

In the last chapter we showed how to find the area between a curve $y = f(x)$ and an interval on the x-axis. Here we will show how to find the area between two curves.

F RIEMANN SUMS

Before we consider the problem of finding the area between two curves it will be helpful to review the basic principle that underlies the calculation of area as a definite integral. Recall that if f is continuous and nonnegative on $[a, b]$, then the definite integral for the area A under $y = f(x)$ over the interval $[a, b]$ is obtained in four steps (Figure 8.1.1):



- Divide the interval $[a, b]$ into n subintervals, and use those subintervals to divide the area under the curve $y = f(x)$ into n strips.
- Assuming that the width of the kth strip is Δx_k, approximate the area of that strip by the area of a rectangle of width Δx_k and height $f(x_k^*)$, where x_k^* is any point in the kth subinterval.
- Add the approximate areas of the strips to approximate the entire area A by the Riemann sum:

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

- Take the limit of the Riemann sums as the number of subintervals increases and their widths approach zero. This causes the error in the approximations to approach zero and produces the following definite integral for the exact area A:

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx$$

Observe the effect that the limit process has on the various parts of the Riemann sum:

- The quantity x_k^* in the Riemann sum becomes the variable x in the definite integral.
- The interval width Δx_k in the Riemann sum becomes the dx in the definite integral.
- The endpoints of the interval $[a, b]$ do not appear in the Riemann sum, but they become the limits of integration in the definite integral.

WEEN y = f(x)
:(x)

We will now consider the following extension of the area problem.

8.1.1 FIRST AREA PROBLEM. Suppose that f and g are continuous functions on an interval $[a, b]$ and

$$f(x) \geq g(x) \quad \text{for} \quad a \leq x \leq b$$

[This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 8.1.2a).

To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of subdividing the region into n strips (Figure 8.1.2b). If we assume that the width of the kth strip is Δx_k, then the area of the strip can be approximated by the area of a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$, where x_k^* is any point in the kth subinterval. Adding these approximations yields the following Riemann sum that approximates the area A:

$$A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the

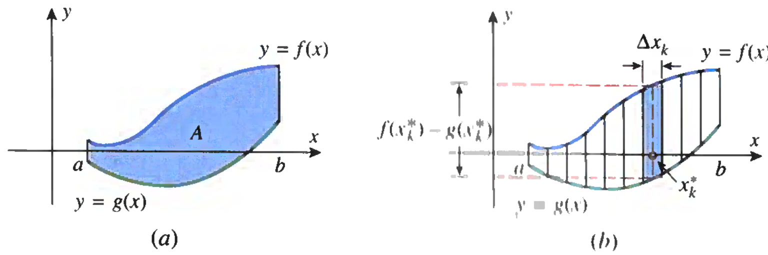


Figure 8.1.2

following definite integral for the area A between the curves:

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k = \int_a^b [f(x) - g(x)]\,dx$$

In summary, we have the following result:

8.1.2 AREA FORMULA. If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx \qquad (1)$$

In the case where f and g are *nonnegative* on the interval $[a, b]$, the formula

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

states that the area A between the curves can be obtained by subtracting the area under $y = g(x)$ from the area under $y = f(x)$ (Figure 8.1.3).

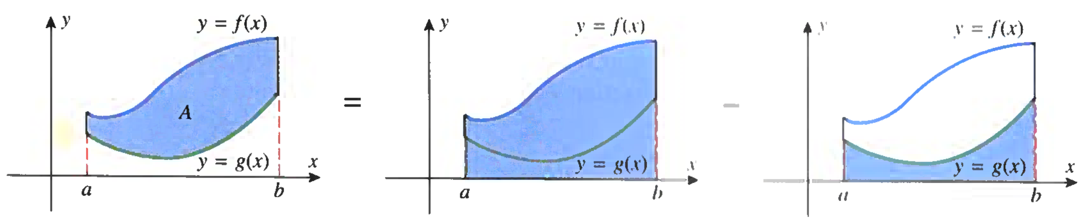


Figure 8.1.3

When the region is complicated, it may require some careful thought to determine the integrand and limits of integration in (1). Here is a systematic procedure that you can follow to set up this formula.

Step 1. Sketch the region and then draw a vertical line segment through the region at an arbitrary point x, connecting the top and bottom boundaries (Figure 8.1.4*a*).

Step 2. The top endpoint of the line segment sketched in Step 1 will be $f(x)$, the bottom one $g(x)$, and the length of the line segment will be $f(x) - g(x)$. This is the integrand in (1).

Step 3. To determine the limits of integration, imagine moving the line segment left and then right. The leftmost position at which the line segment intersects the region is $x = a$ and the rightmost is $x = b$ (Figures 8.1.4*b* and 8.1.4*c*).

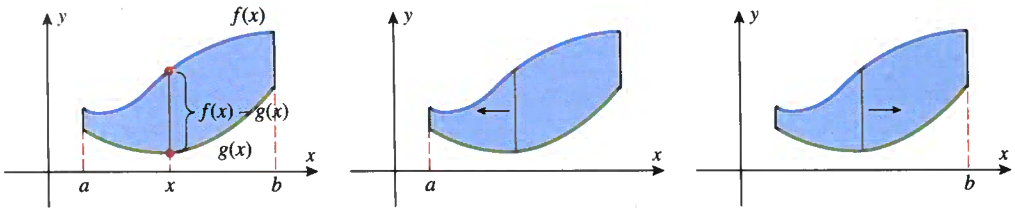


Figure 8.1.4

REMARK. It is not necessary to make an extremely accurate sketch in Step 1; the only purpose of the sketch is to determine which curve is the upper boundary and which is the lower boundary.

REMARK. There is a useful way of thinking about this procedure: If you view the vertical line segment as the "cross section" of the region at the point x, then Formula (1) states that the area between the curves is obtained by integrating the length of the cross section over the interval from a to b.

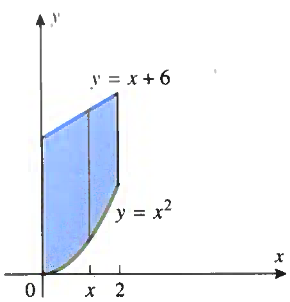


Figure 8.1.5

Example 1

Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution. The region and a cross section are shown in Figure 8.1.5. The cross section extends from $g(x) = x^2$ on the bottom to $f(x) = x + 6$ on the top. If the cross section is moved through the region, then its leftmost position will be $x = 0$ and its rightmost position will be $x = 2$. Thus, from (1)

$$A = \int_0^2 [(x+6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3} \qquad \blacktriangleleft$$

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 8.1.6). When that occurs you will have to determine the points of intersection to obtain the limits of integration.

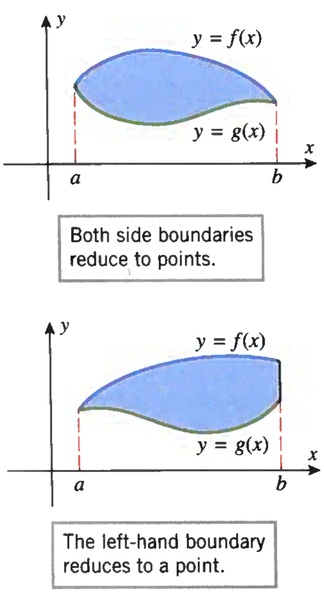


Figure 8.1.6

Example 2

Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution. A sketch of the region (Figure 8.1.7) shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. At the endpoints of the region, the upper and lower boundaries have the same y-coordinates; thus, to find the endpoints we equate

$$y = x^2 \quad \text{and} \quad y = x + 6 \tag{2}$$

This yields

$$x^2 = x + 6 \quad \text{or} \quad x^2 - x - 6 = 0 \quad \text{or} \quad (x+2)(x-3) = 0$$

from which we obtain

$$x = -2 \quad \text{and} \quad x = 3$$

Although the y-coordinates of the endpoints are not essential to our solution, they may be obtained from (2) by substituting $x = -2$ and $x = 3$ in either equation. This yields $y = 4$ and $y = 9$, so the upper and lower boundaries intersect at $(-2, 4)$ and $(3, 9)$.

From (1) with $f(x) = x + 6$, $g(x) = x^2$, $a = -2$, and $b = 3$, we obtain the area

$$A = \int_{-2}^3 [(x+6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^3 = \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6} \qquad \blacktriangleleft$$

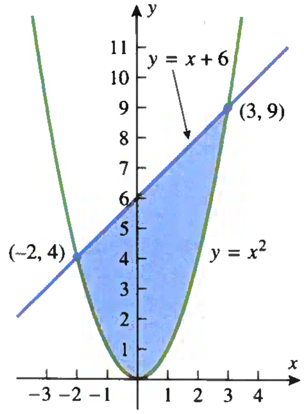


Figure 8.1.7

It is possible for the upper or lower boundary of a region to consist of two or more different curves, in which case it will be necessary to subdivide the region into smaller pieces in order to apply Formula (1). This is illustrated in the next example.

Example 3

Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Solution. To make an accurate sketch of the region, we need to know where the curves $x = y^2$ and $y = x - 2$ intersect. In Example 2 we found intersections by equating the expressions for y. Here it is easier to rewrite the latter equation as $x = y + 2$ and equate the expressions for x, namely

$$x = y^2 \quad \text{and} \quad x = y + 2 \tag{3}$$

This yields

$$y^2 = y + 2 \quad \text{or} \quad y^2 - y - 2 = 0 \quad \text{or} \quad (y + 1)(y - 2) = 0$$

from which we obtain $y = -1$, $y = 2$. Substituting these values in either equation in (3) we see that the corresponding x-values are $x = 1$ and $x = 4$, respectively, so the points of intersection are $(1, -1)$ and $(4, 2)$ (Figure 8.1.8*a*).

To apply Formula (1), the equations of the boundaries must be written so that y is expressed explicitly as a function of x. The upper boundary can be written as $y = \sqrt{x}$ (rewrite $x = y^2$ as $y = \pm\sqrt{x}$ and choose the + for the upper portion of the curve). The lower portion of the boundary consists of two parts: $y = -\sqrt{x}$ for $0 \le x \le 1$ and $y = x - 2$ for $1 \le x \le 4$ (Figure 8.1.8*b*). Because of this change in the formula for the lower boundary, it is necessary to divide the region into two parts and find the area of each part separately.

From (1) with $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x}$, $a = 0$, and $b = 1$, we obtain

$$A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx = 2\int_0^1 \sqrt{x}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3} - 0 = \frac{4}{3}$$

From (1) with $f(x) = \sqrt{x}$, $g(x) = x - 2$, $a = 1$, and $b = 4$, we obtain

$$A_2 = \int_1^4 [\sqrt{x} - (x - 2)]\,dx = \int_1^4 (\sqrt{x} - x + 2)\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{19}{6}$$

Thus, the area of the entire region is

$$A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$

◀

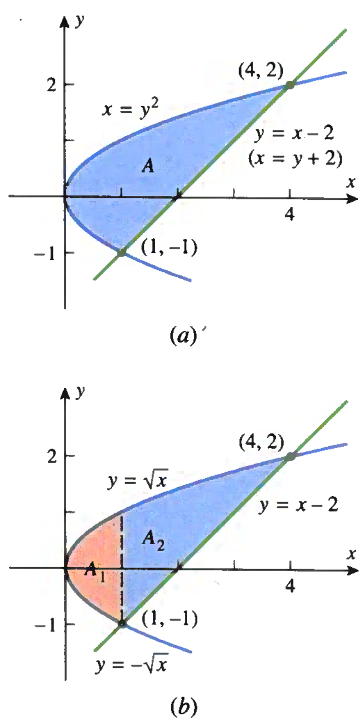


Figure 8.1.8

FOR THE READER. It is assumed in Formula (1) that $f(x) \ge g(x)$ for all x in the interval $[a, b]$. What do you think that the integral represents if this condition is not satisfied, that is, the graphs of f and g cross one another over the interval? Explain your reasoning, and give an example to support your conclusion.

Example 4

Figure 8.1.9 shows velocity versus time curves for two race cars that move along a straight track, starting from rest at the same line. What does the area A between the curves over the interval $0 \le t \le T$ represent?

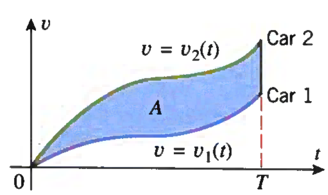


Figure 8.1.9

Solution. From (1)

$$A = \int_0^T [v_2(t) - v_1(t)]\,dt = \int_0^T v_2(t)\,dt - \int_0^T v_1(t)\,dt$$

But from 7.7.4, the first integral is the distance traveled by car 2 during the time interval, and the second integral is the distance traveled by car 1. Thus, A is the distance by which car 2 is ahead of car 1 at time T. ◀

REVERSING THE ROLES OF x AND y

Sometimes it is possible to avoid splitting a region into parts by integrating with respect to y rather than x. We will now show how this can be done.

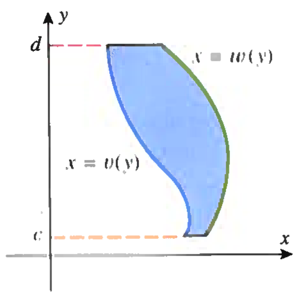


Figure 8.1.10

8.1.3 SECOND AREA PROBLEM. Suppose that w and v are continuous functions of y on an interval $[c, d]$ and that

$$w(y) \geq v(y) \quad \text{for} \quad c \leq y \leq d$$

[This means that the curve $x = w(y)$ lies to the right of the curve $x = v(y)$ and that the two can touch but not cross.] Find the area A of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, and above and below by the lines $y = d$ and $y = c$ (Figure 8.1.10).

Proceeding as in the derivation of (1), but with the roles of x and y reversed, leads to the following analog of 8.1.2.

8.1.4 AREA FORMULA. If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]\, dy \tag{4}$$

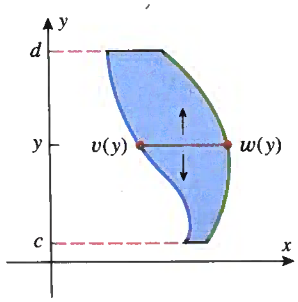


Figure 8.1.11

The guiding principle in applying this formula is the same as with (1): The integrand in (4) can be viewed as the length of the horizontal cross section at the point y, in which case Formula (4) states that the area can be obtained by integrating the length of the horizontal cross section over the interval $[c, d]$ on the y-axis (Figure 8.1.11).

In Example 3, where we integrated with respect to x to find the area of the region enclosed by $x = y^2$ and $y = x - 2$, we had to split the region into parts and evaluate two integrals. In the next example we will see that by integrating with respect to y no splitting of the region is necessary.

Example 5

Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.

Solution. From Figure 8.1.8 the left boundary is $x = y^2$, the right boundary is $y = x - 2$, and the region extends over the interval $-1 \leq y \leq 2$. However, to apply (4) the equations for the boundaries must be written so that x is expressed explicitly as a function of y. Thus, we rewrite $y = x - 2$ as $x = y + 2$. It now follows from (4) that

$$A = \int_{-1}^{2} [(y + 2) - y^2]\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

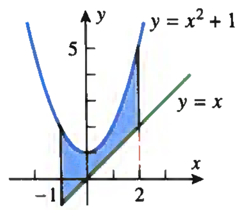
which agrees with the result obtained in Example 3. ◀

REMARK. The choice between Formulas (1) and (4) is generally dictated by the shape of the region, and one would usually choose the formula that requires the least amount of splitting. However, if the integral(s) resulting by one method are difficult to evaluate, then the other method might be preferable, even if it requires more splitting.

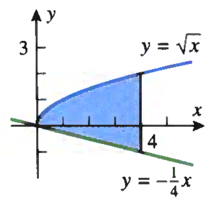
EXERCISE SET 8.1 Graphing Calculator [C] CAS

In Exercises 1–4, find the area of the shaded region.

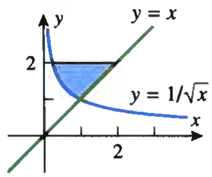
1.



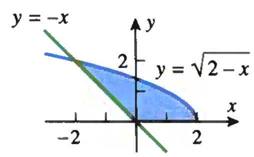
2.



3.



4.



5. Find the area of the region enclosed by the curves $y = x^2$ and $y = 4x$ by integrating
(a) with respect to x (b) with respect to y.

6. Find the area of the region enclosed by the curves $y^2 = 4x$ and $y = 2x - 4$ by integrating
(a) with respect to x (b) with respect to y.

In Exercises 7–16, sketch the region enclosed by the curves, and find its area.

7. $y = x^2, \ y = \sqrt{x}, \ x = 1/4, \ x = 1$

8. $y = x^3 - 4x, \ y = 0, \ x = 0, \ x = 2$

9. $y = \cos 2x, \ y = 0, \ x = \pi/4, \ x = \pi/2$

10. $y = \sec^2 x, \ y = 2, \ x = -\pi/4, \ x = \pi/4$

11. $x = \sin y, \ x = 0, \ y = \pi/4, \ y = 3\pi/4$

12. $x^2 = y, \ x = y - 2$

13. $y = e^x, \ y = e^{2x}, \ x = 0, \ x = \ln 2$

14. $x = 1/y, \ x = 0, \ y = 1, \ y = e$

15. $y = 2 + |x - 1|, \ y = -\frac{1}{5}x + 7$

16. $y = x, \ y = 4x, \ y = -x + 2$

In Exercises 17–22, use a graphing utility, where helpful, to find the area of the region enclosed by the curves.

17. $y = x^3 - 4x^2 + 3x, \ y = 0, \ x = 0, \ x = 3$

18. $y = x^3 - 2x^2, \ y = 2x^2 - 3x, \ x = 0, \ x = 3$

19. $y = \sin x, \ y = \cos x, \ x = 0, \ x = 2\pi$

20. $y = x^3 - 4x, \ y = 0, \ x = -2, \ x = 2$

21. $x = y^3 - y, \ x = 0$

22. $x = y^3 - 4y^2 + 3y, \ x = y^2 - y$

[C] **23.** Use a CAS to find the area enclosed by $y = 3 - 2x$ and $y = x^6 + 2x^5 - 3x^4 + x^2$.

[C] **24.** Use a CAS to find the exact area enclosed by the curves $y = x^5 - 2x^3 - 3x$ and $y = x^3$.

25. Find a horizontal line $y = k$ that divides the area between $y = x^2$ and $y = 9$ into two equal parts.

26. Find a vertical line $x = k$ that divides the area enclosed by $x = \sqrt{y}$, $x = 2$, and $y = 0$ into two equal parts.

27. (a) Find the area of the region enclosed by the parabola $y = 2x - x^2$ and the x-axis.
(b) Find the value of m so that the line $y = mx$ divides the region in part (a) into two regions of equal area.

28. Find the area between the curve $y = \sin x$ and the line segment joining the points $(0, 0)$ and $(5\pi/6, 1/2)$ on the curve.

29. Suppose that f and g are integrable on $[a, b]$, but neither $f(x) \geq g(x)$ nor $g(x) \geq f(x)$ holds for all x in $[a, b]$ [i.e., the curves $y = f(x)$ and $y = g(x)$ are intertwined].
(a) What is the geometric significance of the integral

$$\int_a^b [f(x) - g(x)]\,dx?$$

(b) What is the geometric significance of the integral

$$\int_a^b |f(x) - g(x)|\,dx?$$

30. Let $A(n)$ be the area in the first quadrant enclosed by the curves $y = \sqrt[n]{x}$ and $y = x$.
(a) By considering how the graph of $y = \sqrt[n]{x}$ changes as n increases, make a conjecture about the limit of $A(n)$ as $n \to +\infty$.
(b) Confirm your conjecture by calculating the limit.

In Exercises 31 and 32, use Newton's Method (Section 6.4), where needed, to approximate the x-coordinates of the intersections of the curves to at least four decimal places; and then use those approximations to approximate the area of the region.

31. The region that lies below the curve $y = \sin x$ and above the line $y = 0.2x$, where $x \geq 0$.

32. The region enclosed by the graphs of $y = x^2$ and $y = \cos x$.

33. The accompanying figure shows velocity versus time curves for two cars that move along a straight track, accelerating from rest at a common starting line.
(a) How far apart are the cars after 60 seconds?
(b) How far apart are the cars after T seconds, where $0 \leq T \leq 60$?

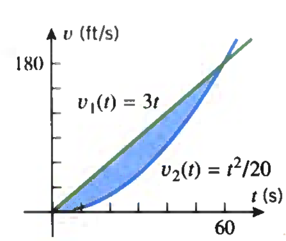


Figure Ex-33

34. The accompanying figure shows acceleration time curves for two cars that move along a straight track, accelerating from rest at the starting line. What does the area A between the curves over the interval $0 \le t \le T$ represent? Justify your answer.

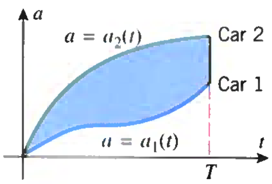


Figure Ex-34

35. Find the area of the region enclosed between the curve $x^{1/2} + y^{1/2} = a^{1/2}$ and the coordinate axes.

36. Show that the area of the ellipse in the accompanying figure is πab. [*Hint:* Use a formula from geometry.]

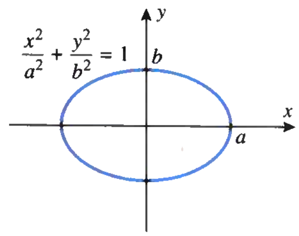


Figure Ex-36

37. A rectangle with edges parallel to the coordinate axes has one vertex at the origin and the diagonally opposite vertex on the curve $y = kx^m$ at the point where $x = b$ $(b > 0, k > 0$, and $m \ge 0)$. Show that the fraction of the area of the rectangle that lies between the curve and the x-axis depends on m but not on k or b.

8.2 VOLUMES BY SLICING; DISKS AND WASHERS

In the last section we showed that the area of a plane region bounded by two curves can be obtained by integrating the length of a general cross section over an appropriate interval. In this section we will see that the same basic principle can be used to find volumes of certain three-dimensional solids.

VOLUMES BY SLICING

Recall that the underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the area. Under appropriate conditions, the same strategy can be used to find the volume of a solid. The idea is to divide the solid into thin slabs, approximate the volume of each slab, add the approximations to form a Riemann sum, and take the limit of the Riemann sums to produce an integral for the volume (Figure 8.2.1).

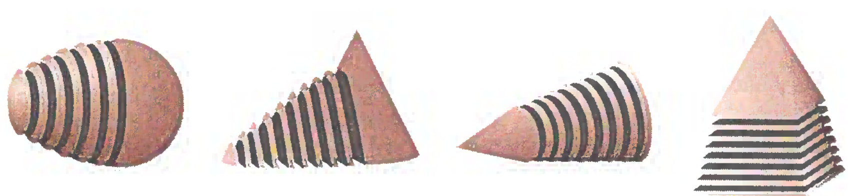
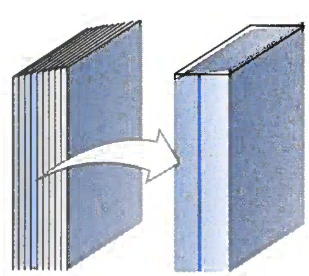


Figure 8.2.1



In a thin slab, the cross sections do not vary much in size and shape.

Figure 8.2.2

What makes this method work is the fact that a *thin* slab has cross sections that do not vary much in size or shape, which, as we will see, makes its volume easy to approximate (Figure 8.2.2). Moreover, the thinner the slab, the less variation in its cross sections and the better the approximation. Thus, once we approximate the volumes of the slabs, we can set up a Riemann sum whose limit is the volume of the entire solid. We will give the details shortly, but first we need to discuss how to find the volume of a solid whose cross sections do not vary in size and shape (i.e., are congruent).

One of the simplest examples of a solid with congruent cross sections is a right circular cylinder of radius r, since all cross sections taken perpendicular to the central axis are circular regions of radius r. The volume V of a right circular cylinder of radius r and height

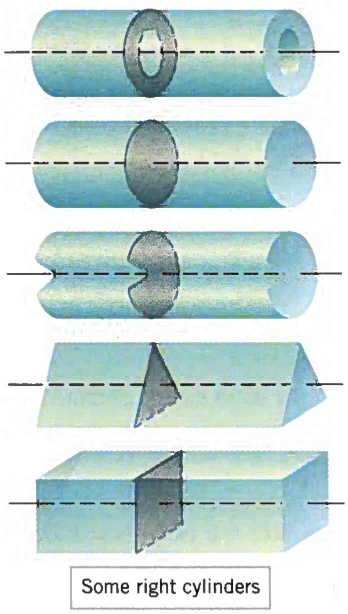


Figure 8.2.3

h can be expressed in terms of the height and the area of a cross section as

$$V = \pi r^2 h = [\text{area of a cross section}] \times [\text{height}] \tag{1}$$

This is a special case of a more general volume formula that applies to solids called *right cylinders*. A ***right cylinder*** is a solid that is generated when a plane region is translated along a line or ***axis*** that is perpendicular to the region (Figure 8.2.3). The distance h that the region is translated is called the ***height*** or sometimes the ***width*** of the cylinder, and each cross section is a duplicate of the translated region. We will assume that the volume V of a right cylinder with cross-sectional area A and height h is given by

$$V = A \cdot h = [\text{area of a cross section}] \times [\text{height}] \tag{2}$$

(Figure 8.2.4). Note that this is consistent with Formula (1) for the volume of a right circular cylinder. We now have all of the tools required to solve the following problem.

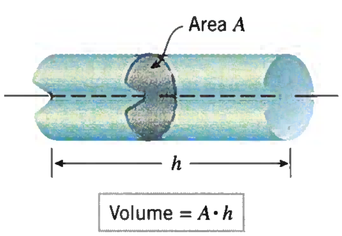


Figure 8.2.4

> **8.2.1 PROBLEM.** Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at $x = a$ and $x = b$ (Figure 8.2.5*a*). Find the volume V of the solid, assuming that its cross-sectional area $A(x)$ is known at each point x in the interval $[a, b]$.

To solve this problem we divide the interval $[a, b]$ into n subintervals, which has the effect of dividing the solid into n slabs (Figure 8.2.5*b*).

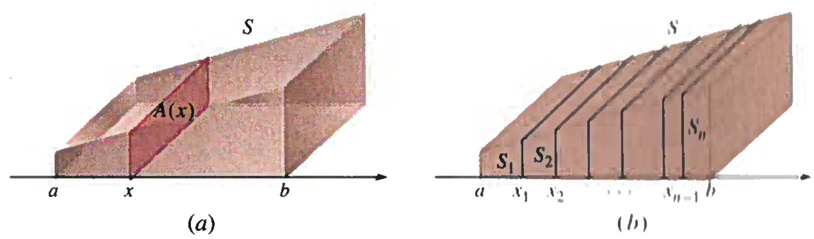


Figure 8.2.5

If we assume that the width of the kth slab is Δx_k, then the volume of the slab can be approximated by the volume of a right cylinder of width (height) Δx_k and cross-sectional area $A(x_k^*)$, where x_k^* is any point in the kth subinterval (Figure 8.2.6). Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} A(x_k^*)\Delta x_k = \int_a^b A(x)\,dx$$

In summary, we have the following result:

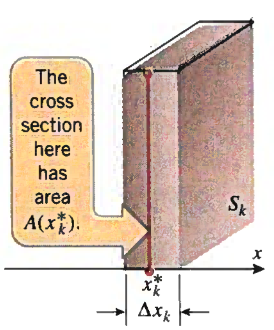


Figure 8.2.6

> **8.2.2 VOLUME FORMULA.** Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is
>
> $$V = \int_a^b A(x)\,dx \tag{3}$$
>
> provided $A(x)$ is integrable.

There is a similar result for cross sections perpendicular to the y-axis.

8.2.3 VOLUME FORMULA. Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\, dy \tag{4}$$

provided $A(y)$ is integrable.

REMARK. In words, these formulas state that the volume of the solid can be obtained by integrating the cross-sectional area from one end of the solid to the other.

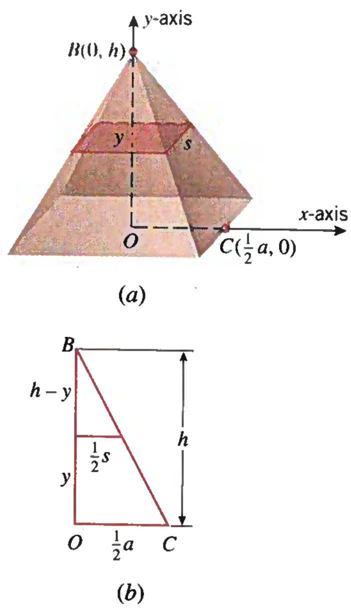


Figure 8.2.7

Example 1

Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

Solution. As illustrated in Figure 8.2.7*a*, we introduce a rectangular coordinate system in which the y-axis passes through the apex and is perpendicular to the base, and the x-axis passes through the base and is parallel to a side of the base.

At any point y in the interval $[0, h]$ on the y-axis, the cross section perpendicular to the y-axis is a square. If s denotes the length of a side of this square, then by similar triangles (Figure 8.2.7*b*)

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h} \quad \text{or} \quad s = \frac{a}{h}(h-y)$$

Thus, the area $A(y)$ of the cross section at y is

$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2$$

and by (4) the volume is

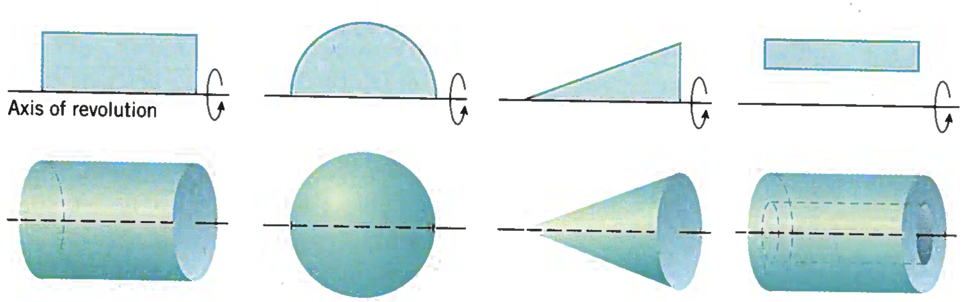
$$V = \int_0^h A(y)\, dy = \int_0^h \frac{a^2}{h^2}(h-y)^2\, dy = \frac{a^2}{h^2}\int_0^h (h-y)^2\, dy$$

$$= \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_{y=0}^{h} = \frac{a^2}{h^2}\left[0 + \frac{1}{3}h^3\right] = \frac{1}{3}a^2h$$

That is, the volume is $\frac{1}{3}$ of the area of the base times the altitude. ◀

SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type (Figure 8.2.8).



Some familiar solids of revolution

Figure 8.2.8

We will be interested in the following general problem:

> **8.2.4 PROBLEM.** Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 8.2.9*a*). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is a circular disk of radius $f(x)$ (Figure 8.2.9*b*). The area of this region is

$$A(x) = \pi[f(x)]^2$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2\,dx \tag{5}$$

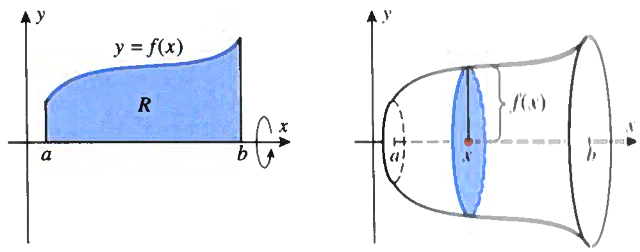


Figure 8.2.9 (*a*) (*b*)

VOLUMES BY DISKS PERPENDICULAR TO THE *x*-AXIS

Because the cross sections are disk shaped, the application of this formula is called the ***method of disks***.

Example 2

Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 8.2.10).

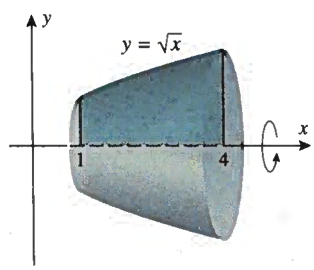


Figure 8.2.10

Solution. From (5), the volume is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_1^4 \pi x\,dx = \frac{\pi x^2}{2}\bigg]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$ ◀

Example 3

Derive the formula for the volume of a sphere of radius r.

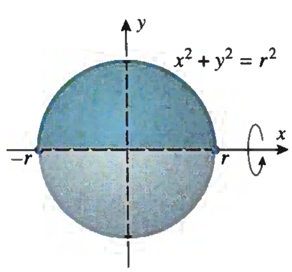


Figure 8.2.11

Solution. As indicated in Figure 8.2.11, a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x-axis and

$$x^2 + y^2 = r^2$$

about the x-axis. Since the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, it follows from (5) that the volume of the sphere is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3$$ ◀

VOLUMES BY WASHERS PERPENDICULAR TO THE *x*-AXIS

Not all solids of revolution have solid interiors; some have holes or channels that create interior surfaces, as in the last part of Figure 8.2.8. Thus, we will be interested in problems of the following type.

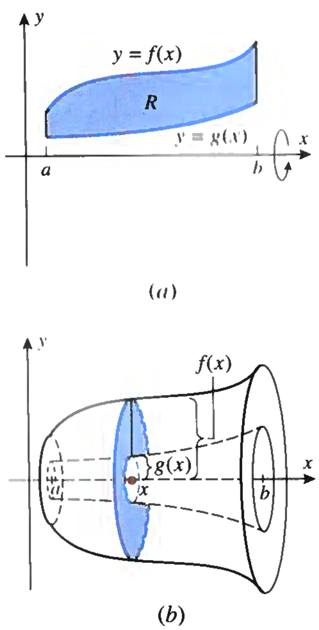


Figure 8.2.12

8.2.5 PROBLEM. Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 8.2.12*a*). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is the annular or "washer-shaped" region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 8.2.12*b*); hence its area is

$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi([f(x)]^2 - [g(x)]^2)$$

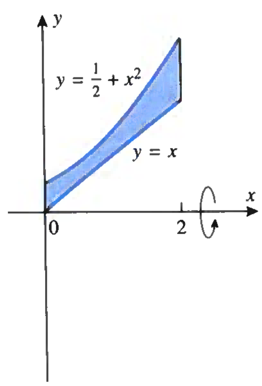
Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx \tag{6}$$

Because the cross sections are washer shaped, the application of this formula is called the ***method of washers***.

Example 4

Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis (Figure 8.2.13).



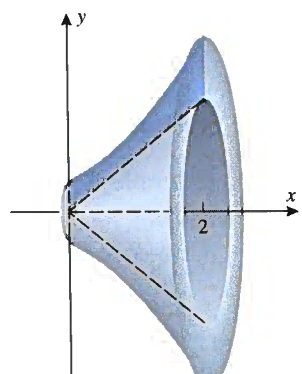


Figure 8.2.13

Not to scale

Solution. From (6) the volume is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx = \int_0^2 \pi\left(\left[\tfrac{1}{2} + x^2\right]^2 - x^2\right)dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right)dx = \pi\left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2 = \frac{69\pi}{10}$$ ◀

VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE y-AXIS

The methods of disks and washers have analogs for regions that are revolved about the y-axis (Figures 8.2.14 and 8.2.15). Using the method of slicing and Formula (4), you should have no trouble deducing the following formulas for the volumes of the solids in the figures.

$$V = \int_c^d \pi[u(y)]^2\,dy \qquad V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy \tag{7–8}$$

Disks   Washers

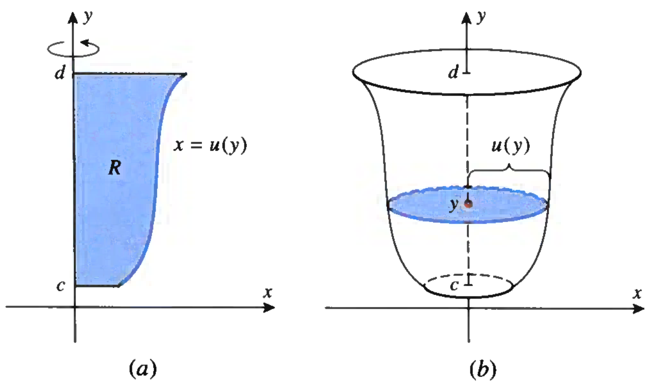


Figure 8.2.14

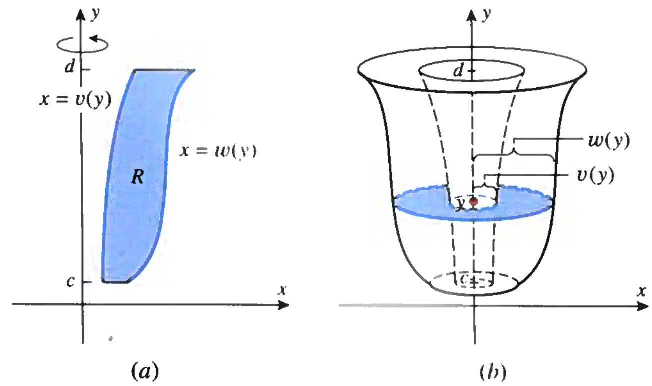


Figure 8.2.15

Example 5

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis (Figure 8.2.16).

Solution. The cross sections taken perpendicular to the y-axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

$$V = \int_c^d \pi[u(y)]^2\,dy = \int_0^2 \pi y^4\,dy = \frac{\pi y^5}{5}\bigg]_0^2 = \frac{32\pi}{5}$$ ◀

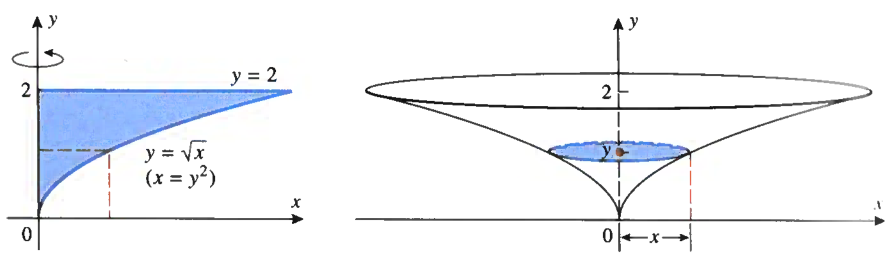
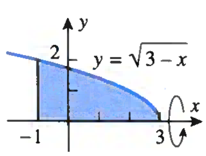


Figure 8.2.16

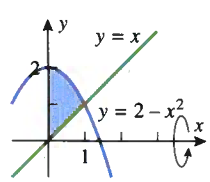
Exercise Set 8.2 C CAS

In Exercises 1–4, find the volume of the solid that results when the shaded region is revolved about the indicated axis.

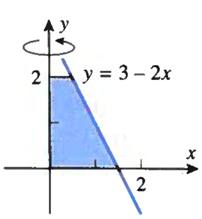
1.



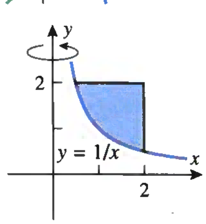
2.



3.



4.



In Exercises 5–14, find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

5. $y = x^2,\ x = 0,\ x = 2,\ y = 0$

6. $y = \sec x,\ x = \pi/4,\ x = \pi/3,\ y = 0$

7. $y = \sqrt{\cos x},\ x = \pi/4,\ x = \pi/2,\ y = 0$

8. $y = x^2,\ y = x^3$

9. $y = \sqrt{25 - x^2},\ y = 3$

10. $y = 9 - x^2,\ y = 0$

11. $y = e^x,\ y = 0,\ x = 0,\ x = \ln 3$

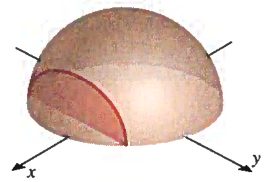
12. $y = e^{-2x},\ y = 0,\ x = 0,\ x = 1$

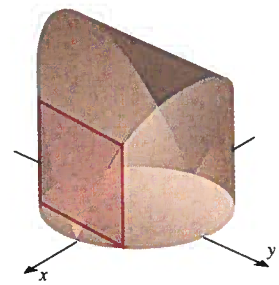
13. $x = \sqrt{y},\ x = y/4$

14. $y = \sin x,\ y = \cos x,\ x = 0,\ x = \pi/4$. [*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

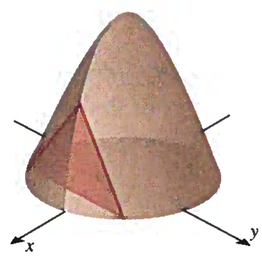
In Exercises 15–22, find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

15. $y = x^3,\ x = 0,\ y = 1$
16. $x = 1 - y^2,\ x = 0$
17. $x = \sqrt{1+y},\ x = 0,\ y = 3$
18. $y = x^2 - 1,\ x = 2,\ y = 0$
19. $x = \csc y,\ y = \pi/4,\ y = 3\pi/4,\ x = 0$
20. $y = x^2,\ x = y^2$
21. $x = y^2,\ x = y + 2$
22. $x = 1 - y^2,\ x = 2 + y^2,\ y = -1,\ y = 1$
23. Find the volume of the solid that results when the region above the x-axis and below the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$
is revolved about the x-axis.
24. Let V be the volume of the solid that results when the region enclosed by $y = 1/x, y = 0, x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.
25. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x+1},\ y = \sqrt{2x}$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]
26. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x},\ y = 6 - x$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]
27. Find the volume of the solid that results when the region enclosed by $y = \sqrt{x},\ y = 0$, and $x = 9$ is revolved about the line $x = 9$.
28. Find the volume of the solid that results when the region in Exercise 27 is revolved about the line $y = 3$.
29. Find the volume of the solid that results when the region enclosed by $x = y^2$ and $x = y$ is revolved about the line $y = -1$.
30. Find the volume of the solid that results when the region in Exercise 29 is revolved about the line $x = -1$.
31. A nose cone for a space reentry vehicle is designed so that a cross section, taken x ft from the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ ft. Find the volume of the nose cone given that its length is 20 ft.
32. A certain solid is 1 ft high, and a horizontal cross section taken x ft above the bottom of the solid is an annulus of inner radius x^2 and outer radius $\sqrt{x}$. Find the volume of the solid.
33. Find the volume of the solid whose base is the region bounded between the curves $y = x$ and $y = x^2$, and whose cross sections perpendicular to the x-axis are squares.
34. The base of a certain solid is the region enclosed by $y = \sqrt{x}$, $y = 0$, and $x = 4$. Every cross section perpendicular to the x-axis is a semicircle with its diameter across the base. Find the volume of the solid.
35. Find the volume of the solid whose base is enclosed by the circle $x^2 + y^2 = 1$ and whose cross sections taken perpendicular to the base are
(a) semicircles
(b) squares





(c) equilateral triangles.



36. Derive the formula for the volume of a right circular cone with radius r and height h.

In Exercises 37 and 38, use a CAS to find the volume of the solid that results when the region enclosed by the curves is revolved about the stated axis.

C 37. $y = \sin^8 x,\ y = 2x/\pi,\ x = 0,\ x = \pi/2$; x-axis
C 38. $y = e^x,\ x = 1,\ y = 1$; y-axis
39. The accompanying figure shows a ***spherical cap*** of radius ρ and height h cut from a sphere of radius r. Show that the volume V of the spherical cap can be expressed as
(a) $V = \frac{1}{3}\pi h^2(3r - h)$ (b) $V = \frac{1}{6}\pi h(3\rho^2 + h^2)$

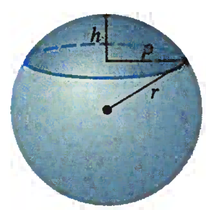


Figure Ex-39

40. If fluid enters a hemispherical vat with a radius of 10 ft at a rate of $\frac{1}{2}$ ft^3/min, how fast will the fluid be rising when the depth is 5 ft? [*Hint:* See Exercise 39.]
41. The accompanying figure shows the dimensions of a small lightbulb at 10 equally spaced points.
(a) Use formulas from geometry to make a rough estimate of the volume enclosed by the glass portion of the bulb.