

# 6

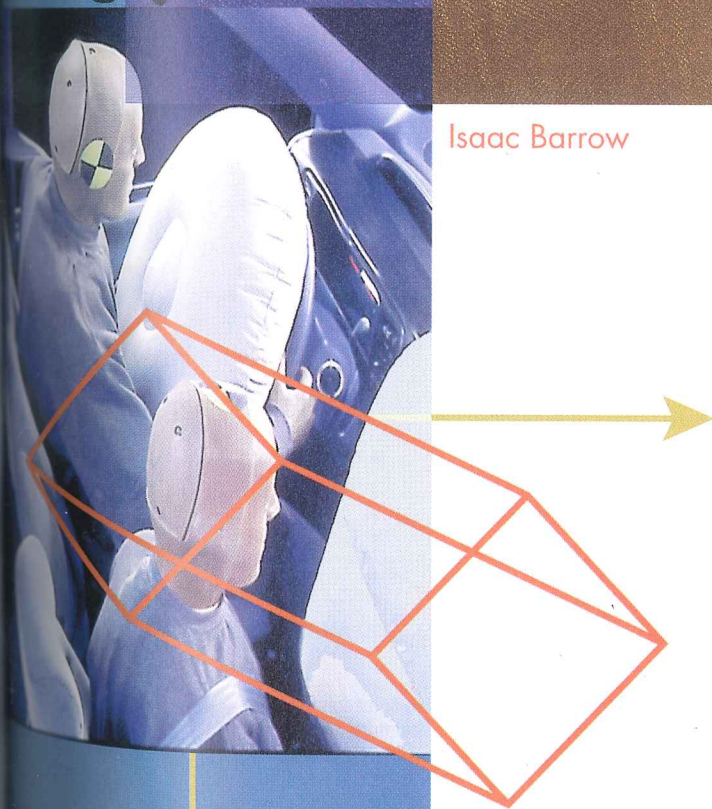
Isaac Barrow



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## APPLICATIONS OF THE DERIVATIVE

In this chapter we will study various applications of the derivative. For example, we will investigate problems that are concerned with finding the “best” way to perform a task—these are called *optimization problems*. Many optimization problems are concerned with time and cost. For example, if time is the main consideration in a problem, we might be interested in finding the quickest way to perform a task, and if cost is the main consideration, we might be interested in finding the most profitable way to perform the task. Mathematically, optimization problems can be reduced to finding the largest or smallest value of a function on some interval and determining where the largest or smallest value occurs; thus, part of our work in this chapter will focus on developing the mathematical tools for solving such problems. We will also use the derivative to study the motion of a particle moving along a line, and we will show how the derivative can be used to approximate solutions of equations.



## 6.1 ABSOLUTE MAXIMA AND MINIMA

At the beginning of Section 5.2 we observed that if the graph of a function  $f$  is viewed as a two-dimensional mountain range (Figure 5.2.1), then the relative maxima and minima correspond to the tops of the hills and the bottoms of the valleys; that is, they are the high and low points in their immediate vicinity. In this section we will be concerned with the more encompassing problem of finding the highest and lowest points over the entire mountain range, that is, we will be looking for the top of the highest hill and the bottom of the deepest valley. In mathematical terms, we will be looking for the largest and smallest values of a function over an interval.

## ABSOLUTE EXTREMA

We will be concerned here with finding the largest and smallest values of a function over a finite or infinite interval  $I$ . We begin with some terminology.

**6.1.1 DEFINITION.** A function  $f$  is said to have an **absolute maximum** on an interval  $I$  at the point  $x_0$  if  $f(x_0)$  is the largest value of  $f$  on  $I$ ; that is,  $f(x_0) \geq f(x)$  for all  $x$  in  $I$ . Similarly,  $f$  is said to have an **absolute minimum** on  $I$  at the point  $x_0$  if  $f(x_0)$  is the smallest value of  $f$  on  $I$ ; that is,  $f(x_0) \leq f(x)$  for all  $x$  in  $I$ . If  $f$  has either an absolute maximum or absolute minimum on  $I$  at  $x_0$ , then  $f$  is said to have an **absolute extremum** on  $I$  at  $x_0$ .

As illustrated in Figure 6.1.1, there is no guarantee that a function  $f$  will have absolute extrema on a given interval.

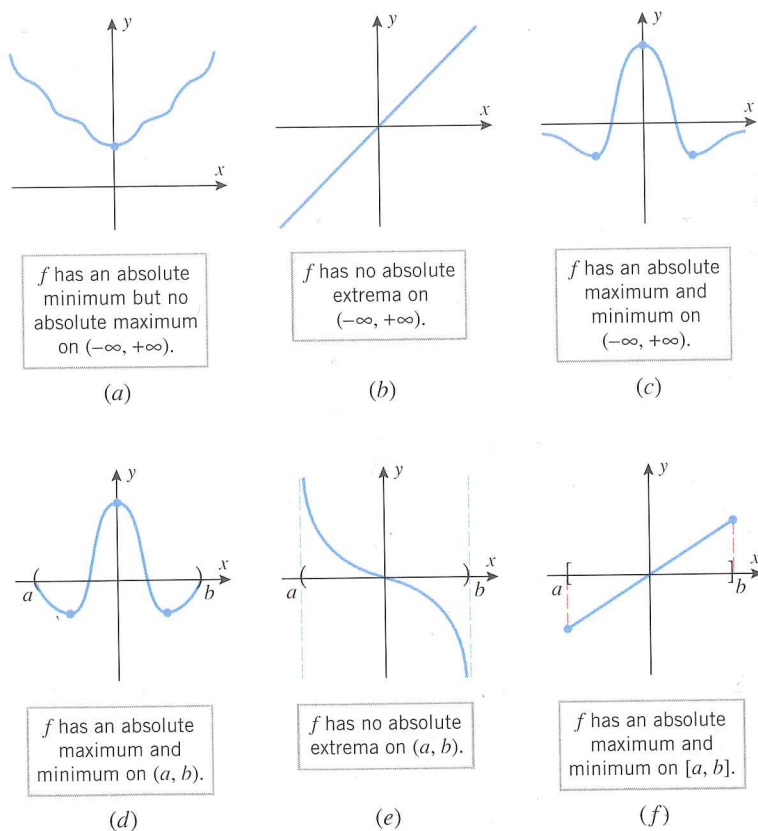


Figure 6.1.1

### EXISTENCE OF ABSOLUTE EXTREMA

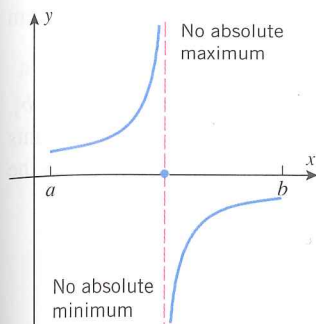


Figure 6.1.2

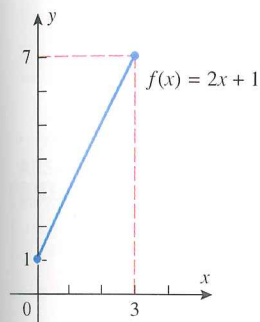


Figure 6.1.3

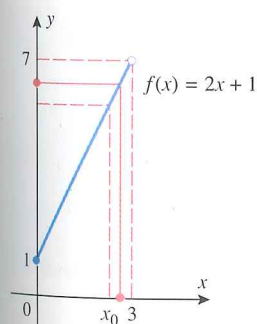


Figure 6.1.4

### FINDING ABSOLUTE EXTREMA ON FINITE CLOSED INTERVALS

The remainder of this section will focus on the following problem.

#### 6.1.2 PROBLEM.

- Determine whether a function  $f$  has any absolute extrema on a given interval  $I$ .
- If there are absolute extrema, determine where they occur and what the absolute maximum and minimum values are.

Parts (a)–(e) of Figure 6.1.1 show that a continuous function may or may not have relative maxima or minima on an infinite interval or on a finite open interval. However, the following theorem shows that a continuous function must have both an absolute maximum and an absolute minimum on every *finite closed* interval [see part (f) of Figure 6.1.1].

**6.1.3 THEOREM (Extreme-Value Theorem).** *If a function  $f$  is continuous on a finite closed interval  $[a, b]$ , then  $f$  has both an absolute maximum and an absolute minimum on  $[a, b]$ .*

**FOR THE READER.** Although the proof of this theorem is too difficult to include here, you should be able to convince yourself of its validity with a little experimentation—try graphing various continuous functions over the interval  $[0, 1]$ , and convince yourself that there is no way to avoid having a highest and lowest point on the graph. As a physical analogy, if you imagine the graph to be a roller coaster track starting at  $x = 0$  and ending at  $x = 1$ , the roller coaster will have to pass through a highest point and a lowest point during the trip.

The hypotheses in the Extreme-Value Theorem are essential; for example, if  $f$  is not continuous, then we could encounter a situation such as that in Figure 6.1.2, and if  $f$  is continuous but the interval is not closed and finite, then we could encounter situations such as those in Figure 6.1.1. This is illustrated further in the following example.

#### Example 1

The function  $f(x) = 2x + 1$  is continuous, and hence is guaranteed to have both an absolute maximum and an absolute minimum on every finite closed interval and, in particular, on the interval  $[0, 3]$ . For this interval an absolute minimum occurs at  $x = 0$  and an absolute maximum occurs at  $x = 3$ , at which points the absolute minimum and maximum values are  $f(0) = 1$  and  $f(3) = 7$  (Figure 6.1.3).

However, if we consider this same function on the half-open interval  $[0, 3)$ , then there is no longer an absolute maximum. To see why this so, observe that  $f(3) = 7$  is no longer the absolute maximum because we have removed the point  $x = 3$  from the interval. However,  $f$  cannot have an absolute maximum in the interval at a point  $x_0$  that is *less* than 3, because  $f$  will have a larger value at any point in the interval to the right of  $x_0$  (Figure 6.1.4). Thus,  $f$  has no absolute maximum on the interval  $[0, 3)$ . ◀

The Extreme-Value Theorem is an example of what mathematicians call an *existence theorem*. Such theorems state conditions under which something exists, in this case absolute extrema. However, knowing that something exists and finding it are two separate things, so we will now address the problem of finding the absolute extrema.

If  $f$  is continuous on the finite closed interval  $[a, b]$ , then the absolute extrema of  $f$  can occur either at the endpoints of the interval or inside on the open interval  $(a, b)$ . If the absolute extrema happen to fall inside, then the following theorem tells us that they must occur at critical points of  $f$ .

**6.1.4 THEOREM.** If  $f$  has an absolute extremum on an open interval  $(a, b)$ , then it must occur at a critical point of  $f$ .

**Proof.** If  $f$  has an absolute maximum on  $(a, b)$  at  $x_0$ , then  $f(x_0)$  is also a relative maximum for  $f$ ; for if  $f(x_0)$  is the largest value of  $f$  on all of  $(a, b)$ , then  $f(x_0)$  is certainly the largest value for  $f$  in the immediate vicinity of  $x_0$ . Thus,  $x_0$  is a critical point of  $f$  by Theorem 5.2.2. The proof for absolute minima is similar. ■

It follows from this theorem, that if  $f$  is continuous on the finite closed interval  $[a, b]$ , then the absolute extrema occur either at the endpoints of the interval or at critical points inside the interval (Figure 6.1.5). Thus, we can use the following procedure to find the absolute extrema of a continuous function on a finite closed interval  $[a, b]$ .

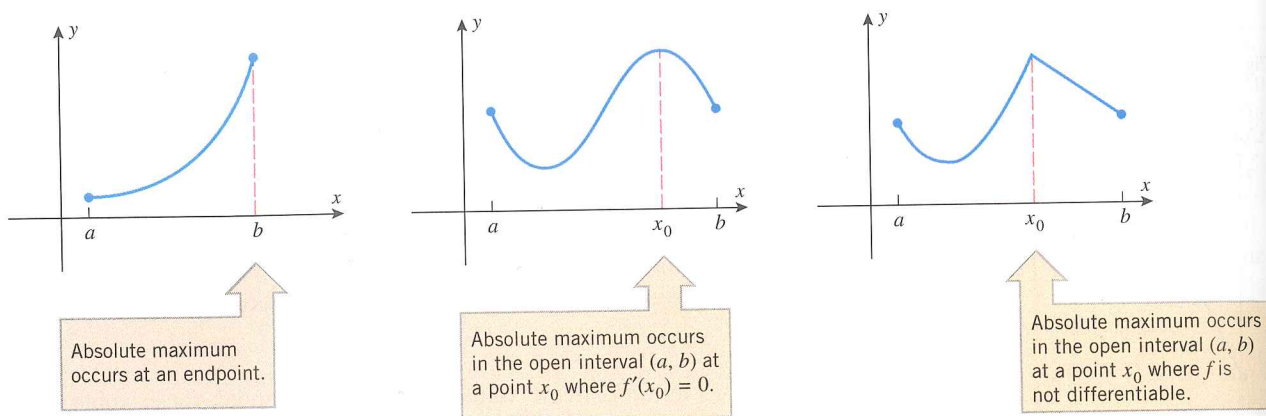


Figure 6.1.5

**A Procedure for Finding the Absolute Extrema of a Continuous Function  $f$  on a Finite Closed Interval  $[a, b]$ .**

- Step 1.** Find the critical points of  $f$  in  $(a, b)$ .
- Step 2.** Evaluate  $f$  at all the critical points and at the endpoints  $a$  and  $b$ .
- Step 3.** The largest of the values in Step 2 is the absolute maximum value of  $f$  on  $[a, b]$  and the smallest value is the absolute minimum.

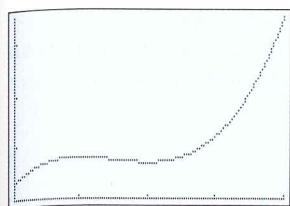
**Example 2**

Find the absolute maximum and minimum values of  $f(x) = 2x^3 - 15x^2 + 36x$  on the interval  $[1, 5]$ , and determine where these values occur.

**Solution.** Since  $f$  is differentiable, the absolute extrema must occur either at the endpoints of the interval  $[1, 5]$  or at stationary points in the open interval  $(1, 5)$ . To find the stationary points, we must solve the equation  $f'(x) = 0$ , which can be written as

$$6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x - 3)(x - 2) = 0$$

Thus, there are stationary points at  $x = 2$  and  $x = 3$ . Evaluating  $f$  at the endpoints and the



$[1, 5] \times [20, 55]$   
 $x\text{Scl} = 1, y\text{Scl} = 10$

Figure 6.1.6

stationary points yields

$$f(1) = 2(1)^3 - 15(1)^2 + 36(1) = 23$$

$$f(2) = 2(2)^3 - 15(2)^2 + 36(2) = 28$$

$$f(3) = 2(3)^3 - 15(3)^2 + 36(3) = 27$$

$$f(5) = 2(5)^3 - 15(5)^2 + 36(5) = 55$$

from which we conclude that an absolute minimum of  $f(x) = 23$  occurs at  $x = 1$  and an absolute maximum of  $f(x) = 55$  occurs at  $x = 5$ . This is consistent with the graph of  $f$  in Figure 6.1.6. ◀

### Example 3

Find the absolute extrema of  $f(x) = 6x^{4/3} - 3x^{1/3}$  on the interval  $[-1, 1]$ , and determine where these values occur.

**Solution.** Differentiating we obtain

$$f'(x) = 8x^{1/3} - x^{-2/3} = x^{-2/3}(8x - 1) = \frac{8x - 1}{x^{2/3}}$$

Thus,  $f'(x) = 0$  at  $x = \frac{1}{8}$ , and there is a point of nondifferentiability at  $x = 0$ . Evaluating  $f$  at these critical points and the endpoints yields Table 6.1.1, from which we conclude that an absolute minimum of  $f(x) = -\frac{9}{8}$  occurs at  $x = \frac{1}{8}$ , and an absolute maximum of  $f(x) = 9$  occurs at  $x = -1$ . ◀

Table 6.1.1

$x$	-1	0	$\frac{1}{8}$	1
$f(x)$	9	0	$-\frac{9}{8}$	3

### ABSOLUTE EXTREMA ON INFINITE INTERVALS

We observed earlier that a continuous function may or may not have absolute extrema on an infinite interval (see Figure 6.1.1). However, certain conclusions about the existence of absolute extrema of a continuous function  $f$  on  $(-\infty, +\infty)$  can be drawn from the behavior of  $f(x)$  as  $x \rightarrow -\infty$  and  $x \rightarrow +\infty$  (Table 6.1.2).

Table 6.1.2

<b>LIMITS</b>	$\lim_{x \rightarrow -\infty} f(x) = +\infty$ $\lim_{x \rightarrow +\infty} f(x) = +\infty$	$\lim_{x \rightarrow -\infty} f(x) = -\infty$ $\lim_{x \rightarrow +\infty} f(x) = -\infty$	$\lim_{x \rightarrow -\infty} f(x) = -\infty$ $\lim_{x \rightarrow +\infty} f(x) = +\infty$	$\lim_{x \rightarrow -\infty} f(x) = +\infty$ $\lim_{x \rightarrow +\infty} f(x) = -\infty$
<b>CONCLUSION IF <math>f</math> IS CONTINUOUS</b>	$f$ has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$ .	$f$ has an absolute maximum but no absolute minimum on $(-\infty, +\infty)$ .	$f$ has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$ .	$f$ has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$ .
<b>GRAPH</b>				

### Example 4

What can you say about the existence of absolute extrema on  $(-\infty, +\infty)$  for polynomials?

**Solution.** If  $p(x)$  is a polynomial of odd degree, then

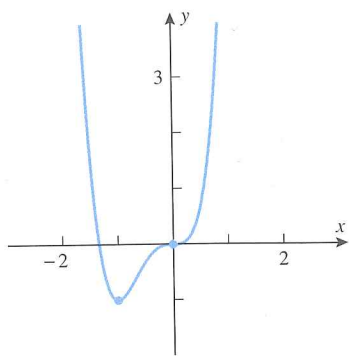
$$\lim_{x \rightarrow +\infty} p(x) \quad \text{and} \quad \lim_{x \rightarrow -\infty} p(x) \quad (1)$$

have opposite signs (one is  $+\infty$  and the other is  $-\infty$ ), so there are no absolute extrema. On

the other hand, if  $p(x)$  has even degree, then the limits in (1) have the same sign (both  $+\infty$  or both  $-\infty$ ). If the leading coefficient is positive, then both limits are  $+\infty$ , and there is an absolute minimum; if the leading coefficient is negative, then both limits are  $-\infty$ , and there is an absolute maximum. ◀

### Example 5

Determine by inspection whether  $p(x) = 3x^4 + 4x^3$  has any absolute extrema. If so, find them, and state where they occur.



$$p(x) = 3x^4 + 4x^3$$

Figure 6.1.7

**Solution.** Since  $p(x)$  has even degree and the leading coefficient is positive,  $p(x) \rightarrow +\infty$  as  $x \rightarrow \pm\infty$ . Thus, there is an absolute minimum but no absolute maximum. From Theorem 6.1.4 [applied to the interval  $(-\infty, +\infty)$ ], the absolute minimum must occur at a critical point of  $p$ . Since  $p$  is differentiable, all critical points are stationary points, so we can find them by solving the equation  $p'(x) = 0$ . This equation is

$$12x^3 + 12x^2 = 12x^2(x + 1) = 0$$

from which we conclude that stationary points occur at  $x = 0$  and  $x = -1$ . Evaluating  $p$  at the stationary points yields

$$p(0) = 0 \quad \text{and} \quad p(-1) = -1$$

from which we conclude that  $p$  has an absolute minimum of  $p(x) = -1$ , and this occurs at  $x = -1$  (Figure 6.1.7). ◀

### ABSOLUTE EXTREMA ON OPEN INTERVALS

We know that a continuous function may or may not have absolute extrema on an open interval. However, certain conclusions about the existence of absolute extrema of a continuous function  $f$  on a finite open interval  $(a, b)$  can be drawn from the behavior of  $f(x)$  as  $x \rightarrow a^+$  and as  $x \rightarrow b^-$  (Table 6.1.3).

Table 6.1.3

<b>LIMITS</b>	$\lim_{x \rightarrow a^+} f(x) = +\infty$ $\lim_{x \rightarrow b^-} f(x) = +\infty$	$\lim_{x \rightarrow a^+} f(x) = -\infty$ $\lim_{x \rightarrow b^-} f(x) = -\infty$	$\lim_{x \rightarrow a^+} f(x) = -\infty$ $\lim_{x \rightarrow b^-} f(x) = +\infty$	$\lim_{x \rightarrow a^+} f(x) = +\infty$ $\lim_{x \rightarrow b^-} f(x) = -\infty$
<b>CONCLUSION IF <math>f</math> IS CONTINUOUS ON <math>(a, b)</math></b>	$f$ has an absolute minimum but no absolute maximum on $(a, b)$ .	$f$ has an absolute maximum but no absolute minimum on $(a, b)$ .	$f$ has neither an absolute maximum nor an absolute minimum on $(a, b)$ .	$f$ has neither an absolute maximum nor an absolute minimum on $(a, b)$ .
<b>GRAPH</b>				

### Example 6

Determine whether the function

$$f(x) = \frac{1}{x^2 - x}$$

has any absolute extrema on the interval  $(0, 1)$ . If so, find them and state where they occur.

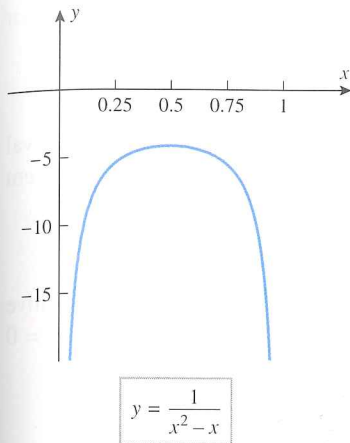


Figure 6.1.8

### ABSOLUTE EXTREMA OF FUNCTIONS WITH ONE RELATIVE EXTREMUM

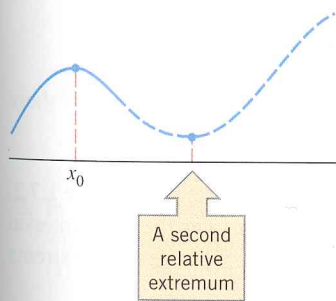


Figure 6.1.9

**Solution.** Since  $f$  is continuous on the interval  $(0, 1)$  and

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x^2 - x} = \lim_{x \rightarrow 0^+} \frac{1}{x(x - 1)} = -\infty$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{1}{x^2 - x} = \lim_{x \rightarrow 1^-} \frac{1}{x(x - 1)} = -\infty$$

the function  $f$  has an absolute maximum but no absolute minimum on the interval  $(0, 1)$ . By Theorem 6.1.4 the absolute maximum must occur at a critical point of  $f$ , so we need to look for stationary points or points of nondifferentiability in the interval  $(0, 1)$ . We have

$$f'(x) = -\frac{2x - 1}{(x^2 - x)^2}$$

so the only solution of the equation  $f'(x) = 0$  is  $x = \frac{1}{2}$ . The denominator of  $f'$  is zero if  $x = 0$  or  $x = 1$ , but these critical points are of no concern here because they fall outside of the open interval  $(0, 1)$ . Thus, the absolute maximum occurs at  $x = \frac{1}{2}$ , and this absolute maximum is

$$f\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2 - \frac{1}{2}} = -4$$

(Figure 6.1.8). ◀

If a continuous function has only one relative extremum on a finite or infinite interval  $I$ , then that relative extremum must of necessity also be an absolute extremum. To understand why this is so, suppose that  $f$  has a relative maximum at a point  $x_0$  on an interval  $I$ , and there are no other relative extrema of  $f$  on  $I$ . If  $f(x_0)$  is not the absolute maximum of  $f$  on  $I$ , then the graph of  $f$  has to make an upward turn somewhere on  $I$  to rise above  $f(x_0)$ . However, this cannot happen because in the process of making an upward turn it would produce a second relative extremum on  $I$  (Figure 6.1.9). Thus,  $f(x_0)$  must be the absolute maximum as well as a relative maximum. This idea is captured in the following theorem, which we state without proof.

**6.1.5 THEOREM.** Suppose that  $f$  is continuous and has exactly one relative extremum on an interval  $I$ , say at  $x_0$ .

- (a) If  $f$  has a relative minimum at  $x_0$ , then  $f(x_0)$  is the absolute minimum of  $f$  on  $I$ .  
 (b) If  $f$  has a relative maximum at  $x_0$ , then  $f(x_0)$  is the absolute maximum of  $f$  on  $I$ .

This theorem is often helpful in situations where other methods are difficult or tedious to apply.

### Example 7

Find all absolute extrema of the function  $f(x) = x^3 - 3x^2 + 4$  on the interval

- (a)  $(-\infty, +\infty)$       (b)  $(0, +\infty)$

**Solution (a).** Because  $f$  is a polynomial of odd degree, it follows from the discussion in Example 4 that there are no absolute extrema on the interval  $(-\infty, +\infty)$ .

**Solution (b).** Since

$$\lim_{x \rightarrow +\infty} (x^3 - 3x^2 + 4) = +\infty$$

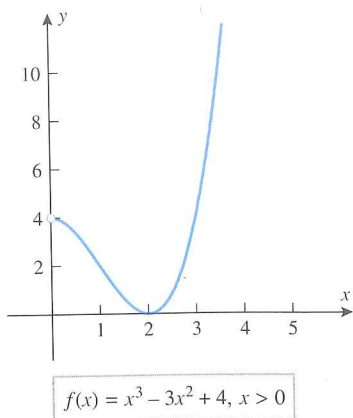


Figure 6.1.10

### ABSOLUTE EXTREMA AND PARAMETRIC CURVES

we know that  $f$  cannot have an absolute maximum on the interval  $(0, +\infty)$ . However, the limit

$$\lim_{x \rightarrow 0^+} (x^3 - 3x^2 + 4) = 4$$

is not infinite, so there is a possibility that  $f$  may have an absolute minimum on this interval. In this case it would have to occur at a stationary point, which suggests that we look for solutions of the equation  $f'(x) = 0$ . But,

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

so  $f$  has stationary points at  $x = 0$  and  $x = 2$ . However,  $x = 0$  falls outside of the interval  $(0, +\infty)$ , so only the stationary point at  $x = 2$  lies in the interval  $(0, +\infty)$ . Thus, Theorem 6.1.5 is applicable here. Since

$$f''(x) = 6x - 6$$

we have  $f''(2) = 6 > 0$ , so a relative minimum occurs at  $x = 2$  by the second derivative test. Thus,  $f(x)$  has an absolute minimum at  $x = 2$ , and this absolute minimum is  $f(2) = 0$  (Figure 6.1.10). ◀

Suppose that a curve  $C$  is given parametrically by the equations

$$x = f(t), \quad y = g(t) \quad (a \leq t \leq b)$$

where  $f$  and  $g$  are *continuous* on the finite closed interval  $[a, b]$ . It follows from the Extreme-Value Theorem that  $f(t)$  and  $g(t)$  have absolute maxima and absolute minima for  $a \leq t \leq b$ ; this means that a particle moving along the curve cannot move away from the origin indefinitely—there must be a smallest and largest  $x$ -coordinate and a smallest and largest  $y$ -coordinate. Geometrically, the entire curve is contained within a box determined by these smallest and largest coordinates.

### Example 8

Suppose that the equations of motion for a paper airplane during its first 10 seconds of flight are

$$x = t - 3 \sin t, \quad y = 4 - 3 \cos t \quad (0 \leq t \leq 10)$$

What are the highest and lowest points in the trajectory, and when is the airplane at those points?

**Solution.** The trajectory, pictured in Figure 6.1.11, is shown in more detail in Figure 1.7.2. We want to find the absolute maximum and minimum values of  $y$  over the time interval  $[0, 10]$  and the values of  $t$  for which these absolute extrema occur. The absolute extrema

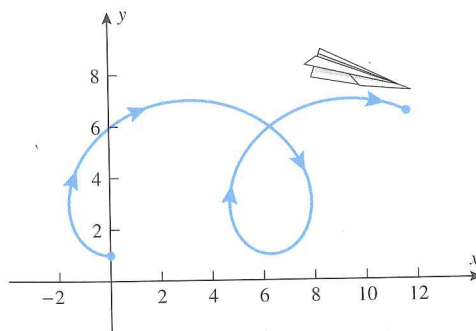


Figure 6.1.11

must occur either at the endpoints of the closed interval  $[0, 10]$  or at stationary points in the open interval  $(0, 10)$ . To find the stationary points, we must solve the equation  $dy/dt = 0$ , which is

$$3 \sin t = 0$$

Thus, there are stationary points in the interval  $(0, 10)$  at  $t = \pi, 2\pi$ , and  $3\pi$ . Evaluating  $y = 4 - 3 \cos t$  at the endpoints and the stationary points yields

$$y = 4 - 3 \cos 0 = 4 - 3 = 1$$

$$y = 4 - 3 \cos \pi = 4 - (-3) = 7$$

$$y = 4 - 3 \cos 2\pi = 4 - 3 = 1$$

$$y = 4 - 3 \cos 3\pi = 4 - (-3) = 7$$

$$y = 4 - 3 \cos 10 \approx 6.517$$

Thus, a high point of  $y = 7$  is reached at times  $t = \pi$  and  $t = 3\pi$ , and a low point of  $y = 1$  is reached at times  $t = 0$  and  $t = 2\pi$ . This is consistent with Figure 1.7.2. ◀

### EXERCISE SET 6.1

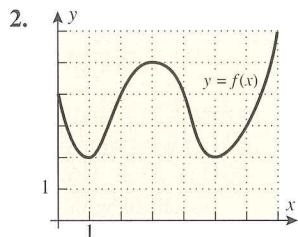
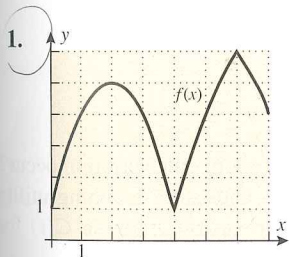


Graphing Calculator



CAS

In Exercises 1–2, use the graph to find  $x$ -coordinates of the relative extrema and absolute extrema of  $f$ .



3. In each part, sketch the graph of a continuous function  $f$  with the stated properties on the interval  $[0, 10]$ .
- $f$  has an absolute minimum at  $x = 0$  and an absolute maximum at  $x = 10$ .
  - $f$  has an absolute minimum at  $x = 2$  and an absolute maximum at  $x = 7$ .
  - $f$  has a relative minima at  $x = 1$  and  $x = 8$ , has relative maxima at  $x = 3$  and  $x = 7$ , has an absolute minimum at  $x = 5$ , and has an absolute maximum at  $x = 10$ .
4. In each part, sketch the graph of a continuous function  $f$  with the stated properties on the interval  $(-\infty, +\infty)$ .
- $f$  has no relative extrema or absolute extrema.
  - $f$  has an absolute minimum at  $x = 0$  but no absolute maximum.
  - $f$  has an absolute maximum at  $x = -5$  and an absolute minimum at  $x = 5$ .

In Exercises 5–14, find the absolute maximum and minimum values of  $f$  on the given closed interval, and state where those values occur.

- $f(x) = 4x^2 - 4x + 1$ ;  $[0, 1]$
- $f(x) = 8x - x^2$ ;  $[0, 6]$
- $f(x) = (x - 1)^3$ ;  $[0, 4]$
- $f(x) = 2x^3 - 3x^2 - 12x$ ;  $[-2, 3]$
- $f(x) = \frac{3x}{\sqrt{4x^2 + 1}}$ ;  $[-1, 1]$
- $f(x) = (x^2 + x)^{2/3}$ ;  $[-2, 3]$
- $f(x) = x - \tan x$ ;  $[-\pi/4, \pi/4]$
- $f(x) = \sin x - \cos x$ ;  $[0, \pi]$
- $f(x) = 1 + |9 - x^2|$ ;  $[-5, 1]$
- $f(x) = |6 - 4x|$ ;  $[-3, 3]$

In Exercises 15–22, find the absolute maximum and minimum values of  $f$ , if any, on the given interval, and state where those values occur.

- $f(x) = x^2 - 3x - 1$ ;  $(-\infty, +\infty)$
- $f(x) = 3 - 4x - 2x^2$ ;  $(-\infty, +\infty)$
- $f(x) = 4x^3 - 3x^4$ ;  $(-\infty, +\infty)$

18.  $f(x) = x^4 + 4x$ ;  $(-\infty, +\infty)$   
 19.  $f(x) = x^3 - 3x - 2$ ;  $(-\infty, +\infty)$   
 20.  $f(x) = x^3 - 9x + 1$ ;  $(-\infty, +\infty)$   
 21.  $f(x) = \frac{x^2}{x+1}$ ;  $(-5, -1)$     22.  $f(x) = \frac{x+3}{x-3}$ ;  $[-5, 5]$

In Exercises 23–34, use a graphing utility to estimate the absolute maximum and minimum values of  $f$ , if any, on the stated interval, and then use calculus methods to find the exact values.

23.  $f(x) = (x^2 - 1)^2$ ;  $(-\infty, +\infty)$   
 24.  $f(x) = (x - 1)^2(x + 2)^2$ ;  $(-\infty, +\infty)$   
 25.  $f(x) = x^{2/3}(20 - x)$ ;  $[-1, 20]$   
 26.  $f(x) = \frac{x}{x^2 + 2}$ ;  $[-1, 4]$   
 27.  $f(x) = 1 + \frac{1}{x}$ ;  $(0, +\infty)$   
 28.  $f(x) = \frac{x}{x^2 + 1}$ ;  $[0, +\infty)$   
 29.  $f(x) = 2 \sec x - \tan x$ ;  $[0, \pi/4]$   
 30.  $f(x) = \sin^2 x + \cos x$ ;  $[-\pi, \pi]$   
 31.  $f(x) = x^3 e^{-2x}$ ;  $[1, 4]$   
 32.  $f(x) = \frac{\ln x}{x}$ ;  $[1, e]$   
 33.  $f(x) = \sin(\cos x)$ ;  $[0, 2\pi]$   
 34.  $f(x) = \cos(\sin x)$ ;  $[0, \pi]$   
 35. Find the absolute maximum and minimum values of

$$f(x) = \begin{cases} 4x - 2, & x < 1 \\ (x - 2)(x - 3), & x \geq 1 \end{cases}$$

on  $[\frac{1}{2}, \frac{7}{2}]$ .

36. Let  $f(x) = x^2 + px + q$ . Find the values of  $p$  and  $q$  such that  $f(1) = 3$  is an extreme value of  $f$  on  $[0, 2]$ . Is this value a maximum or minimum?

If  $f$  is a periodic function, then the locations of all absolute extrema on the interval  $(-\infty, +\infty)$  can be obtained by finding the locations of the absolute extrema for one period and using the periodicity to locate the rest. Use this idea in Exercise 37 and 38 to find the absolute maximum and minimum values of the function, and state the  $x$ -values at which they occur.

37.  $f(x) = 2 \sin 2x + \sin 4x$     38.  $f(x) = 3 \cos \frac{x}{3} + 2 \cos \frac{x}{2}$

One way of proving that  $f(x) \leq g(x)$  for all  $x$  in a given interval is to show that  $0 \leq g(x) - f(x)$  for all  $x$  in the interval; and one way of proving the latter inequality is to show that the absolute minimum value of  $g(x) - f(x)$  on the interval is nonnegative. Use this idea to prove the inequalities in Exercises 39 and 40.

39. Prove that  $\sin x \leq x$  for all  $x$  in the interval  $[0, 2\pi]$ .  
 40. Prove that  $\ln x \leq x - 1$  on the interval  $(0, +\infty)$ .  
 41. What is the smallest possible slope for a tangent to the equation  $y = x^3 - 3x^2 + 5x$ ?  
 42. (a) Show that

$$f(x) = \frac{64}{\sin x} + \frac{27}{\cos x}$$

has a minimum value, but no maximum value on the interval  $(0, \pi/2)$ .

(b) Find the minimum value.

43. Use a CAS to show that the absolute minimum value of

$$f(x) = x^2 + \frac{16x^2}{(8-x)^2}, \quad x > 8$$

occurs at  $x = 4(2 + \sqrt[3]{2})$  by using it to find  $f'(x)$  and to solve the equation  $f'(x) = 0$ .

44. The concentration  $C(t)$  of a drug in the bloodstream  $t$  hours after it has been injected is commonly modeled by an equation of the form

$$C(t) = \frac{K(e^{-bt} - e^{-at})}{a - b}$$

where  $K > 0$  and  $a > b > 0$ .

- (a) At what time does the maximum concentration occur?  
 (b) Let  $K = 1$  for simplicity, and use a graphing utility to check your result in part (a) by graphing  $C(t)$  for various values of  $a$  and  $b$ .  
 45. It can be proved that if  $f$  is differentiable on  $(a, b)$  and  $L$  is a line that does not intersect the curve  $y = f(x)$  over an interval  $(a, b)$ , then the points at which the curve is closest to or farthest from the line  $L$ , if any, occur at points where the tangent line to the curve is parallel to  $L$  (see the accompanying figure). Use this result to find the points on the graph of  $y = -x^2$  that are closest to and farthest from the line  $y = 2 - x$  for  $-1 \leq x \leq \frac{3}{2}$ .

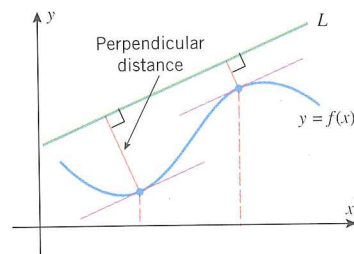


Figure Ex-45

46. Use the idea discussed in Exercise 45 to find the coordinates of all points on the graph of  $y = x^3$  closest to and farthest from the line  $y = \frac{4}{3}x - 1$  for  $-1 \leq x \leq 1$ .
47. Suppose that the equations of motion of a paper airplane during the first 12 seconds of flight are

$$x = t - 2 \sin t, \quad y = 2 - 2 \cos t \quad (0 \leq t \leq 12)$$

What are the highest and lowest points in the trajectory, and when is the airplane at those points?

48. The accompanying figure shows the path of a fly whose equations of motion are

$$x = \frac{\cos t}{2 + \sin t}, \quad y = 3 + \sin(2t) - 2 \sin^2 t \quad (0 \leq t \leq 2\pi)$$

- (a) How high and low does it fly?  
 (b) How far left and right of the origin does it fly?

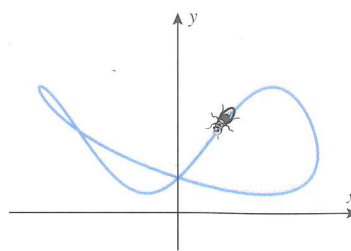


Figure Ex-48

49. Let  $f(x) = ax^2 + bx + c$ , where  $a > 0$ . Prove that  $f(x) \geq 0$  for all  $x$  if and only if  $b^2 - 4ac \leq 0$ . [Hint: Find the minimum of  $f(x)$ .]
50. Prove Theorem 6.1.4 in the case where the extreme value is a minimum.

## 6.2 APPLIED MAXIMUM AND MINIMUM PROBLEMS

*In this section we will show how the methods discussed in the last section can be used to solve various applied optimization problems.*

### CLASSIFICATION OF OPTIMIZATION PROBLEMS

The applied optimization problems that we will consider in this section fall into the following two categories:

- Problems that reduce to maximizing or minimizing a continuous function over a finite closed interval.
- Problems that reduce to maximizing or minimizing a continuous function over an infinite interval or a finite interval that is not closed.

For problems of the first type the Extreme-Value Theorem (6.1.3) guarantees that the problem has a solution, and we know that the solution can be obtained by examining the values of the function at the critical points and the endpoints. However, for problems of the second type there may or may not be a solution. Thus, part of the attack on such problems is to determine whether there actually is a solution. If the function is continuous and has exactly one relative extremum on the interval, then Theorem 6.1.5 guarantees the existence of a solution and provides a method for finding it. In cases where this theorem is not applicable some ingenuity may be required to solve the problem.

### PROBLEMS INVOLVING FINITE CLOSED INTERVALS

#### Example 1

Find the dimensions of a rectangle with perimeter 100 ft whose area is as large as possible.

**Solution.** Let

$x$  = length of the rectangle (ft)

$y$  = width of the rectangle (ft)

$A$  = area of the rectangle (ft<sup>2</sup>)

Then

$$A = xy \tag{1}$$

Since the perimeter of the rectangle is 100 ft, the variables  $x$  and  $y$  are related by the equation

$$2x + 2y = 100 \quad \text{or} \quad y = 50 - x \tag{2}$$

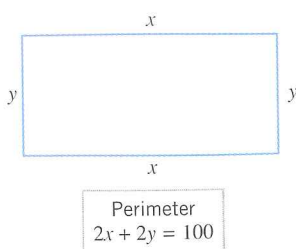


Figure 6.2.1

Table 6.2.1

$x$	0	25	50
$A$	0	625	0

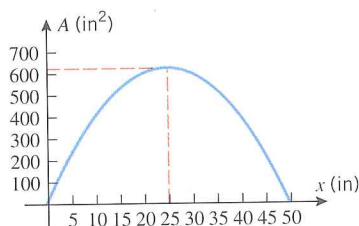


Figure 6.2.2

(See Figure 6.2.1.) Substituting (2) in (1) yields

$$A = x(50 - x) = 50x - x^2 \quad (3)$$

Because  $x$  represents a length it cannot be negative, and because the two sides of length  $x$  cannot have a combined length exceeding the total perimeter of 100 ft, the variable  $x$  must satisfy

$$0 \leq x \leq 50 \quad (4)$$

Thus, we have reduced the problem to that of finding the value (or values) of  $x$  in  $[0, 50]$ , for which  $A$  is maximum. Since  $A$  is a polynomial in  $x$ , it is continuous on  $[0, 50]$ , and so the maximum must occur at an endpoint of this interval or at a stationary point.

From (3) we obtain

$$\frac{dA}{dx} = 50 - 2x$$

Setting  $dA/dx = 0$  we obtain

$$50 - 2x = 0$$

or  $x = 25$ . Thus, the maximum occurs at one of the points

$$x = 0, \quad x = 25, \quad x = 50$$

Substituting these values in (3) yields Table 6.2.1, which tells us that the maximum area of  $625 \text{ ft}^2$  occurs at  $x = 25$ , which is consistent with the graph of (3) in Figure 6.2.2. From (2) the corresponding value of  $y = 25$ , so the rectangle of perimeter 100 ft with greatest area is a square with sides of length 25 ft. ◀

**REMARK.** In this example we included  $x = 0$  and  $x = 50$  as possible values for  $x$ , even though both values lead to rectangles with two sides of length zero. Whether or not these values should be allowed will depend on our objective in the problem. If we view this purely as a mathematical problem, then there is nothing wrong with allowing sides of length zero. However, if we view this as an applied problem in which the rectangle will be formed from physical material, then these values should be excluded.

Example 1 illustrates the following five-step procedure that can be used for solving many applied maximum and minimum problems.

- Step 1.** Draw an appropriate figure and label the quantities relevant to the problem.
- Step 2.** Find a formula for the quantity to be maximized or minimized.
- Step 3.** Using the conditions stated in the problem to eliminate variables, express the quantity to be maximized or minimized as a function of one variable.
- Step 4.** Find the interval of possible values for this variable from the physical restrictions in the problem.
- Step 5.** If applicable, use the techniques of the preceding section to obtain the maximum or minimum.

### Example 2

An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 6.2.3). What size should the squares be to obtain a box with largest possible volume?

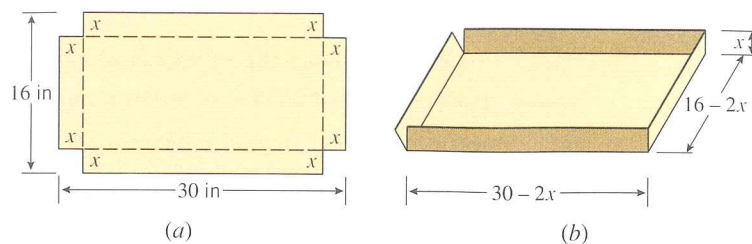


Figure 6.2.3

**Solution.** Let

$x$  = length (in inches) of the sides of the squares to be cut out

$V$  = volume (in cubic inches) of the resulting box

Because we are removing a square of side  $x$  from each corner, the resulting box will have dimensions  $16 - 2x$  by  $30 - 2x$  by  $x$  (Figure 6.2.3b). Since the volume of a box is the product of its dimensions, we have

$$V = (16 - 2x)(30 - 2x)x = 480x - 92x^2 + 4x^3 \quad (5)$$

The variable  $x$  in this expression is subject to certain restrictions. Because  $x$  represents a length it cannot be negative, and because the width of the cardboard is 16 inches we cannot cut out squares whose sides are more than 8 inches long. Thus, the variable  $x$  in (5) must satisfy

$$0 \leq x \leq 8$$

and hence we have reduced our problem to finding the value (or values) of  $x$  in the interval  $[0, 8]$  for which (5) is maximum. From (5) we obtain

$$\frac{dV}{dx} = 480 - 184x + 12x^2 = 4(120 - 46x + 3x^2)$$

Setting  $dV/dx = 0$  yields

$$120 - 46x + 3x^2 = 0$$

which can be solved by the quadratic formula to obtain the critical points

$$x = \frac{10}{3} \quad \text{and} \quad x = 12$$

Since  $x = 12$  falls outside the interval  $[0, 8]$ , the maximum value of  $V$  occurs either at the critical point  $x = \frac{10}{3}$  or at one of the endpoints  $x = 0$ ,  $x = 8$ . Substituting these values in (5) yields Table 6.2.2, which tells us that the greatest possible volume  $V = \frac{19600}{27} \text{ in}^3 \approx 726 \text{ in}^3$  occurs when we cut out squares whose sides have length  $\frac{10}{3}$  inches. This is consistent with the graph of (5) shown in Figure 6.2.4. ◀

In Example 2 of Section 1.1 we used approximate graphical methods to solve a problem of piping oil from an offshore well to a point on the shore with minimal cost. We will now show how to solve that problem exactly using calculus.

### Example 3

Figure 6.2.5 shows an offshore oil well located at a point  $W$  that is 5 km from the closest point  $A$  on a straight shoreline. Oil is to be piped from  $W$  to a shore point  $B$  that is 8 km from  $A$  by piping it on a straight line under water from  $W$  to some shore point  $P$  between  $A$  and  $B$  and then on to  $B$  via pipe along the shoreline. If the cost of laying pipe is \$1,000,000/km under water and \$500,000/km over land, where should the point  $P$  be located to minimize the cost of laying the pipe?

Table 6.2.2

$x$	0	$\frac{10}{3}$	8
$V$	0	$\frac{19600}{27} \approx 726$	0

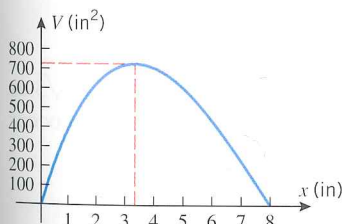


Figure 6.2.4

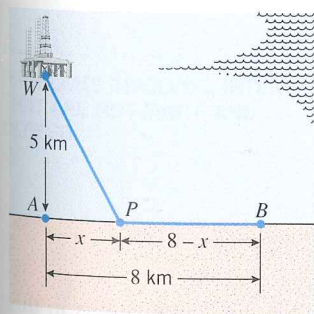


Figure 6.2.5

**Solution.** Let

$x$  = distance (in kilometers) between  $A$  and  $P$

$c$  = cost (in millions of dollars) for the entire pipeline

From Figure 6.2.5 the length of pipe under water is the distance between  $W$  and  $P$ . By the Theorem of Pythagoras, that length is

$$\sqrt{x^2 + 25} \quad (6)$$

Also from Figure 6.2.5, the length of pipe over land is the distance between  $P$  and  $B$ , which is

$$8 - x \quad (7)$$

From (6) and (7) it follows that the total cost  $c$  (in millions of dollars) for the pipeline is

$$c = 1(\sqrt{x^2 + 25}) + \frac{1}{2}(8 - x) = \sqrt{x^2 + 25} + \frac{1}{2}(8 - x) \quad (8)$$

Because the distance between  $A$  and  $B$  is 8 km, the distance  $x$  between  $A$  and  $P$  must satisfy

$$0 \leq x \leq 8$$

We have thus reduced our problem to finding the value (or values) of  $x$  in the interval  $[0, 8]$  for which (8) is a minimum. Since  $c$  is a continuous function of  $x$  on the closed interval  $[0, 8]$ , we can use the methods developed in the preceding section to find the minimum.

From (8) we obtain

$$\frac{dc}{dx} = \frac{x}{\sqrt{x^2 + 25}} - \frac{1}{2}$$

Setting  $dc/dx = 0$  and solving for  $x$  yields

$$\frac{x}{\sqrt{x^2 + 25}} = \frac{1}{2} \quad (9)$$

$$x^2 = \frac{1}{4}(x^2 + 25)$$

$$x = \pm \frac{5}{\sqrt{3}}$$

The number  $-5/\sqrt{3}$  is not a solution of (9) and must be discarded, leaving  $x = 5/\sqrt{3}$  as the only critical point. Since this point lies in the interval  $[0, 8]$ , the minimum must occur at one of the points

$$x = 0, \quad x = 5/\sqrt{3}, \quad x = 8$$

Substituting these values in (8) yields Table 6.2.3, which tells us that the least possible cost of the pipeline (to the nearest dollar) is  $c = \$8,330,127$ , and this occurs when the point  $P$  is located at a distance of  $5/\sqrt{3} \approx 2.89$  km from  $A$ . This is consistent with the graph in Figure 1.1.9c. ◀

**Table 6.2.3**

$x$	0	$\frac{5}{\sqrt{3}}$	8
$c$	9	$\frac{10}{\sqrt{3}} + \left(4 - \frac{5}{2\sqrt{3}}\right) \approx 8.330127$	$\sqrt{89} \approx 9.433981$

**FOR THE READER.** If you have a CAS, use it to check all of the computations in this example. Specifically, differentiate  $c$  with respect to  $x$ , solve the equation  $dc/dx = 0$ , and perform all of the numerical calculations.

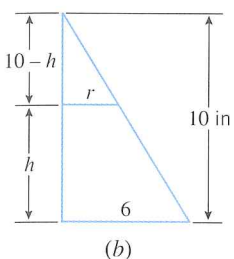
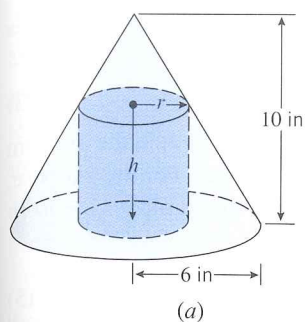


Figure 6.2.6

**Example 4**

Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches (Figure 6.2.6a).

**Solution.** Let

$r$  = radius (in inches) of the cylinder

$h$  = height (in inches) of the cylinder

$V$  = volume (in cubic inches) of the cylinder

The formula for the volume of the inscribed cylinder is

$$V = \pi r^2 h \quad (10)$$

To eliminate one of the variables in (10) we need a relationship between  $r$  and  $h$ . Using similar triangles (Figure 6.2.6b) we obtain

$$\frac{10 - h}{r} = \frac{10}{6} \quad \text{or} \quad h = 10 - \frac{5}{3}r \quad (11)$$

Substituting (11) into (10) we obtain

$$V = \pi r^2 \left(10 - \frac{5}{3}r\right) = 10\pi r^2 - \frac{5}{3}\pi r^3 \quad (12)$$

which expresses  $V$  in terms of  $r$  alone. Because  $r$  represents a radius it cannot be negative, and because the radius of the inscribed cylinder cannot exceed the radius of the cone, the variable  $r$  must satisfy

$$0 \leq r \leq 6$$

Thus, we have reduced the problem to that of finding the value (or values) of  $r$  in  $[0, 6]$  for which (12) is a maximum. Since  $V$  is a continuous function of  $r$  on  $[0, 6]$ , the methods developed in the preceding section apply.

From (12) we obtain

$$\frac{dV}{dr} = 20\pi r - 5\pi r^2 = 5\pi r(4 - r)$$

Setting  $dV/dr = 0$  gives

$$5\pi r(4 - r) = 0$$

so  $r = 0$  and  $r = 4$  are critical points. Since these lie in the interval  $[0, 6]$ , the maximum must occur at one of the points

$$r = 0, \quad r = 4, \quad r = 6$$

Substituting these values in (12) yields Table 6.2.4, which tells us that the maximum volume  $V = \frac{160}{3}\pi \approx 168 \text{ in}^3$  occurs when the inscribed cylinder has radius 4 in. When  $r = 4$  it follows from (11) that  $h = \frac{10}{3}$ . Thus, the inscribed cylinder of largest volume has radius  $r = 4$  in and height  $h = \frac{10}{3}$  in. ◀

Table 6.2.4

$r$	0	4	6
$V$	0	$\frac{160}{3}\pi$	0

**PROBLEMS INVOLVING INTERVALS THAT ARE NOT FINITE AND CLOSED**

**Example 5**

A closed cylindrical can is to hold 1 liter ( $1000 \text{ cm}^3$ ) of liquid. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?

**Solution.** Let

$h$  = height (in cm) of the can

$r$  = radius (in cm) of the can

$S$  = surface area (in  $\text{cm}^2$ ) of the can

Assuming there is no waste or overlap, the amount of material needed for manufacture will be the same as the surface area of the can. Since the can consists of two circular disks of

radius  $r$  and a rectangular sheet with dimensions  $h$  by  $2\pi r$  (Figure 6.2.7), the surface area will be

$$S = 2\pi r^2 + 2\pi r h \quad (13)$$

Since  $S$  depends on two variables,  $r$  and  $h$ , we will look for some condition in the problem that will allow us to express one of these variables in terms of the other. For this purpose, observe that the volume of the can is  $1000 \text{ cm}^3$ , so it follows from the formula  $V = \pi r^2 h$  for the volume of a cylinder that

$$1000 = \pi r^2 h \quad \text{or} \quad h = \frac{1000}{\pi r^2} \quad (14-15)$$

Substituting (15) in (13) yields

$$S = 2\pi r^2 + \frac{2000}{r} \quad (16)$$

Thus, we have reduced the problem to finding a value of  $r$  in the interval  $(0, +\infty)$  for which  $S$  is minimum, provided there actually is a minimum.\* However,  $S$  is a continuous function of  $r$  on the interval  $(0, +\infty)$  and

$$\lim_{r \rightarrow 0^+} \left( 2\pi r^2 + \frac{2000}{r} \right) = +\infty \quad \text{and} \quad \lim_{r \rightarrow +\infty} \left( 2\pi r^2 + \frac{2000}{r} \right) = +\infty$$

so the analysis in Table 6.1.3 implies that  $S$  does have a minimum on the interval  $(0, +\infty)$ . Since this minimum must occur at a critical point, we calculate

$$\frac{dS}{dr} = 4\pi r - \frac{2000}{r^2} \quad (17)$$

Setting  $dS/dr = 0$  gives

$$4\pi r - \frac{2000}{r^2} = 0 \quad \text{or} \quad r = \frac{10}{\sqrt[3]{2\pi}} \quad (18)$$

Since (18) is the only critical point in the interval  $(0, +\infty)$ , this value of  $r$  yields the minimum value of  $S$ . From (15) the value of  $h$  corresponding to this  $r$  is

$$h = \frac{1000}{\pi(10/\sqrt[3]{2\pi})^2} = \frac{20}{\sqrt[3]{2\pi}} = 2r$$

It is not accidental here that the minimum occurs when the height of the can is equal to the diameter of its base (Exercise 27).

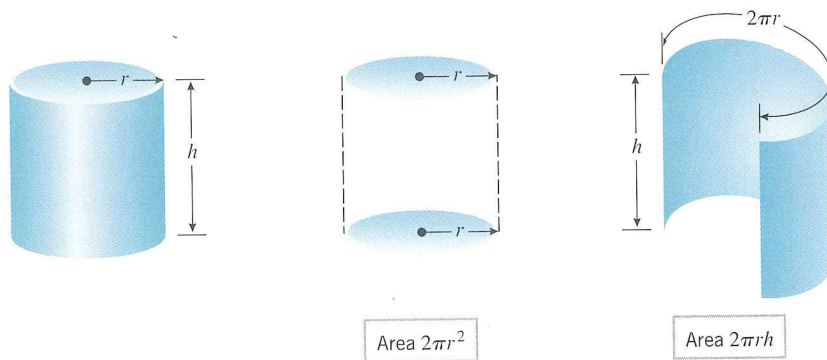


Figure 6.2.7

**Second Solution.** The conclusion that a minimum occurs at the value of  $r$  in (18) can be deduced from Theorem 6.1.5 and the second derivative test by noting that

\*The value  $r = 0$  must be excluded because a cylindrical can of radius 0 cm cannot have a volume of  $1000 \text{ cm}^3$  [see (14)].

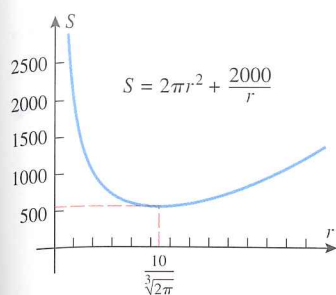


Figure 6.2.8

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive if  $r > 0$  and hence is positive if  $r = 10/\sqrt[3]{2\pi}$ . This implies that a relative minimum, and therefore a minimum, occurs at the critical point  $r = 10/\sqrt[3]{2\pi} \approx 5.4$ .

**Third Solution.** The existence of a minimum is implied by the graph of  $S$  versus  $r$  in Figure 6.2.8. As shown in (18), this minimum occurs at  $r = 10/\sqrt[3]{2\pi}$ . ◀

**REMARK.** Note that  $S$  has no maximum on  $(0, +\infty)$ . Thus, had we asked for the dimensions of the can requiring the maximum amount of material for its manufacture, there would have been no solution to the problem. Optimization problems with no solution are sometimes called *ill posed*.

### Example 6

Find a point on the curve  $y = x^2$  that is closest to the point  $(18, 0)$ .

**Solution.** The distance  $L$  between  $(18, 0)$  and an arbitrary point  $(x, y)$  on the curve  $y = x^2$  (Figure 6.2.9) is given by

$$L = \sqrt{(x - 18)^2 + (y - 0)^2}$$

Since  $(x, y)$  lies on the curve,  $x$  and  $y$  satisfy  $y = x^2$ ; thus,

$$L = \sqrt{(x - 18)^2 + x^4} \quad (19)$$

Because there are no restrictions on  $x$ , the problem reduces to finding a value of  $x$  in  $(-\infty, +\infty)$  for which (19) is minimum, provided such a value exists.

In problems of minimizing or maximizing a distance, there is a trick that is helpful for simplifying the computations. It is based on the observation that the distance and the square of the distance have their maximum or minimum at the same point (see Exercise 60). Thus, the minimum value of  $L$  in (19) and the minimum value of

$$S = L^2 = (x - 18)^2 + x^4 \quad (20)$$

occur at the same  $x$ -value.

From (20),

$$\frac{dS}{dx} = 2(x - 18) + 4x^3 = 4x^3 + 2x - 36 \quad (21)$$

so that the critical points satisfy  $4x^3 + 2x - 36 = 0$  or, equivalently,

$$2x^3 + x - 18 = 0 \quad (22)$$

To solve for  $x$  we will begin by checking the divisors of  $-18$  to see whether the polynomial on the left side has any integer roots (see Appendix F). These divisors are  $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9,$  and  $\pm 18$ . A check of these values shows that  $x = 2$  is a root, so that  $x - 2$  is a factor of the polynomial. After dividing the polynomial by this factor we can rewrite (22) as

$$(x - 2)(2x^2 + 4x + 9) = 0$$

Thus, the remaining solutions of (22) satisfy the quadratic equation

$$2x^2 + 4x + 9 = 0$$

But these solutions are imaginary numbers (use the quadratic formula), so that  $x = 2$  is the only real solution of (22) and consequently the only critical point of  $S$ . To determine the nature of this critical point we will use the second derivative test. From (21),

$$\frac{d^2S}{dx^2} = 12x^2 + 2, \quad \text{so} \quad \left. \frac{d^2S}{dx^2} \right|_{x=2} = 50 > 0$$

which shows that a relative minimum occurs at  $x = 2$ . Since  $x = 2$  is the only relative

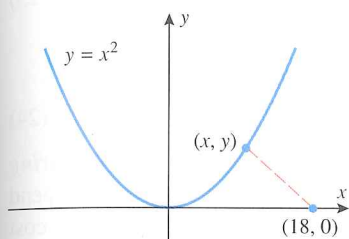


Figure 6.2.9

extremum for  $L$ , it follows from Theorem 6.1.5 that an absolute minimum value of  $L$  also occurs at  $x = 2$ . Thus, the point on the curve  $y = x^2$  closest to  $(18, 0)$  is

$$(x, y) = (x, x^2) = (2, 4)$$

### AN APPLICATION TO ECONOMICS

Three functions of importance to an economist or a manufacturer are

$C(x)$  = total cost of producing  $x$  units of a product during some time period

$R(x)$  = total revenue from selling  $x$  units of the product during the time period

$P(x)$  = total profit obtained by selling  $x$  units of the product during the time period

These are called, respectively, the **cost function**, **revenue function**, and **profit function**. If all units produced are sold, then these are related by

$$P(x) = R(x) - C(x) \quad (23)$$

[profit] = [revenue] - [cost]

The total cost  $C(x)$  of producing  $x$  units can be expressed as a sum

$$C(x) = a + M(x) \quad (24)$$

where  $a$  is a constant, called **overhead**, and  $M(x)$  is a function representing **manufacturing cost**. The overhead, which includes such fixed costs as rent and insurance, does not depend on  $x$ ; it must be paid even if nothing is produced. On the other hand, the manufacturing cost  $M(x)$ , which includes such items as cost of materials and labor, depends on the number of items manufactured. It is shown in economics that with suitable simplifying assumptions,  $M(x)$  can be expressed in the form

$$M(x) = bx + cx^2$$

where  $b$  and  $c$  are constants. Substituting this in (24) yields

$$C(x) = a + bx + cx^2 \quad (25)$$

If a manufacturing firm can sell all the items it produces for  $p$  dollars apiece, then its total revenue  $R(x)$  (in dollars) will be

$$R(x) = px \quad (26)$$

and its total profit  $P(x)$  (in dollars) will be

$$P(x) = [\text{total revenue}] - [\text{total cost}] = R(x) - C(x) = px - C(x)$$

Thus, if the cost function is given by (25),

$$P(x) = px - (a + bx + cx^2) \quad (27)$$

Depending on such factors as number of employees, amount of machinery available, economic conditions, and competition, there will be some upper limit  $l$  on the number of items a manufacturer is capable of producing and selling. Thus, during a fixed time period the variable  $x$  in (27) will satisfy

$$0 \leq x \leq l$$

By determining the value or values of  $x$  in  $[0, l]$  that maximize (27), the firm can determine how many units of its product must be manufactured and sold to yield the greatest profit. This is illustrated in the following numerical example.

#### Example 7

A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the total production cost (in dollars) for  $x$  units is

$$C(x) = 500,000 + 80x + 0.003x^2$$

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize the profit?

**Solution.** Since the total revenue for selling  $x$  units is  $R(x) = 200x$ , the profit  $P(x)$  on  $x$  units will be

$$P(x) = R(x) - C(x) = 200x - (500,000 + 80x + 0.003x^2) \quad (28)$$

Since the production capacity is at most 30,000 units,  $x$  must lie in the interval  $[0, 30,000]$ . From (28)

$$\frac{dP}{dx} = 200 - (80 + 0.006x) = 120 - 0.006x$$

Setting  $dP/dx = 0$  gives

$$120 - 0.006x = 0 \quad \text{or} \quad x = 20,000$$

Since this critical point lies in the interval  $[0, 30,000]$ , the maximum profit must occur at one of the points

$$x = 0, \quad x = 20,000, \quad \text{or} \quad x = 30,000$$

Substituting these values in (28) yields Table 6.2.5, which tells us that the maximum profit  $P = \$700,000$  occurs when  $x = 20,000$  units are manufactured and sold in the specified time. ◀

Table 6.2.5

$x$	0	20,000	30,000
$P(x)$	-500,000	700,000	400,000

### MARGINAL ANALYSIS

Economists call  $P'(x)$ ,  $R'(x)$ , and  $C'(x)$  the *marginal profit*, *marginal revenue*, and *marginal cost*, respectively; and they interpret these quantities as the *additional* profit, revenue, and cost that result from producing and selling one additional unit of the product when the production and sales levels are at  $x$  units. These interpretations follow from the local linear approximations of the profit, revenue, and cost functions. For example, it follows from Formula (7) of Section 3.6 that when the production and sales levels are at  $x$  units the local linear approximation of the profit function is

$$P(x + \Delta x) \approx P(x) + P'(x)\Delta x$$

Thus, if  $\Delta x = 1$  (one additional unit produced and sold), this formula implies

$$P(x + 1) \approx P(x) + P'(x)$$

and hence the *additional* profit that results from producing and selling one additional unit can be approximated as

$$P(x + 1) - P(x) \approx P'(x)$$

It follows from (23) that  $P'(x) = 0$  at those points where  $C'(x) = R'(x)$ , and this implies that the maximum profit must occur at a point where the marginal revenue is equal to the marginal cost; that is:

*The maximum profit occurs at a point where the cost of manufacturing and selling an additional unit of a product is exactly equal to the revenue generated by the additional unit.*

This is one of the basic principles of economics.

### A BASIC PRINCIPLE OF ECONOMICS

## EXERCISE SET 6.2

- Express the number 10 as a sum of two nonnegative numbers whose product is as large as possible.
- How should two nonnegative numbers be chosen so that their sum is 1 and the sum of their squares is
  - as large as possible
  - as small as possible?
- Find a number in the closed interval  $[\frac{1}{2}, \frac{3}{2}]$  such that the sum of the number and its reciprocal is
  - as small as possible
  - as large as possible.
- A rectangular field is to be bounded by a fence on three sides and by a straight stream on the fourth side. Find the dimensions of the field with maximum area that can be enclosed with 1000 feet of fence.
- A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy-duty fencing selling for \$3 a foot, while the remaining two sides will use standard fencing selling for \$2 a foot. What are the dimensions of the rectangular plot of greatest area that can be fenced in at a cost of \$6000?
- A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in. Find the dimensions of the rectangle with greatest area assuming the rectangle is positioned as in the accompanying figure.
- Solve the problem in Exercise 6 assuming the rectangle is positioned as in the accompanying figure.

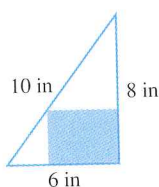


Figure Ex-6

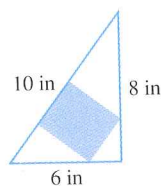


Figure Ex-7

- A rectangle has its two lower corners on the  $x$ -axis and its two upper corners on the curve  $y = 16 - x^2$ . For all such rectangles, what are the dimensions of the one with largest area?
- Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.
- Find the dimensions of the rectangle of greatest area that can be inscribed in a semicircle of radius  $R$  as shown in the accompanying figure.

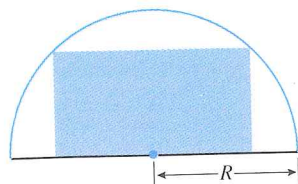


Figure Ex-10

- A rectangular area of 3200 ft<sup>2</sup> is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.
- Show that among all rectangles with perimeter  $p$ , the square has the maximum area.
- Show that among all rectangles with area  $A$ , the square has the minimum perimeter.
- A wire of length 12 in can be bent into a circle, bent into a square, or cut into two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be
  - a maximum
  - a minimum?
- Suppose that the number of bacteria in a culture at time  $t$  is given by  $N = 5000(25 + te^{-t/20})$ .
  - Find the largest and smallest number of bacteria in the culture during the time interval  $0 \leq t \leq 100$ .
  - At what time during the time interval in part (a) is the number of bacteria decreasing most rapidly?
- A church window consisting of a rectangle topped by a semicircle is to have a perimeter  $p$ . Find the radius of the semicircle if the area of the window is to be maximum.
- A sheet of cardboard 12 in square is used to make an open box by cutting squares of equal size from the four corners and folding up the sides. What size squares should be cut to obtain a box with largest possible volume?
- A square sheet of cardboard of side  $k$  is used to make an open box by cutting squares of equal size from the four corners and folding up the sides. What size squares should be cut from the corners to obtain a box with largest possible volume?
- An open box is to be made from a 3-ft by 8-ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the box can have.
- A closed rectangular container with a square base is to have a volume of 2250 in<sup>3</sup>. The material for the top and bottom of the container will cost \$2 per in<sup>2</sup>, and the material for the sides will cost \$3 per in<sup>2</sup>. Find the dimensions of the container of least cost.
- A closed rectangular container with a square base is to have a volume of 2000 cm<sup>3</sup>. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.
- A container with square base, vertical sides, and open top is to be made from 1000 ft<sup>2</sup> of material. Find the dimensions of the container with greatest volume.
- A rectangular container with two square sides and an open top is to have a volume of  $V$  cubic units. Find the dimensions of the container with minimum surface area.

24. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius  $R$ .
25. Find the dimensions of the right circular cylinder of greatest surface area that can be inscribed in a sphere of radius  $R$ .
26. Show that the right circular cylinder of greatest volume that can be inscribed in a right circular cone has volume that is  $\frac{4}{9}$  the volume of the cone (Figure Ex-26).

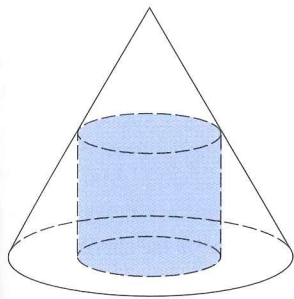


Figure Ex-26

27. A closed, cylindrical can is to have a volume of  $V$  cubic units. Show that the can of minimum surface area is achieved when the height is equal to the diameter of the base.
28. A closed cylindrical can is to have a surface area of  $S$  square units. Show that the can of maximum volume is achieved when the height is equal to the diameter of the base.
29. A cylindrical can, open at the top, is to hold  $500 \text{ cm}^3$  of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.
30. A soup can in the shape of a right circular cylinder of radius  $r$  and height  $h$  is to have a prescribed volume  $V$ . The top and bottom are cut from squares as shown in the accompanying figure. If the shaded corners are wasted, but there is no other waste, find the ratio  $r/h$  for the can requiring the least material (including waste).
31. A box-shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length (Figure Ex-31). If the frame is to be made from a wire of length  $L$ , what should the dimensions be to obtain a box of greatest volume?

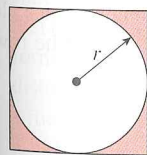


Figure Ex-30

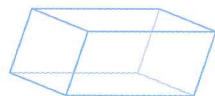


Figure Ex-31

32. Suppose that the sum of the surface areas of a sphere and a cube is a constant.
- (a) Show that the sum of their volumes is smallest when the diameter of the sphere is equal to the length of an edge of the cube.
- (b) When will the sum of their volumes be greatest?

33. Find the height and radius of the cone of slant height  $L$  whose volume is as large as possible.
34. A cone is made from a circular sheet of radius  $R$  by cutting out a sector and gluing the cut edges of the remaining piece together (Figure Ex-34). What is the maximum volume attainable for the cone?

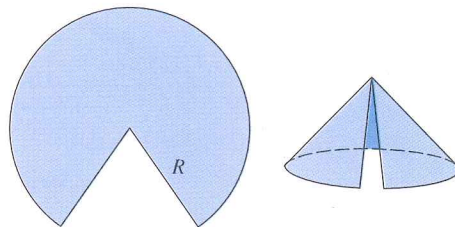


Figure Ex-34

35. A cone-shaped paper drinking cup is to hold  $10 \text{ cm}^3$  of water. Find the height and radius of the cup that will require the least amount of paper.
36. Find the dimensions of the isosceles triangle of least area that can be circumscribed about a circle of radius  $R$ .
37. Find the height and radius of the right circular cone with least volume that can be circumscribed about a sphere of radius  $R$ .
38. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter (Figure Ex-38). Find the maximum possible area for the trapezoid. [Hint: Express the area of the trapezoid in terms of  $\theta$ .]
39. A drainage channel is to be made so that its cross section is a trapezoid with equally sloping sides (Figure Ex-39). If the sides and bottom all have a length of 5 ft, how should the angle  $\theta$  ( $0 \leq \theta \leq \pi/2$ ) be chosen to yield the greatest cross-sectional area?

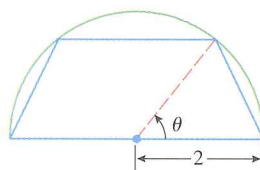


Figure Ex-38

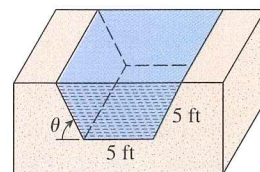


Figure Ex-39

40. A lamp is suspended above the center of a round table of radius  $r$ . How high above the table should the lamp be placed to achieve maximum illumination at the edge of the table? [Assume that the illumination  $I$  is directly proportional to the cosine of the angle of incidence  $\phi$  of the light rays and inversely proportional to the square of the distance  $l$  from the light source (Figure Ex-40).]
41. A plank is used to reach over a fence 8 ft high to support a wall that is 1 ft behind the fence (Figure Ex-41). What is the length of the shortest plank that can be used? [Hint: Express

the length of the plank in terms of the angle  $\theta$  shown in the figure.]

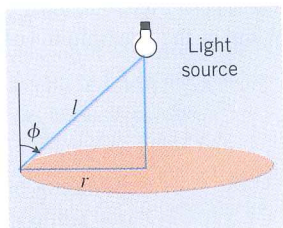


Figure Ex-40

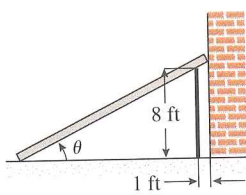


Figure Ex-41

42. A commercial cattle ranch currently allows 20 steers per acre of grazing land; on the average its steers weigh 2000 lb at market. Estimates by the Agriculture Department indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should be allowed in order for the ranch to get the largest possible total market weight for its cattle?
43. (a) A chemical manufacturer sells sulfuric acid in bulk at a price of \$100 per unit. If the daily total production cost in dollars for  $x$  units is
- $$C(x) = 100,000 + 50x + 0.0025x^2$$
- and if the daily production capacity is at most 7000 units, how many units of sulfuric acid must be manufactured and sold daily to maximize the profit?
- (b) Would it benefit the manufacturer to expand the daily production capacity?
44. A firm determines that  $x$  units of its product can be sold daily at  $p$  dollars per unit, where
- $$x = 1000 - p$$
- The cost of producing  $x$  units per day is
- $$C(x) = 3000 + 20x$$
- (a) Find the revenue function  $R(x)$ .
- (b) Find the profit function  $P(x)$ .
- (c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.
- (d) Find the maximum profit.
- (e) What price per unit must be charged to obtain the maximum profit?
45. In a certain chemical manufacturing process, the daily weight  $y$  of defective chemical output depends on the total weight  $x$  of all output according to the empirical formula
- $$y = 0.01x + 0.00003x^2$$
- where  $x$  and  $y$  are in pounds. If the profit is \$100 per pound

of nondefective chemical produced and the loss is \$20 per pound of defective chemical produced, how many pounds of chemical should be produced daily to maximize the total daily profit?

46. The cost  $c$  (in dollars per hour) to run an ocean liner at a constant speed  $v$  (in miles per hour) is given by  $c = a + bv^n$ , where  $a$ ,  $b$ , and  $n$  are positive constants with  $n > 1$ . Find the speed needed to make the cheapest 3000-mi run.
47. Two particles,  $A$  and  $B$ , are in motion in the  $xy$ -plane. Their coordinates at each instant of time  $t$  ( $t \geq 0$ ) are given by  $x_A = t$ ,  $y_A = 2t$ ,  $x_B = 1 - t$ , and  $y_B = t$ . Find the minimum distance between  $A$  and  $B$ .
48. Follow the directions of Exercise 47, with  $x_A = t$ ,  $y_A = t^2$ ,  $x_B = 2t$ , and  $y_B = 2$ .
49. Prove that  $(1, 0)$  is the closest point on the curve  $x^2 + y^2 = 1$  to  $(2, 0)$ .
50. Find all points on the curve  $y = \sqrt{x}$  for  $0 \leq x \leq 3$  that are closest to, and at the greatest distance from, the point  $(2, 0)$ .
51. Find all points on the curve  $x^2 - y^2 = 1$  closest to  $(0, 2)$ .
52. Find a point on the curve  $x = 2y^2$  closest to  $(0, 9)$ .
53. Find the coordinates of the point  $P$  on the curve
- $$y = \frac{1}{x^2} \quad (x > 0)$$
- where the segment of the tangent line at  $P$  that is cut off by the coordinate axes has its shortest length.
54. Find the  $x$ -coordinate of the point  $P$  on the parabola
- $$y = 1 - x^2 \quad (0 < x \leq 1)$$
- where the triangle that is enclosed by the tangent line at  $P$  and the coordinate axes has the smallest area.
55. Where on the curve  $y = (1 + x^2)^{-1}$  does the tangent line have the greatest slope?
56. A man is on the bank of a river that is 1 mile wide. He wants to travel to a town on the opposite bank, but 1 mile upstream. He intends to row on a straight line to some point  $P$  on the opposite bank and then walk the remaining distance along the bank (Figure Ex-56). To what point should he row in order to reach his destination in the least time if
- (a) he can walk 5 mi/h and row 3 mi/h
- (b) he can walk 5 mi/h and row 4 mi/h?
57. A pipe of negligible diameter is to be carried horizontally around a corner from a hallway 8 ft wide into a hallway 4 ft wide (Figure Ex-57). What is the maximum length that the pipe can have? [An interesting discussion of this problem in the case where the diameter of the pipe is not neglected is given by Norman Miller in the *American Mathematical Monthly*, Vol. 56, 1949, pp. 177–179.]

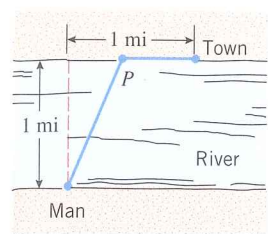


Figure Ex-56

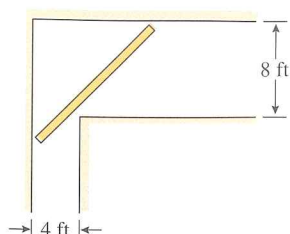


Figure Ex-57

58. In an unknown physical quantity  $x$  is measured  $n$  times, the measurements  $x_1, x_2, \dots, x_n$  often vary because of uncontrollable factors such as temperature, atmospheric pressure, and so forth. Thus, a scientist is often faced with the problem of using  $n$  different observed measurements to obtain an estimate  $\bar{x}$  of an unknown quantity  $x$ . One method for making such an estimate is based on the *least squares principle*, which states that the estimate  $\bar{x}$  should be chosen to minimize

$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$$

which is the sum of the squares of the deviations between the estimate  $\bar{x}$  and the measured values. Show that the estimate resulting from the least squares principle is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

that is,  $\bar{x}$  is the arithmetic average of the observed values.

59. Suppose that the intensity of a point light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two point light sources with strengths of  $S$  and  $8S$  are separated by a distance of 90 cm. Where on the line segment between the two sources is the intensity a minimum?
60. Prove: If  $f(x) \geq 0$  on an interval  $I$  and if  $f(x)$  has a maximum value on  $I$  at  $x_0$ , then  $\sqrt{f(x)}$  also has a maximum value at  $x_0$ . Similarly for minimum values. [Hint: Use the fact that  $\sqrt{x}$  is an increasing function on the interval  $[0, +\infty)$ .]
61. Fermat's\* (biography on pp. 352–353) principle in optics states that light traveling from one point to another follows that path for which the total travel time is minimum. In a uniform medium, the paths of “minimum time” and “shortest distance” turn out to be the same, so that light, if unobstructed, travels along a straight line. Assume that we have a light source, a flat mirror, and an observer in a uniform medium. If a light ray leaves the source, bounces off the mirror, and travels on to the observer, then its path will consist of two line segments, as shown in Figure Ex-61. According to Fermat's principle, the path will be such that the total travel time  $t$  is minimum or, since the medium is uniform, the path will be such that the total distance traveled from  $A$  to  $P$  to  $B$  is as small as possible. Assuming the minimum

occurs when  $dt/dx = 0$ , show that the light ray will strike the mirror at the point  $P$  where the “angle of incidence”  $\theta_1$  equals the “angle of reflection”  $\theta_2$ .

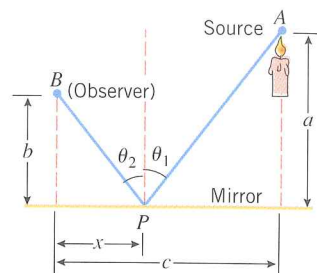


Figure Ex-61

62. Fermat's principle (Exercise 61) also explains why light rays traveling between air and water undergo bending (refraction). Imagine that we have two uniform media (such as air and water) and a light ray traveling from a source  $A$  in one medium to an observer  $B$  in the other medium (Figure Ex-62). It is known that light travels at a constant speed in a uniform medium, but more slowly in a dense medium (such as water) than in a thin medium (such as air). Consequently, the path of shortest time from  $A$  to  $B$  is not necessarily a straight line, but rather some broken line path  $A$  to  $P$  to  $B$  allowing the light to take greatest advantage of its higher speed through the thin medium. Snell's† (biography on p. 353) law of refraction states that the path of the light ray will be such that

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where  $v_1$  is the speed of light in the first medium,  $v_2$  is the speed of light in the second medium, and  $\theta_1$  and  $\theta_2$  are the angles shown in Figure Ex-62. Show that this follows from the assumption that the path of minimum time occurs when  $dt/dx = 0$ .

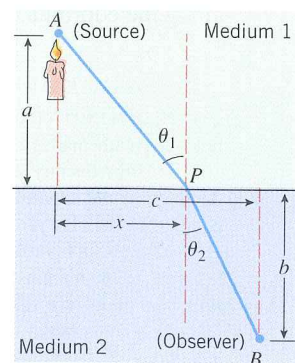


Figure Ex-62

63. A farmer wants to walk at a constant rate from her barn to a straight river, fill her pail, and carry it to her house in the least time.

- (a) Explain how this problem relates to Fermat's principle and the light-reflection problem in Exercise 61.
- (b) Use the result of Exercise 61 to describe geometrically the best path for the farmer to take.
- (c) Use part (b) to determine where the farmer should fill her pail if her house and barn are located as in Figure Ex-63.

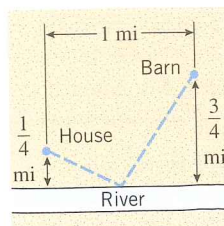


Figure Ex-63

## 6.3 RECTILINEAR MOTION (MOTION ALONG A LINE)

In Section 1.5 we discussed the motion of a particle moving with constant velocity in one direction along a line, and in Section 3.1 we discussed the motion of a particle moving with variable velocity in one direction along a line. In this section we will investigate the more general situation in which a particle may move back and forth with variable velocity along a line. Some examples are a piston moving up and down in a cylinder, a buoy bobbing up and down in the waves, or an object attached to a vibrating spring.

### TERMINOLOGY

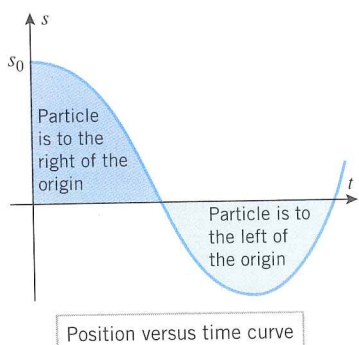


Figure 6.3.1

In this section we will assume that a particle representing some object is allowed to move in either direction along a coordinate line. This is called *rectilinear motion*. The coordinate line might be an  $x$ -axis, a  $y$ -axis, or an axis that is inclined at some angle. To avoid being specific, we will denote the coordinate line as the  $s$ -axis. We will assume that units are chosen for measuring distance and time and that we begin observing the particle at time  $t = 0$ . As the particle moves along the  $s$ -axis, its coordinate is some function of the elapsed time  $t$ , say  $s = s(t)$ . We call  $s(t)$  the *position function* of the particle, and we call the graph of  $s$  versus  $t$  the *position versus time curve*.

Figure 6.3.1 shows a typical position versus time curve for a particle in rectilinear motion. We can tell from that graph that the coordinate of the particle at time  $t = 0$  is  $s_0$ , and we can tell from the sign of  $s$  when the particle is to the left or right of the origin as it moves along the coordinate line.

\* **PIERRE DE FERMAT** (1601–1665). Fermat, the son of a successful French leather merchant, was a lawyer who practiced mathematics as a hobby. He received a Bachelor of Civil Laws degree from the University of Orleans in 1631 and subsequently held various government positions, including a post as councillor to the Toulouse parliament. Although he was apparently financially successful, confidential documents of that time suggest that his performance in office and as a lawyer was poor, perhaps because he devoted so much time to mathematics. Throughout his life, Fermat fought all his efforts to have his mathematical results published. He had the unfortunate habit of scribbling his work in the margins of books and often sent his results to friends without keeping copies for himself. As a result, he never received credit for many major achievements until his name was raised from obscurity in the mid-nineteenth century. It is now known that Fermat, simultaneously and independently of Descartes, developed analytic geometry. Unfortunately, Descartes and Fermat argued bitterly over various problems so that there was never any real cooperation between these two great geniuses.

Fermat solved many fundamental calculus problems. He obtained the first procedure to differentiating polynomials, and solved many important maximization, minimization, area, and tangent problems. His work served to inspire Isaac Newton. Fermat is best known for his work in number theory, the study of properties and relationships between whole numbers. He was the first mathematician to make substantial contributions to this field after the ancient Greek mathematician Diophantus. Unfortunately, none of Fermat's contemporaries appreciated his work in this area, a fact that eventually pushed Fermat into isolation and obscurity in later life. In addition to his work

### Example 1

Figure 6.3.2 shows the position versus time curve for a jackrabbit moving along an  $s$ -axis. In words, describe how the position of the rabbit changes with time.

**Solution.** The rabbit is at the point  $s = -3$  at time  $t = 0$ . It moves in the positive direction until time  $t = 4$ , since  $s$  is increasing. At time  $t = 4$  the rabbit is at the point  $s = 3$ . At that time it turns around and travels in the negative direction until time  $t = 7$ , since  $s$  is decreasing. At time  $t = 7$  the rabbit is at the point  $s = -1$ , and it remains stationary at that point thereafter, since  $s$  is constant for  $t > 7$ . ◀

#### INSTANTANEOUS VELOCITY

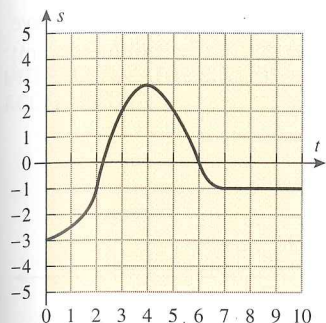


Figure 6.3.2

In rectilinear motion, the rate at which the coordinate of a particle changes with time is called the *velocity* of the particle. More precisely, we make the following definition.

**6.3.1 DEFINITION.** If  $s(t)$  is the position function of a particle moving on a coordinate line, then the *instantaneous velocity* of the particle at time  $t$  is defined by

$$v(t) = s'(t) = \frac{ds}{dt} \quad (1)$$

Geometrically, the instantaneous velocity at a given time is the slope of the tangent line to the position versus time curve at that time, and hence the sign of the velocity tells which way the particle is moving—a positive velocity means that  $s$  is increasing with time, so the particle is moving in the positive direction; a negative velocity means that  $s$  is decreasing with time, so the particle is moving in the negative direction (Figure 6.3.3). For example, in Figure 6.3.2 the rabbit is moving in the positive direction between times  $t = 0$  and  $t = 4$  and is moving in the negative direction between times  $t = 4$  and  $t = 7$ .

#### SPEED VERSUS VELOCITY

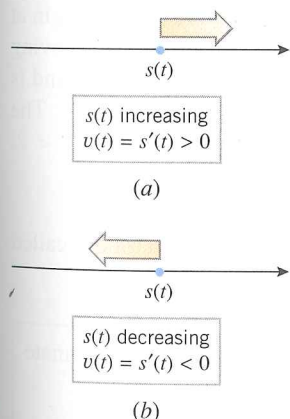


Figure 6.3.3

Recall from our discussion of uniform rectilinear motion in Section 1.5 that there is a distinction between the terms *speed* and *velocity*—speed describes how fast an object is moving without regard to direction, whereas velocity describes how fast it is moving and in what direction. Mathematically, we define the *instantaneous speed* of a particle to be the absolute value of its instantaneous velocity; that is,

$$\left[ \begin{array}{l} \text{instantaneous} \\ \text{speed} \end{array} \right] = |v(t)| = \left| \frac{ds}{dt} \right| \quad (2)$$

in calculus and number theory, Fermat was one of the founders of probability theory and made major contributions to the theory of optics. Outside mathematics, Fermat was a classical scholar of some note, was fluent in French, Italian, Spanish, Latin, and Greek, and he composed a considerable amount of Latin poetry.

One of the great mysteries of mathematics is shrouded in Fermat's work in number theory. In the margin of a book by Diophantus, Fermat scribbled that for integer values of  $n$  greater than 2, the equation  $x^n + y^n = z^n$  has no nonzero integer solutions for  $x$ ,  $y$ , and  $z$ . He stated, "I have discovered a truly marvelous proof of this, which however the margin is not large enough to contain." This result, which became known as "Fermat's last theorem," appeared to be true, but its proof evaded the greatest mathematical geniuses for 300 years until Professor Andrew Wiles of Princeton University presented a proof in June 1993 in a dramatic series of three lectures that drew international media attention (see *New York Times*, June 27, 1993). A prize of 100,000 German marks was offered in 1908 for the solution, but it is worthless today because of inflation.

† WILLEBRORD VAN ROIJEN SNELL (1591–1626). Dutch mathematician. Snell, who succeeded his father to the post of Professor of Mathematics at the University of Leiden in 1613, is most famous for the result of light refraction that bears his name. Although this phenomenon was studied as far back as the ancient Greek astronomer Ptolemy, until Snell's work the relationship was incorrectly thought to be  $\theta_1/v_1 = \theta_2/v_2$ . Snell's law was published by Descartes in 1638 without giving proper credit to Snell. Snell also discovered a method for determining distances by triangulation that founded the modern technique of mapmaking.

For example, if two particles on the same coordinate line are moving with velocities  $v = 5$  m/s and  $v = -5$  m/s, respectively, then the particles are moving in opposite directions, but they both have a speed of  $|v| = 5$  m/s.

### Example 2

Let  $s(t) = t^3 - 6t^2$  be the position function of a particle moving along an  $s$ -axis, where  $s$  is in meters and  $t$  is in seconds. Find the instantaneous velocity and speed, and show the graphs of position, velocity, and speed versus time.

**Solution.** From (1) and (2), the instantaneous velocity and speed are given by

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t \quad \text{and} \quad |v(t)| = |3t^2 - 12t|$$

The graphs of position, velocity, and speed versus time are shown in Figure 6.3.4. Observe that velocity and speed both have units of meters per second (m/s), since  $s$  is in meters (m) and time is in seconds (s). ◀

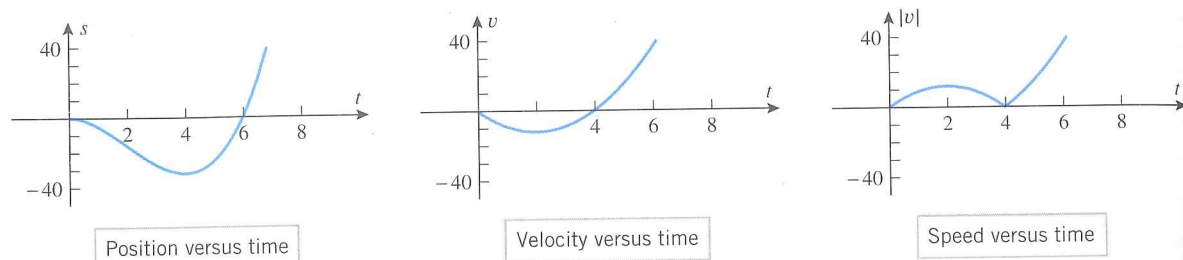


Figure 6.3.4

The graphs in Figure 6.3.4 provide a wealth of visual information about the motion of the particle. For example, the position versus time curve tells us that the particle is to the left of the origin for  $0 < t < 6$ , is to the right of the origin for  $t > 6$ , and is at the origin at times  $t = 0$  and  $t = 6$ . The velocity versus time curve tells us that the particle is moving in the negative direction if  $0 < t < 4$ , is moving in the positive direction if  $t > 4$ , and is momentarily stopped at times  $t = 0$  and  $t = 4$  (the velocity is zero at those times). The speed versus time curve tells us that the speed of the particle is increasing for  $0 < t < 2$ , decreasing for  $2 < t < 4$ , and increasing again for  $t > 4$ .

### ACCELERATION

In rectilinear motion, the rate at which the velocity of a particle changes with time is called its *acceleration*. More precisely, we make the following definition.

**6.3.2 DEFINITION.** If  $s(t)$  is the position function of a particle moving on a coordinate line, then the instantaneous acceleration of the particle at time  $t$  is defined by

$$a(t) = v'(t) = \frac{dv}{dt} \quad (3)$$

or alternatively, since  $v(t) = s'(t)$ ,

$$a(t) = s''(t) = \frac{d^2s}{dt^2} \quad (4)$$

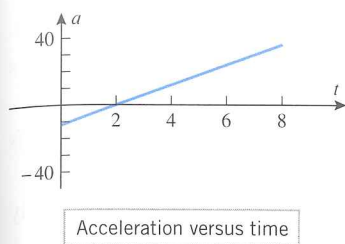


Figure 6.3.5

### INTERPRETING THE SIGN OF ACCELERATION

### Example 3

Let  $s(t) = t^3 - 6t^2$  be the position function of a particle moving along an  $s$ -axis, where  $s$  is in meters and  $t$  is in seconds. Find the instantaneous acceleration  $a(t)$ , and show the graph of acceleration versus time.

**Solution.** From Example 2, the instantaneous velocity of the particle is  $v(t) = 3t^2 - 12t$ , so the instantaneous acceleration is

$$a(t) = \frac{dv}{dt} = 6t - 12$$

and the acceleration versus time curve is the line shown in Figure 6.3.5. Note that in this example the acceleration has units of  $\text{m/s}^2$ , since  $v$  is in meters per second ( $\text{m/s}$ ) and time is in seconds (s). ◀

We will say that a particle in rectilinear motion is **speeding up** when its instantaneous speed is increasing and is **slowing down** when its instantaneous speed is decreasing. In everyday language an object that is speeding up is said to be “accelerating” and an object that is slowing down is said to be “decelerating”; thus, one might expect that a particle in rectilinear motion will be speeding up when its instantaneous acceleration is positive and slowing down when it is negative. Although this is true for a particle moving in the positive direction, it is *not* true for a particle moving in the negative direction—a particle with negative velocity is speeding up when its acceleration is negative and slowing down when its acceleration is positive. This is because a positive acceleration implies an increasing velocity, and increasing a negative velocity decreases its absolute value; similarly, a negative acceleration implies a decreasing velocity, and decreasing a negative velocity increases its absolute value. In summary:

**6.3.3 INTERPRETING THE SIGN OF ACCELERATION.** A particle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.

**FOR THE READER.** For a particle in rectilinear motion, what is happening when  $v(t) = 0$ ? When  $a(t) = 0$ ?

### Example 4

In Examples 2 and 3 we found the velocity versus time curve and the acceleration versus time curve for a particle with position function  $s(t) = t^3 - 6t^2$ . Use those curves to determine when the particle is speeding up and slowing down, and confirm that your results are consistent with the speed versus time curve obtained in Example 2.

**Solution.** Over the time interval  $0 < t < 2$  the velocity and acceleration are negative, so the particle is speeding up. This is consistent with the speed versus time curve, since the speed is increasing over this time interval. Over the time interval  $2 < t < 4$  the velocity is negative and the acceleration is positive, so the particle is slowing down. This is also consistent with the speed versus time curve, since the speed is decreasing over this time interval. Finally, on the time interval  $t > 4$  the velocity and acceleration are positive, so the particle is speeding up, which again is consistent with the speed versus time curve. ◀

The position versus time curve contains all of the significant information about the position and velocity of a particle in rectilinear motion:

- Where  $s(t) > 0$ , the particle is on the positive side of the  $s$ -axis.
- Where  $s(t) < 0$ , the particle is on the negative side of the  $s$ -axis.

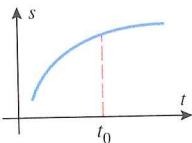
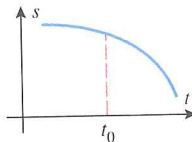
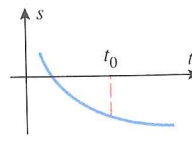
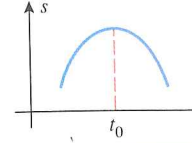
### ANALYZING THE POSITION VERSUS TIME CURVE

- The slope of the tangent line at a point in time is the instantaneous velocity at that time.
- Where the tangent line has positive slope, the velocity is positive and the particle is moving in the positive direction.
- Where the tangent line has negative slope, the velocity is negative, and the particle is moving in the negative direction.
- Where the tangent line is horizontal, the velocity is zero, and the particle is momentarily stopped.

Information about the acceleration of a particle in rectilinear motion can also be deduced from the position versus time curve by examining its concavity. To see why this is so, observe that the position versus time curve will be concave up on intervals where  $s''(t) > 0$ , and it will be concave down on intervals where  $s''(t) < 0$ . But we know from (4) that  $s''(t)$  is the instantaneous acceleration, so that on intervals where the position versus time curve is concave up the particle has a positive acceleration, and on intervals where it is concave down the particle has a negative acceleration.

Table 6.3.1 summarizes our observations about the position versus time curve.

Table 6.3.1

POSITION VERSUS TIME CURVE	CHARACTERISTICS OF THE CURVE AT $t = t_0$	BEHAVIOR OF THE PARTICLE AT TIME $t = t_0$
	<ul style="list-style-type: none"> <li>• <math>s(t_0) &gt; 0</math></li> <li>• Tangent line has positive slope.</li> <li>• Curve is concave down.</li> </ul>	<ul style="list-style-type: none"> <li>• Particle is on the positive side of the origin.</li> <li>• Particle is moving in the positive direction.</li> <li>• Velocity is decreasing.</li> <li>• Particle is slowing down.</li> </ul>
	<ul style="list-style-type: none"> <li>• <math>s(t_0) &gt; 0</math></li> <li>• Tangent line has negative slope.</li> <li>• Curve is concave down.</li> </ul>	<ul style="list-style-type: none"> <li>• Particle is on the positive side of the origin.</li> <li>• Particle is moving in the negative direction.</li> <li>• Velocity is decreasing.</li> <li>• Particle is speeding up.</li> </ul>
	<ul style="list-style-type: none"> <li>• <math>s(t_0) &lt; 0</math></li> <li>• Tangent line has negative slope.</li> <li>• Curve is concave up.</li> </ul>	<ul style="list-style-type: none"> <li>• Particle is on the negative side of the origin.</li> <li>• Particle is moving in the negative direction.</li> <li>• Velocity is increasing.</li> <li>• Particle is slowing down.</li> </ul>
	<ul style="list-style-type: none"> <li>• <math>s(t_0) &gt; 0</math></li> <li>• Tangent line has zero slope.</li> <li>• Curve is concave down.</li> </ul>	<ul style="list-style-type: none"> <li>• Particle is on the positive side of the origin.</li> <li>• Particle is momentarily stopped.</li> <li>• Velocity is decreasing.</li> </ul>

### Example 5

Use the position versus time curve in Figure 6.3.2 to determine when the jackrabbit in Example 1 is speeding up and slowing down.

**Solution.** From  $t = 0$  to  $t = 2$ , the acceleration and velocity are positive, so the rabbit is speeding up. From  $t = 2$  to  $t = 4$ , the acceleration is negative and the velocity is positive.

so the rabbit is slowing down. At  $t = 4$ , the velocity is zero, so the rabbit has momentarily stopped. From  $t = 4$  to  $t = 6$ , the acceleration is negative and the velocity is negative, so the rabbit is speeding up. From  $t = 6$  to  $t = 7$ , the acceleration is positive and the velocity is negative, so the rabbit is slowing down. Thereafter, the velocity is zero, so the rabbit has stopped. ◀

### Example 6

Suppose that the position function of a particle moving on a coordinate line is given by  $s(t) = 2t^3 - 21t^2 + 60t + 3$ . Analyze the motion of the particle for  $t \geq 0$ .

**Solution.** The velocity and acceleration at time  $t$  are

$$v(t) = s'(t) = 6t^2 - 42t + 60 = 6(t - 2)(t - 5)$$

$$a(t) = v'(t) = 12t - 42 = 12\left(t - \frac{7}{2}\right)$$

At each instant we can determine the direction of motion from the sign of  $v(t)$  and whether the particle is speeding up or slowing down from the signs of  $v(t)$  and  $a(t)$  together (Figures 6.3.6a and 6.3.6b). The motion of the particle is described schematically by the curved line in Figure 6.3.6c. At time  $t = 0$  the particle is at the point  $s(0) = 3$  moving right with velocity  $v(0) = 60$  ft/s, but slowing down with acceleration  $a(0) = -42$  ft/s<sup>2</sup>. The particle continues moving right until time  $t = 2$ , when it stops at the point  $s(2) = 55$ , reverses

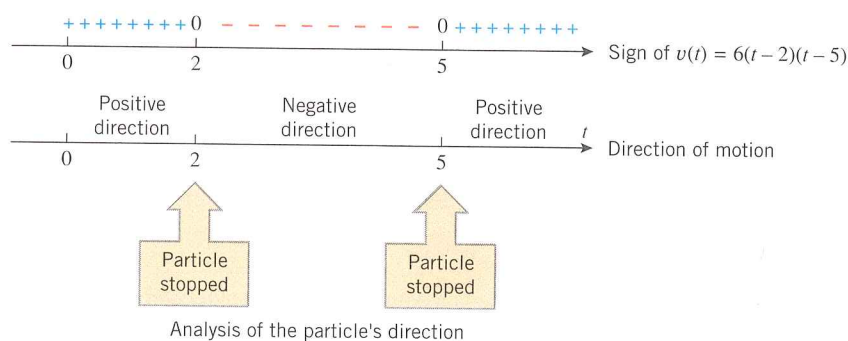


Figure 6.3.6a

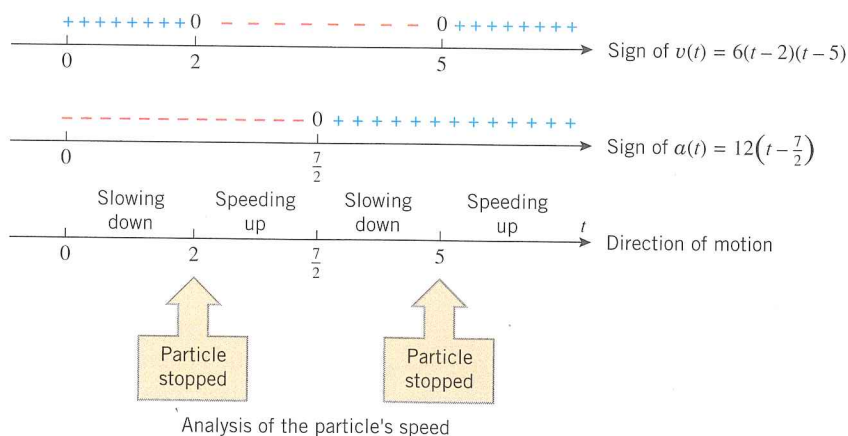


Figure 6.3.6b

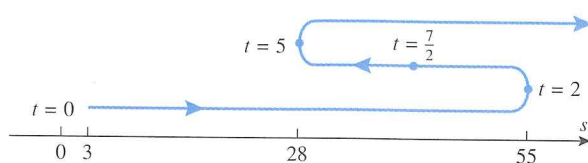


Figure 6.3.6c

direction, and begins to speed up with an acceleration of  $a(2) = -18 \text{ ft/s}^2$ . At time  $t = \frac{7}{2}$  the particle begins to slow down, but continues moving left until time  $t = 5$ , when it stops at the point  $s(5) = 28$ , reverses direction again, and begins to speed up with acceleration  $a(5) = 18 \text{ ft/s}^2$ . The particle then continues moving right thereafter with increasing speed. ◀

**REMARK.** The curved line in Figure 6.3.6c is descriptive only. The actual path of the particle is back and forth on the coordinate line.

### FREE-FALL MOTION

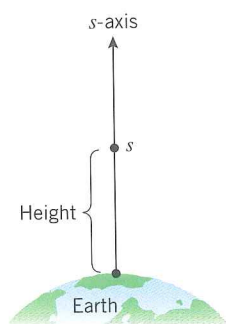


Figure 6.3.7

We will now discuss how some of the ideas in this section can be applied to the study of *free-fall motion*, which is the motion that occurs when an object near the Earth is imparted some initial vertical velocity (up or down), and thereafter moves on a vertical line. In modeling free-fall motion it is assumed that the only force acting on the object is the Earth's gravity and that the object stays sufficiently close to the Earth's surface so that the gravitational force is constant. In particular, air resistance and the gravitational pull of other celestial bodies are neglected.

In our study of free-fall motion, we will ignore the physical size of the object by treating it as a particle, and we will assume that the object moves along an  $s$ -axis whose origin is at the surface of the Earth and whose positive direction is up. With this convention, the  $s$ -coordinate of the particle is the height of the particle above the Earth's surface (Figure 6.3.7). The following result will be derived later using calculus and some basic principles of physics.

**6.3.4 THE FREE-FALL MODEL.** Suppose that at time  $t = 0$  an object at a height of  $s_0$  above the Earth's surface is imparted an upward or downward velocity of  $v_0$  and thereafter moves vertically subject only to the force of the Earth's gravity. If the positive direction of the  $s$ -axis is up, and if the origin is at the surface of the Earth, then at any time  $t$  the height  $s = s(t)$  of the object is given by the formula

$$s = s_0 + v_0 t - \frac{1}{2} g t^2 \quad (5)$$

where  $g$  is a constant, called the *acceleration due to gravity*. In this text we will use the following approximations for  $g$ , depending on the units of measurement:

$$g = 9.8 \text{ m/s}^2 \quad [\text{distance in meters and time in seconds}]$$

$$g = 32 \text{ ft/s}^2 \quad [\text{distance in feet and time in seconds}]$$

It follows from (5) that the instantaneous velocity and acceleration of an object in free-fall motion are

$$v = \frac{ds}{dt} = v_0 - gt \quad (6)$$

$$a = \frac{dv}{dt} = -g \quad (7)$$

**REMARK.** Because we have chosen the positive direction of the  $s$ -axis to be up, a positive velocity implies an upward motion and a negative velocity a downward motion. Thus, it makes sense that instantaneous acceleration  $-g$  is negative, since an upward-moving object has positive velocity and negative acceleration, which implies that it is slowing down; and a downward-moving object has negative velocity and negative acceleration, which implies that it is speeding up. (It is a little confusing that the positive constant  $g$  is called the *acceleration due to gravity* in 6.3.4, given that the instantaneous acceleration is actually the negative constant  $-g$ . This mismatch in terminology is caused by the upward orientation of the  $s$ -axis in Figure 6.3.7; had we chosen the positive direction to be down, then the instantaneous acceleration would have turned out to be  $g$ . However, our orientation has the advantage of allowing us to interpret  $s$  as the height of the object.)



Nolan Ryan's rookie baseball card

**Example 7**

Nolan Ryan, one of the fastest baseball pitchers of all time, was capable of throwing a baseball 150 ft/s (over 102 mi/h). Could Nolan Ryan have hit the 208-ft ceiling of the Houston Astrodome if he were capable of giving a baseball an upward velocity of 100 ft/s from a height of 7 ft?

**Solution.** Taking  $g = 32 \text{ ft/s}^2$ ,  $v_0 = 100 \text{ ft/s}$ , and  $s_0 = 7 \text{ ft}$  in (5) and (6) yields the equations

$$s = 7 + 100t - 16t^2 \quad \text{and} \quad v = 100 - 32t \quad (8-9)$$

whose graphs are shown in Figure 6.3.8. It is evident from the graph of  $s$  versus  $t$  that the maximum height of the baseball is less than 208 ft, so Ryan could not have hit the ceiling. However, let us go a step further and determine exactly how high the ball will go. The maximum height  $s$  occurs at the stationary point obtained by solving the equation  $ds/dt = 0$ . However,  $ds/dt = v$ , which means that the maximum height occurs when  $v = 0$ , which from (9) can be expressed as

$$100 - 32t = 0 \quad (10)$$

Solving this equation yields  $t = 25/8$ . To find the height  $s$  at this time we substitute this value of  $t$  in (8), from which we obtain

$$s = 7 + 100(25/8) - 16(25/8)^2 = 163.25 \text{ ft}$$

which is roughly 45 ft short of hitting the ceiling. ◀

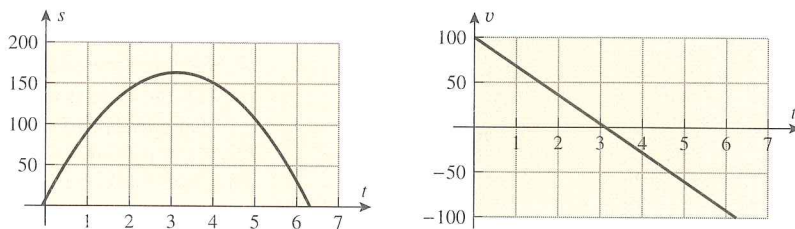


Figure 6.3.8

**REMARK.** Equation (10) can also be deduced by physical reasoning: The ball is moving up when the velocity is positive and moving down when the velocity is negative, so it makes sense that the velocity is zero when the ball reaches its peak.

**EXERCISE SET 6.3**  Graphing Calculator

1. The graphs of three position functions are shown in the accompanying figure. In each case determine the sign of the velocity and acceleration, then determine whether the particle is speeding up or slowing down.

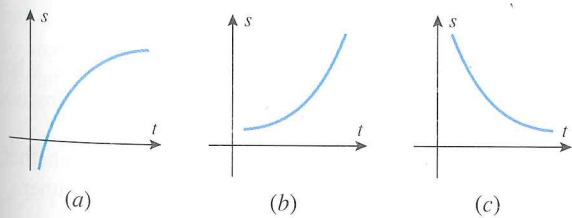


Figure Ex-1

2. The graphs of three velocity functions are shown in the accompanying figure. In each case determine the sign of the acceleration, then determine whether the particle is speeding up or slowing down.

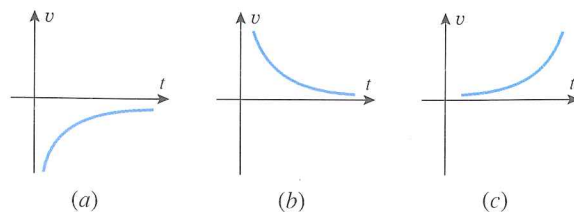


Figure Ex-2

3. The position function of a particle moving on a horizontal  $x$ -axis is shown in the accompanying figure.

- Is the particle moving left or right at time  $t_0$ ?
- Is the acceleration positive or negative at time  $t_0$ ?
- Is the particle speeding up or slowing down at time  $t_0$ ?
- Is the particle speeding up or slowing down at time  $t_1$ ?

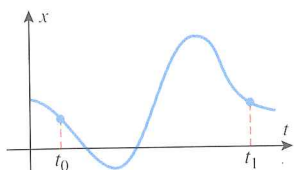


Figure Ex-3

4. For the graphs in the accompanying figure, match the position functions with their corresponding velocity functions.

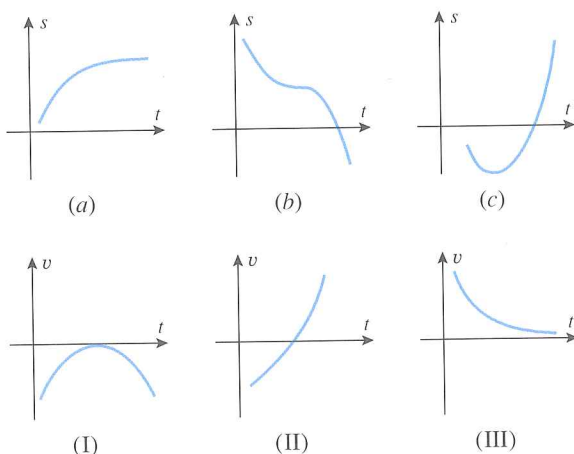


Figure Ex-4

- Sketch a reasonable graph of  $s$  versus  $t$  for a mouse that is trapped in a narrow corridor (an  $s$ -axis with the positive direction to the right) and scurries back and forth as follows. It runs right with a constant speed of 1.2 m/s for awhile, then gradually slows down to 0.6 m/s, then quickly speeds up to 2.0 m/s, then gradually slows to a stop but immediately reverses direction and quickly speeds up to 1.2 m/s.
- The accompanying figure shows the graph of  $s$  versus  $t$  for an ant that moves along a narrow vertical pipe (an  $s$ -axis with the positive direction up).
  - When, if ever, is the ant above the origin?
  - When, if ever, does the ant have velocity zero?
  - When, if ever, is the ant moving down the pipe?
- The accompanying figure shows the graph of velocity versus time for a particle moving along a coordinate line. Make

a rough sketch of the graphs of speed versus time and acceleration versus time.

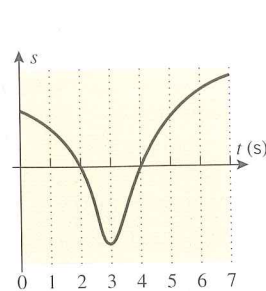


Figure Ex-6

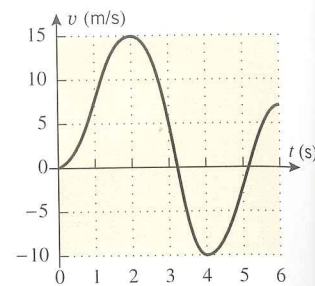


Figure Ex-7

8. The accompanying figure shows the position versus time graph for an elevator that ascends 40 m from one stop to the next.

- Estimate the velocity when the elevator is halfway up.
- Sketch rough graphs of the velocity versus time curve and the acceleration versus time curve.

9. The accompanying figure shows the velocity versus time graph for a test run on the Grand Prix GTP. Using this graph, estimate

- the acceleration at 60 mi/h (in units of  $\text{ft/s}^2$ )
- the time at which the maximum acceleration occurs.

[Data from *Car and Driver Magazine*, October 1990.]

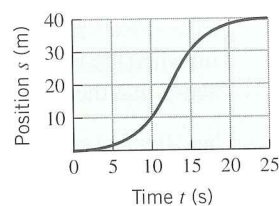


Figure Ex-8

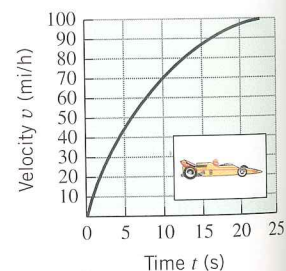


Figure Ex-9

10. Let  $s(t) = \sin(\pi t/4)$  be the position function of a particle moving along a coordinate line, where  $s$  is in meters and  $t$  is in seconds.
- Make a table showing the position, velocity, and acceleration to two decimal places at times  $t = 1, 2, 3, 4,$  and  $5$ .
  - At each of the times in part (a), determine whether the particle is stopped; if it is not, state its direction of motion.
  - At each of the times in part (a), determine whether the particle is speeding up, slowing down, or neither.

In Exercises 11–14, the position function of a particle moving along a coordinate line is given, where  $s$  is in feet and  $t$  is in seconds.

- Find the velocity and acceleration functions.
- Find the position, velocity, speed, and acceleration at time  $t = 1$ .
- At what times is the particle stopped?
- When is the particle speeding up? Slowing down?
- Find the total distance traveled by the particle from time  $t = 0$  to time  $t = 5$ .

11.  $s(t) = t^3 - 6t^2, \quad t \geq 0$

12.  $s(t) = t^4 - 4t + 2, \quad t \geq 0$

13.  $s(t) = 3 \cos(\pi t/2), \quad 0 \leq t \leq 5$

14.  $s(t) = \frac{t}{t^2 + 4}, \quad t \geq 0$

15. Let  $s(t) = t/(t^2 + 5)$  be the position function of a particle moving along a coordinate line, where  $s$  is in meters and  $t$  is in seconds. Use a graphing utility to generate the graphs of  $s(t)$ ,  $v(t)$ , and  $a(t)$  for  $t \geq 0$ , and use those graphs where needed.

- Use the appropriate graph to make a rough estimate of the time at which the particle first reverses the direction of its motion; and then find the time exactly.
- Find the exact position of the particle when it first reverses the direction of its motion.
- Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

16. Let  $s(t) = t/e^t$  be the position function of a particle moving along a coordinate line, where  $s$  is in meters and  $t$  is in seconds. Use a graphing utility to generate the graphs of  $s(t)$ ,  $v(t)$ , and  $a(t)$  for  $t \geq 0$ , and use those graphs where needed.

- Use the appropriate graph to make a rough estimate of the time at which the particle first reverses the direction of its motion; and then find the time exactly.
- Find the exact position of the particle when it first reverses the direction of its motion.
- Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

In Exercises 17–22, the position function of a particle moving along a coordinate line is given. Use the method of Example 6 to analyze the motion of the particle for  $t \geq 0$ , and give a schematic picture of the motion (as in Figure 6.3.6).

17.  $s = -3t + 2$

18.  $s = t^3 - 6t^2 + 9t + 1$

19.  $s = t^3 - 9t^2 + 24t$

20.  $s = t + \frac{9}{t+1}$

21.  $s = \begin{cases} \cos t, & 0 \leq t \leq 2\pi \\ 1, & t > 2\pi \end{cases}$

22.  $s = \sqrt{t}(4 - 4t + 2t^2)$

23. Let  $s(t) = 5t^2 - 22t$  be the position function of a particle moving along a coordinate line, where  $s$  is in feet and  $t$  is in seconds.

- Find the maximum speed of the particle during the time interval  $1 \leq t \leq 3$ .
- When, during the time interval  $1 \leq t \leq 3$ , is the particle farthest from the origin? What is its position at that instant?

24. Let  $s = 100/(t^2 + 12)$  be the position function of a particle moving along a coordinate line, where  $s$  is in feet and  $t$  is in seconds. Find the maximum speed of the particle for  $t \geq 0$ , and find the direction of motion of the particle when it has its maximum speed.

In Exercises 25–29, assume that the free-fall model applies and that the positive direction is up, so that Formulas (5), (6), and (7) can be used. In those problems stating that an object is “dropped” or “released from rest,” you should interpret that to mean that the initial velocity of the object is zero. Take  $g = 32 \text{ ft/s}^2$  or  $g = 9.8 \text{ m/s}^2$ , depending on the units.

25. A wrench is accidentally dropped at the top of an elevator shaft in a tall building.

- How many meters does the wrench fall in 1.5 s?
- What is the velocity of the wrench at that time?
- How long does it take for the wrench to reach a speed of 12 m/s?
- How long does it take for the wrench to fall 100 m?

26. In 1939, Joe Sprinz of the San Francisco Seals Baseball Club attempted to catch a ball dropped from a blimp at a height of 800 ft (for the purpose of breaking the record for catching a ball dropped from the greatest height set the preceding year by members of the Cleveland Indians).

- How long does it take for a ball to drop 800 ft?
- What is the velocity of a ball in miles per hour after an 800-ft drop ( $88 \text{ ft/s} = 60 \text{ mi/h}$ )?

[Note: As a practical matter, it is unrealistic to ignore wind resistance in this problem; however, even with the slowing effect of wind resistance, the impact of the ball slammed Sprinz’s glove hand into his face, fractured his upper jaw in 12 places, broke five teeth, and knocked him unconscious. He dropped the ball!]

27. A projectile is launched upward from ground level with an initial speed of 60 m/s.

- How long does it take for the projectile to reach its highest point?
- How high does the projectile go?
- How long does it take for the projectile to drop back to the ground from its highest point?
- What is the speed of the projectile when it hits the ground?

28. (a) Use the results in Exercise 27 to make a conjecture about the relationship between the initial and final speeds of a projectile that is launched upward from ground level and returns to ground level.  
 (b) Prove your conjecture.
29. In Example 7, how fast would Nolan Ryan have to throw a ball upward from a height of 7 feet in order to hit the ceiling of the Astrodome?
30. The free-fall formulas (5) and (6) can be combined and rearranged in various useful ways. Derive the following variations of those formulas.  
 (a)  $v^2 = v_0^2 - 2g(s - s_0)$     (b)  $s = s_0 + \frac{1}{2}(v_0 + v)t$   
 (c)  $s = s_0 + vt + \frac{1}{2}gt^2$
31. A rock, dropped from an unknown height, strikes the ground with a speed of 24 m/s. Use the formula in part (a) of Exercise 30 to find the unknown height.
32. A rock thrown downward with an unknown initial velocity from a height of 1000 ft reaches the ground in 5 s. Use the formula in part (c) of Exercise 30 to find the velocity of the rock when it hits the ground.
33. (a) A ball is thrown upward from a height  $s_0$  with an initial velocity of  $v_0$ . Use the formula in part (a) of Exercise 30 to show that the maximum height of the ball is  $s_{\max} = s_0 + v_0^2/2g$ .  
 (b) Use this result to solve Exercise 29.
34. Let  $s = t^3 - 6t^2 + 1$ .  
 (a) Find  $s$  and  $v$  when  $a = 0$ .  
 (b) Find  $s$  and  $a$  when  $v = 0$ .
35. Let  $s = \sqrt{2t^2 + 1}$  be the position function of a particle moving along a coordinate line.  
 (a) Use a graphing utility to generate the graph of  $v$  versus  $t$ , and make a conjecture about the velocity of the particle as  $t \rightarrow +\infty$ .  
 (b) Check your conjecture by finding  $\lim_{t \rightarrow +\infty} v$ .
36. (a) Use the chain rule to show that for a particle in rectilinear motion  $a = v(dv/ds)$ .  
 (b) Let  $s = \sqrt{3t + 7}$ ,  $t \geq 0$ . Find a formula for  $v$  in terms of  $s$  and use the equation in part (a) to find the acceleration when  $s = 5$ .
37. Suppose that the position function of two particles,  $P_1$  and  $P_2$ , in motion along the same line are
- $$s_1 = \frac{1}{2}t^2 - t + 3 \quad \text{and} \quad s_2 = -\frac{1}{4}t^2 + t + 1$$

respectively, for  $t \geq 0$ .

- (a) Prove that  $P_1$  and  $P_2$  do not collide.  
 (b) How close can  $P_1$  and  $P_2$  get to one another?  
 (c) During what intervals of time are they moving in opposite directions?
38. Let  $s_A = 15t^2 + 10t + 20$  and  $s_B = 5t^2 + 40t$ ,  $t \geq 0$ , be the position functions of cars  $A$  and  $B$  that are moving along parallel straight lanes of a highway.  
 (a) How far is car  $A$  ahead of car  $B$  when  $t = 0$ ?  
 (b) At what instants of time are the cars next to one another?  
 (c) At what instant of time do they have the same velocity? Which car is ahead at this instant?
39. The accompanying figure shows the velocity versus distance graph for a 222 Remington Magnum 55 grain pointed soft point bullet.  
 (a) Use the graph to estimate the value of  $dv/ds$  when the velocity is 2000 ft/s.  
 (b) Use the result in part (a) and the chain rule to approximate the acceleration when the velocity is 2000 ft/s. [Hint: See Exercise 36.]  
 [Data from the *Shooter's Bible*, No. 82, Stoeger Publishing Co., 1991.]

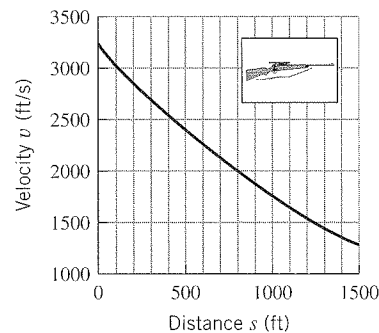


Figure Ex-39

40. Prove that a particle is speeding up if the velocity and acceleration have the same sign, and slowing down if they have opposite signs. [Hint: Let  $r(t) = |v(t)| = \sqrt{v^2(t)}$ , and find  $r'(t)$ .]

## 6.4 NEWTON'S METHOD

In Section 2.4 we showed how to approximate the roots of an equation  $f(x) = 0$  by using the Intermediate-Value Theorem and also by zooming in on the  $x$ -intercepts of  $y = f(x)$  with a graphing utility. In this section we will study a technique, called Newton's Method, that is usually more efficient than either of those methods. Newton's Method is the technique used by many commercial and scientific computer programs for finding roots.

### NEWTON'S METHOD

In beginning algebra one learns that the solution of a first-degree equation  $ax + b = 0$  is given by the formula  $x = -b/a$ , and the solutions of a second-degree equation

$$ax^2 + bx + c = 0$$

are given by the quadratic formula. Formulas also exist for the solutions of all third- and fourth-degree equations, although they are too complicated to be of practical use. In 1826 it was shown by the Norwegian mathematician Niels Henrik Abel\* that it is impossible to construct a similar formula for the solutions of a *general* fifth-degree equation or higher. Thus, for a *specific* fifth-degree polynomial equation such as

$$x^5 - 9x^4 + 2x^3 - 5x^2 + 17x - 8 = 0$$

it may be difficult or impossible to find exact values for all of the solutions. Similar difficulties occur for trigonometric equations such as

$$x - \cos x = 0$$

as well as equations of other types. For such equations the solutions are generally approximated in some way, often by the method we will now discuss.

Suppose that we are trying to find a root  $r$  of the equation  $f(x) = 0$ , and suppose that by some method we are able to obtain a rough initial estimate of  $r$ , say by generating the graph of  $y = f(x)$  with a graphing utility and examining the  $x$ -intercepts. If we let  $x_1$  denote our

\* **NIELS HENRIK ABEL** (1802–1829). Norwegian mathematician. Abel was the son of a poor Lutheran minister and a remarkably beautiful mother from whom he inherited strikingly good looks. In his brief life of 26 years Abel lived in virtual poverty and suffered a succession of adversities; yet he managed to prove major results that altered the mathematical landscape forever. At the age of thirteen he was sent away from home to a school whose better days had long passed. By a stroke of luck the school had just hired a teacher named Bernt Michael Holmboe, who quickly discovered that Abel had extraordinary mathematical ability. Together, they studied the calculus texts of Euler and works of Newton and the later French mathematicians. By the time he graduated, Abel was familiar with most of the great mathematical literature. In 1820 his father died, leaving the family in dire financial straits. Abel was able to enter the University of Christiania in Oslo only because he was granted a free room and several professors supported him directly from their salaries. The University had no advanced courses in mathematics, so Abel took a preliminary degree in 1822 and then continued to study mathematics on his own. In 1824 he published at his own expense the proof that it is impossible to solve the general fifth-degree polynomial equation algebraically. With the hope that this landmark paper would lead to his recognition and acceptance by the European mathematical community, Abel sent the paper to the great German mathematician Gauss, who casually declared it to be a “monstrosity” and tossed it aside. However, in 1826 Abel's paper on the fifth-degree equation and other work was published in the first issue of a new journal, founded by his friend, Leopold Crelle. In the summer of 1826 he completed a landmark work on transcendental functions, which he submitted to the French Academy of Sciences in the hope of establishing himself as a major mathematician, for many young mathematicians had gained quick distinction by having their work accepted by the Academy. However, Abel waited in vain because the paper was either ignored or misplaced by one of the referees, and it did not surface again until two years after his death. That paper was later described by one major mathematician as “. . . the most important mathematical discovery that has been made in our century. . . .” After submitting his paper, Abel returned to Norway, ill with tuberculosis and in heavy debt. While eking out a meager living as a tutor, he continued to produce great work and his fame spread. Soon great efforts were being made to secure a suitable mathematical position for him. Fearing that his great work had been lost by the Academy, he mailed a proof of the main results to Crelle in January of 1829. In April he suffered a violent hemorrhage and died. Two days later Crelle wrote to inform him that an appointment had been secured for him in Berlin and his days of poverty were over! Abel's great paper was finally published by the Academy twelve years after his death.

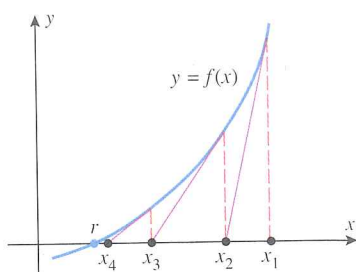


Figure 6.4.1

initial approximation to  $r$ , then we can generally improve on this approximation by moving along the tangent line to  $y = f(x)$  at  $x_1$  until we meet the  $x$ -axis at a point  $x_2$  (Figure 6.4.1). Usually,  $x_2$  will be closer to  $r$  than  $x_1$ . To improve the approximation further, we can repeat the process by moving along the tangent line to  $y = f(x)$  at  $x_2$  until we meet the  $x$ -axis at a point  $x_3$ . Continuing in this way we can generate a succession of values  $x_1, x_2, x_3, x_4, \dots$  that will usually get closer and closer to  $r$ . This procedure for approximating  $r$  is called **Newton's Method**.

To implement Newton's Method analytically, we must derive a formula that will tell us how to calculate each improved approximation from the preceding approximation. For this purpose, we note that the point-slope form of the tangent line to  $y = f(x)$  at the initial approximation  $x_1$  is

$$y - f(x_1) = f'(x_1)(x - x_1) \quad (1)$$

If  $f'(x_1) \neq 0$ , then this line is not parallel to the  $x$ -axis and consequently it crosses the  $x$ -axis at some point  $(x_2, 0)$ . Substituting the coordinates of this point in (1) yields

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

Solving for  $x_2$  we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad (2)$$

The next approximation can be obtained more easily. If we view  $x_2$  as the starting approximation and  $x_3$  the new approximation, we can simply apply (2) with  $x_2$  in place of  $x_1$  and  $x_3$  in place of  $x_2$ . This yields

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \quad (3)$$

provided  $f'(x_2) \neq 0$ . In general, if  $x_n$  is the  $n$ th approximation, then it is evident from the pattern in (2) and (3) that the improved approximation  $x_{n+1}$  is given by

#### Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \dots \quad (4)$$

#### Example 1

Use Newton's Method to approximate the real solutions of

$$x^3 - x - 1 = 0$$

**Solution.** Let  $f(x) = x^3 - x - 1$ , so  $f'(x) = 3x^2 - 1$  and (4) becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} \quad (5)$$

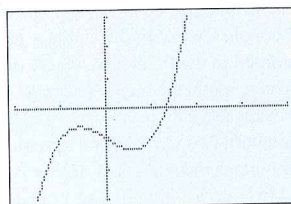
From the graph of  $f$  in Figure 6.4.2, we see that the given equation has only one real solution. This solution lies between 1 and 2 because  $f(1) = -1 < 0$  and  $f(2) = 5 > 0$ . We will use  $x_1 = 1.5$  as our first approximation ( $x_1 = 1$  or  $x_1 = 2$  would also be reasonable choices).

Letting  $n = 1$  in (5) and substituting  $x_1 = 1.5$  yields

$$x_2 = 1.5 - \frac{(1.5)^3 - 1.5 - 1}{3(1.5)^2 - 1} = 1.34782609$$

(We used a calculator that displays nine digits.) Next, we let  $n = 2$  in (5) and substitute  $x_2 = 1.34782609$  to obtain

$$x_3 = 1.34782609 - \frac{(1.34782609)^3 - (1.34782609) - 1}{3(1.34782609)^2 - 1} = 1.32520040$$



$[-2, 4] \times [-3, 3]$   
 $x\text{Scl} = 1, y\text{Scl} = 1$

$$y = x^3 - x - 1$$

Figure 6.4.2

If we continue this process until two identical approximations are generated in succession, we obtain

$$\begin{aligned}x_1 &= 1.5 \\x_2 &= 1.34782609 \\x_3 &= 1.32520040 \\x_4 &= 1.32471817 \\x_5 &= 1.32471796 \\x_6 &= 1.32471796\end{aligned}$$

At this stage there is no need to continue further because we have reached the accuracy limit of our calculator, and all subsequent approximations that the calculator generates will be the same. Thus, the solution is approximately  $x \approx 1.32471796$ . ◀

### Example 2

It is evident from Figure 6.4.3 that if  $x$  is in radians, then the equation

$$\cos x = x$$

has a solution between 0 and 1. Use Newton's Method to approximate it.

**Solution.** Rewrite the equation as

$$x - \cos x = 0$$

and apply (4) with  $f(x) = x - \cos x$ . Since  $f'(x) = 1 + \sin x$ , (4) becomes

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n} \quad (6)$$

From Figure 6.4.3, the solution seems closer to  $x = 1$  than  $x = 0$ , so we will use  $x_1 = 1$  (radian) as our initial approximation. Letting  $n = 1$  in (6) and substituting  $x_1 = 1$  yields

$$x_2 = 1 - \frac{1 - \cos 1}{1 + \sin 1} = .750363868$$

Next, letting  $n = 2$  in (6) and substituting this value of  $x_2$  yields

$$x_3 = .750363868 - \frac{.750363868 - \cos(.750363868)}{1 + \sin(.750363868)} = .739112891$$

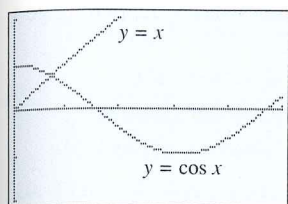
If we continue this process until two identical approximations are generated in succession, we obtain

$$\begin{aligned}x_1 &= 1 \\x_2 &= .750363868 \\x_3 &= .739112891 \\x_4 &= .739085133 \\x_5 &= .739085133\end{aligned}$$

Thus, to the accuracy limit of our calculator, the solution of the equation  $\cos x = x$  is  $x \approx .739085133$ . ◀

When Newton's Method works, the approximations usually converge toward the solution with dramatic speed. However, there are situations in which the method fails. For example, if  $f'(x_n) = 0$  for some  $n$ , then (4) involves a division by zero, making it impossible to generate  $x_{n+1}$ . However, this is to be expected because the tangent line to  $y = f(x)$  is parallel to the  $x$ -axis where  $f'(x_n) = 0$ , and hence this tangent line does not cross the  $x$ -axis to generate the next approximation (Figure 6.4.4).

Newton's Method can fail for other reasons as well; sometimes it may overlook the root you are trying to find and converge to a different root, and sometimes it may fail to converge



$$\begin{aligned}[0, 5] \times [-2, 2] \\x\text{Scl} = 1, y\text{Scl} = 1\end{aligned}$$

Figure 6.4.3

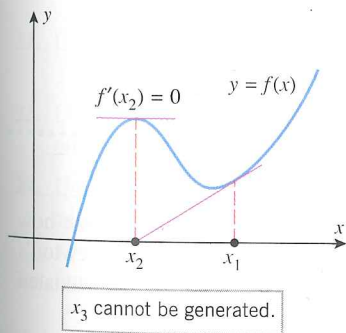


Figure 6.4.4

### SOME DIFFICULTIES WITH NEWTON'S METHOD

altogether. For example, consider the equation

$$x^{1/3} = 0$$

which has  $x = 0$  as its only solution, and try to approximate this solution by Newton's Method with a starting value of  $x_0 = 1$ . Letting  $f(x) = x^{1/3}$ , Formula (4) becomes

$$x_{n+1} = x_n - \frac{(x_n)^{1/3}}{\frac{1}{3}(x_n)^{-2/3}} = x_n - 3x_n = -2x_n$$

Beginning with  $x_1 = 1$ , the successive values generated by this formula are

$$x_1 = 1, \quad x_2 = -2, \quad x_3 = 4, \quad x_4 = -8, \dots$$

which obviously do not converge to  $x = 0$ . Figure 6.4.5 illustrates what is happening geometrically in this situation.

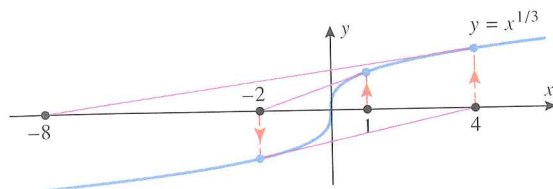


Figure 6.4.5

To learn more about the conditions under which Newton's Method converges and for a discussion of error questions, you should consult a book on numerical analysis. For a more in-depth discussion of Newton's Method and its relationship to contemporary studies of chaos and fractals, you may want to read the article, "Newton's Method and Fractal Patterns," by Phillip Straffin, which appears in *Applications of Calculus*, MAA Notes, Vol. 3, No. 29, 1993, published by the Mathematical Association of America.

#### EXERCISE SET 6.4 Graphing Calculator

In this exercise set, use a calculator, and keep as many decimal places as it can display.

1. Approximate  $\sqrt{2}$  by applying Newton's Method to the equation  $x^2 - 2 = 0$ .
2. Approximate  $\sqrt{7}$  by applying Newton's Method to the equation  $x^2 - 7 = 0$ .
3. Approximate  $\sqrt[3]{6}$  by applying Newton's Method to the equation  $x^3 - 6 = 0$ .
4. To what equation would you apply Newton's Method to approximate the  $n$ th root of  $a$ ?

In Exercises 5–8, the equation has one real solution. Approximate it by Newton's Method.


5.  $x^3 - x + 3 = 0$


6.  $x^3 + x - 1 = 0$

7.  $x^5 + x^4 - 5 = 0$


8.  $x^5 - x + 1 = 0$


In Exercises 9–14, use a graphing utility to determine how many solutions the equation has, and then use Newton's Method to approximate the solution that satisfies the stated condition.

 9.  $x^4 + x - 3 = 0$ ;  $x < 0$

 10.  $x^5 - 5x^3 - 2 = 0$ ;  $x > 0$

 11.  $2 \sin x = x$ ;  $x > 0$        12.  $\sin x = x^2$ ;  $x > 0$

 13.  $x - \tan x = 0$ ;  $\pi/2 < x < 3\pi/2$

 14.  $1 - e^x \cos x = 0$ ;  $0 < x < \pi$

In Exercises 15–18, use a graphing utility to determine the number of times the curves intersect; and then apply Newton's Method, where needed, to approximate the  $x$ -coordinates of all intersections.

15.  $y = x^3$  and  $y = \frac{1}{2}x - 1$
16.  $y = e^{-x}$  and  $y = \ln x$
17.  $y = x^2$  and  $y = \sqrt{2x + 1}$
18.  $y = \frac{1}{8}x^3 + 1$  and  $y = \cos 2x$
19. The *mechanic's rule* for approximating square roots states that  $\sqrt{a} \approx x_{n+1}$ , where

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right), \quad n = 1, 2, 3, \dots$$

and  $x_1$  is any positive approximation to  $\sqrt{a}$ .

- (a) Apply Newton's Method to

$$f(x) = x^2 - a$$

to derive the mechanic's rule.

- (b) Use the mechanic's rule to approximate  $\sqrt{10}$ .

20. Many calculators compute reciprocals using the approximation  $1/a \approx x_{n+1}$ , where

$$x_{n+1} = x_n(2 - ax_n), \quad n = 1, 2, 3, \dots$$

and  $x_1$  is an initial approximation to  $1/a$ . This formula makes it possible to perform divisions using multiplications and subtractions, which is a faster procedure than dividing directly.

- (a) Apply Newton's Method to

$$f(x) = (1/x) - a$$

to derive this approximation.

- (b) Use the formula to approximate  $\frac{1}{17}$ .

21. Use Newton's Method to find the absolute minimum of

$$f(x) = \frac{1}{4}x^4 + x^2 + 5x$$

22. Use Newton's Method to find the absolute maximum of  $f(x) = x \sin x$  on the interval  $[0, \pi]$ .

23. Use Newton's Method to find the coordinates of the point on the parabola  $y = x^2$  that is closest to the point  $(1, 0)$ .

24. Use Newton's Method to find the dimensions of the rectangle of largest area that can be inscribed under the curve  $y = \cos x$  for  $0 \leq x \leq \pi/2$ , as shown in the accompanying figure.

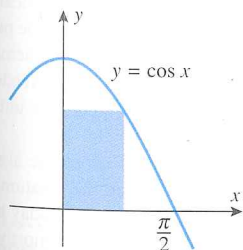


Figure Ex-24

25. (a) Show that on a circle of radius  $r$ , the central angle  $\theta$  that subtends an arc whose length is 1.5 times the length  $L$

of its chord satisfies the equation  $\theta = 3 \sin(\theta/2)$  (see the accompanying figure).

- (b) Use Newton's Method to approximate  $\theta$ .

26. A *segment* of a circle is the region enclosed by an arc and its chord (see the accompanying figure). If  $r$  is the radius of the circle and  $\theta$  the angle subtended at the center of the circle, then it can be shown that the area  $A$  of the segment is  $A = \frac{1}{2}r^2(\theta - \sin \theta)$ , where  $\theta$  is in radians. Find the value of  $\theta$  for which the area of the segment is one-fourth the area of the circle. Give  $\theta$  to the nearest degree.

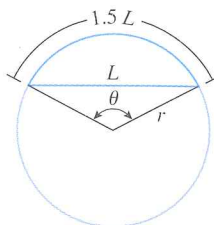


Figure Ex-25

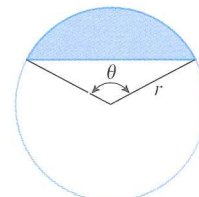


Figure Ex-26

In Exercises 27 and 28, use Newton's Method to approximate all real values of  $y$  satisfying the given equation for the indicated value of  $x$ .

27.  $xy^4 + x^3y = 1$ ;  $x = 1$

28.  $xy - \cos(\frac{1}{2}xy) = 0$ ;  $x = 2$

29. An *annuity* is a sequence of equal payments that are paid or received at regular time intervals. For example, you may want to deposit equal amounts at the end of each year into an interest-bearing account for the purpose of accumulating a lump sum at some future time. If, at the end of each year, interest of  $i \times 100\%$  on the account balance for that year is added to the account, then the account is said to pay  $i \times 100\%$  interest, *compounded annually*. It can be shown that if payments of  $Q$  dollars are deposited at the end of each year into an account that pays  $i \times 100\%$  compounded annually, then at the time when the  $n$ th payment and the accrued interest for the past year are deposited, the amount  $S(n)$  in the account is given by the formula

$$S(n) = \frac{Q}{i} [(1+i)^n - 1]$$

Suppose that you can invest \$5000 in an interest-bearing account at the end of each year, and your objective is to have \$250,000 on the 25th payment. What annual compound interest rate must the account pay for you to achieve your goal? [Hint: Show that the interest rate  $i$  satisfies the equation  $50i = (1+i)^{25} - 1$ , and solve it using Newton's Method.]

30. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{x}{x^2 + 1}$$

and use it to explain what happens if you apply Newton's Method with a starting value of  $x_1 = 2$ . Check your conclusion by computing  $x_2, x_3, x_4,$  and  $x_5$ .

- (b) Use the graph generated in part (a) to explain what happens if you apply Newton's Method with a starting value of  $x_1 = 0.5$ . Check your conclusion by computing  $x_2$ ,  $x_3$ ,  $x_4$ , and  $x_5$ .

31. (a) Apply Newton's Method to the function  $f(x) = x^2 + 1$  with a starting value of  $x_1 = 0.5$ , and determine if the values of  $x_2, \dots, x_{10}$  appear to converge.  
(b) Explain what is happening.

### ROLLE'S THEOREM

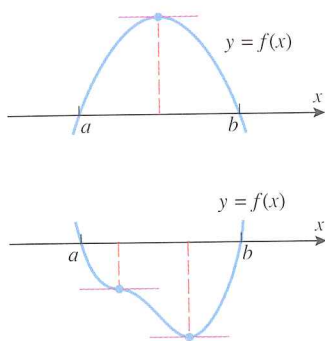


Figure 6.5.1

## 6.5 ROLLE'S THEOREM; MEAN-VALUE THEOREM

In this section we will discuss a result called the Mean-Value Theorem. This theorem has so many important consequences that it is regarded as one of the major principles in calculus.

We will begin with a special case of the Mean-Value Theorem, called Rolle's Theorem, in honor of the mathematician Michel Rolle.\* This theorem states the geometrically obvious fact that if the graph of a differentiable function crosses the  $x$ -axis at two points,  $a$  and  $b$ , then somewhere between those points there must be at least one place where the tangent line is horizontal (Figure 6.5.1). The precise statement of the theorem is as follows:

**6.5.1 THEOREM (Rolle's Theorem).** Let  $f$  be differentiable on  $(a, b)$  and continuous on  $[a, b]$ . If  $f(a) = f(b) = 0$ , then there is at least one point  $c$  in  $(a, b)$  where  $f'(c) = 0$ .

**Proof.** Either  $f(x)$  is equal to zero for all  $x$  in  $[a, b]$  or it is not. If it is, then  $f'(x) = 0$  for all  $x$  in  $(a, b)$ , since  $f$  is constant on  $(a, b)$ . Thus, for any  $c$  in  $(a, b)$

$$f'(c) = 0$$

If  $f(x)$  is not equal to zero for all  $x$  in  $[a, b]$ , then there must be a point  $x$  in  $(a, b)$  where  $f(x) > 0$  or  $f(x) < 0$ . We will consider the first case and leave the second as an exercise.

Since  $f$  is continuous on  $[a, b]$ , it follows from the Extreme-Value Theorem (6.1.3) that  $f$  has a maximum value at some point  $c$  in  $[a, b]$ . Since  $f(a) = f(b) = 0$  and  $f(x) > 0$  at some point in  $(a, b)$ , the point  $c$  cannot be an endpoint; it must lie in  $(a, b)$ . By hypothesis,  $f$  is differentiable everywhere on  $(a, b)$ . In particular, it is differentiable at  $c$  so that  $f'(c) = 0$  by Theorem 6.1.4. ■

\* MICHEL ROLLE (1652-1719), French mathematician. Rolle, the son of a shopkeeper, received only an elementary education. He married early and as a young man struggled hard to support his family on the meager wages of a transcriber for notaries and attorneys. In spite of his financial problems and minimal education, Rolle studied algebra and Diophantine analysis (a branch of number theory) on his own. Rolle's fortune changed dramatically in 1682 when he published an elegant solution of a difficult, unsolved problem in Diophantine analysis. The public recognition of his achievement led to a patronage under minister Louvois, a job as an elementary mathematics teacher, and eventually to a short-term administrative post in the Ministry of War. In 1685 he joined the Académie des Sciences in a low-level position for which he received no regular salary until 1699. He stayed there until he died of apoplexy in 1719.

While Rolle's forte was always Diophantine analysis, his most important work was a book on the algebra of equations, called *Traité d'algèbre*, published in 1690. In that book Rolle firmly established the notation  $\sqrt[n]{a}$  [earlier written as  $\sqrt[n]{a}$ ] for the  $n$ th root of  $a$ , and proved a polynomial version of the theorem that today bears his name. (Rolle's Theorem was named by Giusto Bellavitis in 1846.) Ironically, Rolle was one of the most vocal early antagonists of calculus. He strove intently to demonstrate that it gave erroneous results and was based on unsound reasoning. He quarreled so vigorously on the subject that the Académie des Sciences was forced to intervene on several occasions. Among his several achievements, Rolle helped advance the currently accepted size order for negative numbers. Descartes, for example, viewed  $-2$  as smaller than  $-5$ . Rolle preceded most of his contemporaries by adopting the current convention in 1691.

**Example 1**

The function  $f(x) = \sin x$  has roots at  $x = 0$  and  $x = 2\pi$ . Moreover,  $f$  is continuous and differentiable everywhere, so it is differentiable on  $(0, 2\pi)$  and continuous on  $[0, 2\pi]$ . Thus, Rolle's Theorem guarantees that there is at least one point  $c$  in the interval  $(0, 2\pi)$  where the tangent line to the graph of  $y = \sin x$  is horizontal. Since  $dy/dx = \cos x$ , we can find  $c$  by solving the equation  $\cos c = 0$  on the interval  $(0, 2\pi)$ . This yields two values for  $c$ , namely  $c_1 = \pi/2$  and  $c_2 = 3\pi/2$  (Figure 6.5.2). ◀

**REMARK.** In the preceding example, we were able to find the values of  $c$  because the equation  $f'(c) = 0$  was easy to solve. However, if this equation cannot be solved, then you will not be able to find values of  $c$ , even though you know they exist. This will rarely cause problems because usually one is more interested in knowing that the values of  $c$  exist than in finding them.

The hypotheses in Rolle's Theorem are critical—if  $f$  fails to be differentiable at even one point in the interval, then the theorem may fail. For example, the function  $f(x) = |x| - 1$  has roots at  $x = \pm 1$ , yet there is no horizontal tangent line to the graph of  $f$  over the interval  $(-1, 1)$  (Figure 6.5.3).

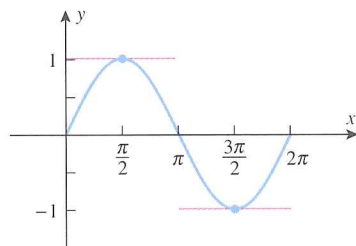


Figure 6.5.2

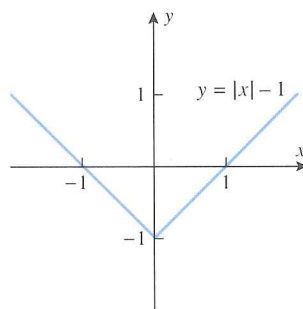


Figure 6.5.3

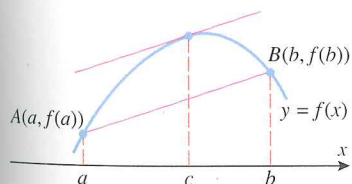
**THE MEAN-VALUE THEOREM**

Figure 6.5.4

Rolle's Theorem is a special case of the *Mean-Value Theorem*, which states that between any two points  $A$  and  $B$  on the graph of a differentiable function, there must be at least one place where the tangent line to the curve is parallel to the secant line joining  $A$  and  $B$  (Figure 6.5.4).

Noting that the slope of the secant line joining  $A(a, f(a))$  and  $B(b, f(b))$  is

$$\frac{f(b) - f(a)}{b - a}$$

and the slope of the tangent at  $c$  is  $f'(c)$ , the Mean-Value Theorem can be stated precisely as follows.

**6.5.2 THEOREM (Mean-Value Theorem).** Let  $f$  be differentiable on  $(a, b)$  and continuous on  $[a, b]$ . Then there is at least one point  $c$  in  $(a, b)$  where

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad (1)$$

**VELOCITY INTERPRETATION OF THE MEAN-VALUE THEOREM**

There is a nice interpretation of the Mean-Value Theorem in the situation where  $x = f(t)$  is the position versus time curve for a car moving along a straight road. In this case, the right side of (1) is the average velocity of the car over the time interval from  $a \leq t \leq b$ , and the left side is the instantaneous velocity at time  $t = c$ . Thus, the Mean-Value Theorem implies that at least once during the time interval the instantaneous velocity must equal the

average velocity. This agrees with our real-world experience—if the average velocity for a trip is 40 mi/h, then sometime during the trip the speedometer has to read 40 mi/h.

### Example 2

You are driving on a straight highway on which the speed limit is 55 mi/h. At 8:05 A.M. a police car clocks your velocity at 50 mi/h and at 8:10 A.M. a second police car posted 5 mi down the road clocks your velocity at 55 mi/h. Explain why the police have a right to charge you with a speeding violation.

**Solution.** You traveled 5 mi in 5 min ( $= \frac{1}{12}$  h), so your average velocity was 60 mi/h. However, the Mean-Value Theorem guarantees the police that your instantaneous velocity was 60 mi/h at least once over the 5-mi section of highway. ◀

### PROOF OF THE MEAN-VALUE THEOREM

**Motivation for the Proof of Theorem 6.5.2.** Figure 6.5.4 suggests that (1) will hold (i.e., the tangent line will be parallel to the secant line) at a point  $c$  where the vertical distance between the curve and the secant line is maximum. Thus, to prove the Mean-Value Theorem it is natural to begin by looking for a formula for the vertical distance  $v(x)$  between the curve  $y = f(x)$  and the secant line joining  $(a, f(a))$  and  $(b, f(b))$ .

**Proof of Theorem 6.5.2.** Since the two-point form of the equation of the secant line joining  $(a, f(a))$  and  $(b, f(b))$  is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or equivalently,

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

the difference  $v(x)$  between the height of the graph of  $f$  and the height of the secant line is

$$v(x) = f(x) - \left[ \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right] \quad (2)$$

Since  $f(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , so is  $v(x)$ . Moreover,

$$v(a) = 0 \quad \text{and} \quad v(b) = 0$$

so that  $v(x)$  satisfies the hypotheses of Rolle's Theorem on the interval  $[a, b]$ . Thus, there is a point  $c$  in  $(a, b)$  such that  $v'(c) = 0$ . But from Equation (2)

$$v'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so

$$v'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Thus, at the point  $c$  in  $(a, b)$ , where  $v'(c) = 0$ , we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

### Example 3

- Generate the graph of  $f(x) = (x^3/4) + 1$  over the interval  $[0, 2]$ , and use it to determine the number of tangent lines to the graph of  $f$  over the interval  $(0, 2)$  that are parallel to the secant line joining the endpoints of the graph.
- Show that  $f$  satisfies the hypotheses of the Mean-Value Theorem on the interval  $[0, 2]$ , and find all values of  $c$  in the interval  $(0, 2)$  whose existence is guaranteed by the Mean-Value Theorem. Confirm that these values of  $c$  are consistent with your graph in part (a).

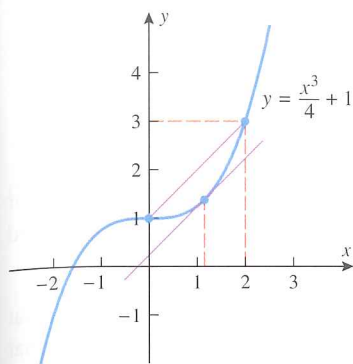


Figure 6.5.5

**Solution (a).** The graph of  $f$  in Figure 6.5.5 suggests that there is only one tangent line over the interval  $(0, 2)$  that is parallel to the secant line joining the endpoints.

**Solution (b).** The function  $f$  is continuous and differentiable everywhere because it is a polynomial. In particular,  $f$  is continuous on  $[0, 2]$  and differentiable on  $(0, 2)$ , so the hypotheses of the Mean-Value Theorem are satisfied with  $a = 0$  and  $b = 2$ . But

$$f(a) = f(0) = 1, \quad f(b) = f(2) = 3$$

$$f'(x) = \frac{3x^2}{4}, \quad f'(c) = \frac{3c^2}{4}$$

so in this case Formula (1) becomes

$$\frac{3c^2}{4} = \frac{3 - 1}{2 - 0} \quad \text{or} \quad 3c^2 = 4$$

which has the two solutions  $c = \pm 2/\sqrt{3} \approx \pm 1.15$ . However, only the positive solution lies in the interval  $[0, 2]$ ; this value of  $c$  is consistent with Figure 6.5.5. ◀

We stated at the beginning of this section that the Mean-Value Theorem is the starting point for many important results in calculus. As an example of this, we will use it to prove Theorem 5.1.2, which was one of our fundamental tools for analyzing graphs of functions.

### CONSEQUENCES OF THE MEAN-VALUE THEOREM

**5.1.2 THEOREM (Revisited).** Let  $f$  be a function that is continuous on a closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$ .

- (a) If  $f'(x) > 0$  for every value of  $x$  in  $(a, b)$ , then  $f$  is increasing on  $[a, b]$ .
- (b) If  $f'(x) < 0$  for every value of  $x$  in  $(a, b)$ , then  $f$  is decreasing on  $[a, b]$ .
- (c) If  $f'(x) = 0$  for every value of  $x$  in  $(a, b)$ , then  $f$  is constant on  $[a, b]$ .

**Proof (a).** Suppose that  $x_1$  and  $x_2$  are points in  $[a, b]$  such that  $x_1 < x_2$ . We must show that  $f(x_1) < f(x_2)$ . Because the hypotheses of the Mean-Value Theorem are satisfied on the entire interval  $[a, b]$ , they are satisfied on the subinterval  $[x_1, x_2]$ . Thus, there is some point  $c$  in the open interval  $(x_1, x_2)$  such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

or equivalently,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \quad (3)$$

Since  $c$  is in the open interval  $(x_1, x_2)$ , it follows that  $a < c < b$ ; thus,  $f'(c) > 0$ . However,  $x_2 - x_1 > 0$  since we assumed that  $x_1 < x_2$ . It follows from (3) that  $f(x_2) - f(x_1) > 0$  or, equivalently,  $f(x_1) < f(x_2)$ , which is what we were to prove. The proofs of parts (b) and (c) are similar and are left as exercises. ■

### THE CONSTANT DIFFERENCE THEOREM

We know from our earliest study of derivatives that the derivative of a constant is zero. Part (c) of Theorem 5.1.2 is the converse of that result; that is, a function whose derivative is zero on an interval must be constant on that interval. If we apply this to the difference of two functions, we obtain the following useful theorem.

**6.5.3 THEOREM (The Constant Difference Theorem).** If  $f$  and  $g$  are continuous on a closed interval  $[a, b]$ , and if  $f'(x) = g'(x)$  for all  $x$  in the open interval  $(a, b)$ , then  $f$  and  $g$  differ by a constant on  $[a, b]$ ; that is, there is a constant  $k$  such that  $f(x) - g(x) = k$  for all  $x$  in  $[a, b]$ .

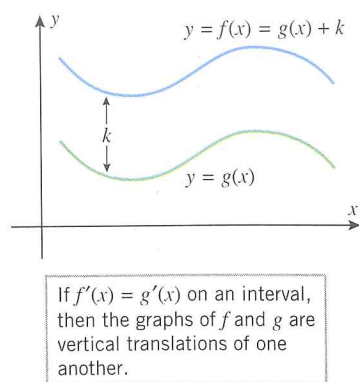


Figure 6.5.6

**Proof.** Let  $h(x) = f(x) - g(x)$ . Then for every  $x$  in  $(a, b)$

$$h'(x) = f'(x) - g'(x) = 0$$

Thus,  $h(x) = f(x) - g(x)$  is constant on  $[a, b]$  by Theorem 5.1.2(c). ■

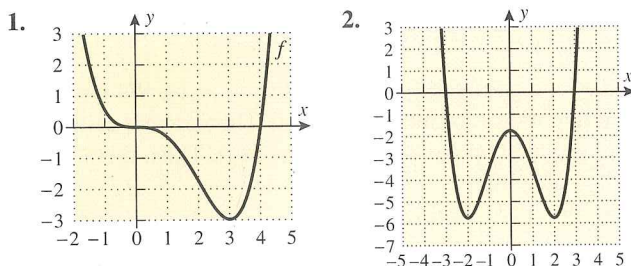
**REMARK.** This theorem remains true if the closed interval  $[a, b]$  is replaced by a finite or infinite interval  $(a, b)$ ,  $[a, b)$ , or  $(a, b]$ , provided  $f$  and  $g$  are differentiable on  $(a, b)$  and continuous on the entire interval.

The Constant Difference Theorem has a simple geometric interpretation—it tells us that if  $f$  and  $g$  have the same derivative on an interval, then there is a constant  $k$  such that  $f(x) = g(x) + k$  for each  $x$  in the interval; that is, the graphs of  $f$  and  $g$  can be obtained from one another by a vertical translation (Figure 6.5.6).

### EXERCISE SET 6.5



In Exercises 1 and 2, use the graph of  $f$  to find an interval  $[a, b]$  on which Rolle's Theorem applies, and find all values of  $c$  in that interval that satisfy the conclusion of the theorem.



In Exercises 3–8, verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of  $c$  in that interval that satisfy the conclusion of the theorem.

3.  $f(x) = x^2 - 6x + 8$ ;  $[2, 4]$
4.  $f(x) = x^3 - 3x^2 + 2x$ ;  $[0, 2]$
5.  $f(x) = \cos x$ ;  $[\pi/2, 3\pi/2]$
6.  $f(x) = \frac{x^2 - 1}{x - 2}$ ;  $[-1, 1]$
7.  $f(x) = \frac{1}{2}x - \sqrt{x}$ ;  $[0, 4]$
8.  $f(x) = \frac{1}{x^2} - \frac{4}{3x} + \frac{1}{3}$ ;  $[1, 3]$
9. Use the graph of  $f$  in the accompanying figure to estimate all values of  $c$  that satisfy the conclusion of the Mean-Value Theorem on the interval  $[0, 8]$ .

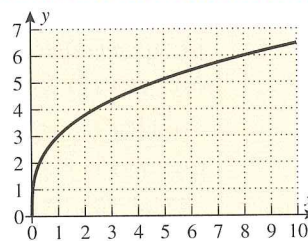


Figure Ex-9

10. Use the graph of  $f$  in Exercise 9 to estimate all values of  $c$  that satisfy the conclusion of the Mean-Value Theorem on the interval  $[0, 4]$ .

In Exercises 11–16, verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of  $c$  in that interval that satisfy the conclusion of the theorem.

11.  $f(x) = x^2 + x$ ;  $[-4, 6]$
12.  $f(x) = x^3 + x - 4$ ;  $[-1, 2]$
13.  $f(x) = \sqrt{x+1}$ ;  $[0, 3]$
14.  $f(x) = x + \frac{1}{x}$ ;  $[3, 4]$
15.  $f(x) = \sqrt{25 - x^2}$ ;  $[-5, 3]$
16.  $f(x) = \frac{1}{x-1}$ ;  $[2, 5]$
17. (a) Find an interval  $[a, b]$  on which

$$f(x) = x^4 + x^3 - x^2 + x - 2$$

satisfies the hypotheses of Rolle's Theorem.

- (b) Generate the graph of  $f'(x)$ , and use it to make rough estimates of all values of  $c$  in the interval obtained in part (a) that satisfy the conclusion of Rolle's Theorem.
- (c) Use Newton's Method to improve on the rough estimates obtained in part (b).

18. Let  $f(x) = x^3 + 4x$ .
- (a) Find the equation of the secant line through the points  $(-2, f(-2))$  and  $(1, f(1))$ .
- (b) Show that there is only one number  $c$  in the interval  $(-2, 1)$  that satisfies the conclusion of the Mean-Value Theorem for the secant line in part (a).
- (c) Find the equation of the tangent line to the graph of  $f$  at the point  $(c, f(c))$ .
- (d) Use a graphing utility to generate the secant line in part (a) and the tangent line in part (c) in the same coordinate system, and confirm visually that the two lines seem parallel.

19. Let  $f(x) = \tan x$ .
- (a) Show that there is no point  $c$  in the interval  $(0, \pi)$  such that  $f'(c) = 0$ , even though  $f(0) = f(\pi) = 0$ .
- (b) Explain why the result in part (a) does not violate Rolle's Theorem.

20. Let  $f(x) = x^{2/3}$ ,  $a = -1$ , and  $b = 8$ .
- (a) Show that there is no point  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- (b) Explain why the result in part (a) does not violate the Mean-Value Theorem.
21. (a) Show that if  $f$  is differentiable on  $(-\infty, +\infty)$ , and if  $y = f(x)$  and  $y = f'(x)$  are graphed in the same coordinate system, then between any two  $x$ -intercepts of  $f$  there is at least one  $x$ -intercept of  $f'$ .
- (b) Give some examples that illustrate this.

22. Review Definitions 3.1.3 and 3.1.4 of average and instantaneous rate of change of  $y$  with respect to  $x$ , and use the Mean-Value Theorem to show that if  $f$  is differentiable on  $(-\infty, +\infty)$ , then in any interval  $[x_0, x_1]$  there is at least one point where the instantaneous rate of change of  $y$  with respect to  $x$  is equal to the average rate of change over the interval.

In Exercises 23–25, use the result of Exercise 22.

23. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer read exactly 48 mi/h at least once during the trip.
24. At 11 A.M. on a certain morning the outside temperature was  $76^\circ\text{F}$ . At 11 P.M. that evening it had dropped to  $52^\circ\text{F}$ .
- (a) Show that at some instant during this period the temperature was decreasing at the rate of  $2^\circ\text{F/h}$ .
- (b) Suppose that you know that the temperature reached a high of  $88^\circ\text{F}$  sometime between 11 A.M. and 11 P.M.

Show that at some instant during this period the temperature was decreasing at a rate greater than  $3^\circ\text{F/h}$ .

25. Suppose that two runners in a 100-m dash finish in a tie. Show that they had the same velocity at least once during the race.
26. Use the fact that

$$\frac{d}{dx}(x^6 - 2x^2 + x) = 6x^5 - 4x + 1$$

to show that the equation  $6x^5 - 4x + 1 = 0$  has at least one solution in the interval  $(0, 1)$ .

27. (a) Use the Constant Difference Theorem (6.5.3) to show that if  $f'(x) = g'(x)$  for all  $x$  in the interval  $(-\infty, +\infty)$ , and if  $f$  and  $g$  have the same value at any point  $x_0$ , then  $f(x) = g(x)$  for all  $x$  in  $(-\infty, +\infty)$ .
- (b) Use the result in part (a) to prove the trigonometric identity  $\sin^2 x + \cos^2 x = 1$ .
28. (a) Use the Constant Difference Theorem (6.5.3) to show that if  $f'(x) = g'(x)$  for all  $x$  in  $(-\infty, +\infty)$ , and if  $f(x_0) - g(x_0) = c$  at some point  $x_0$ , then

$$f(x) - g(x) = c$$

for all  $x$  in  $(-\infty, +\infty)$ .

- (b) Use the result in part (a) to show that the function

$$h(x) = (x - 1)^3 - (x^2 + 3)(x - 3)$$

is constant for all  $x$  in  $(-\infty, +\infty)$ , and find the constant.

- (c) Check the result in part (b) by multiplying out and simplifying the formula for  $h(x)$ .
29. (a) Use the Mean-Value Theorem to show that if  $f$  is differentiable on an interval  $I$ , and if  $|f'(x)| \leq M$  for all values of  $x$  in  $I$ , then

$$|f(x) - f(y)| \leq M|x - y|$$

for all values of  $x$  and  $y$  in  $I$ .

- (b) Use the result in part (a) to show that

$$|\sin x - \sin y| \leq |x - y|$$

for all real values of  $x$  and  $y$ .

30. (a) Use the Mean-Value Theorem to show that if  $f$  is differentiable on an open interval  $I$ , and if  $|f'(x)| \geq M$  for all values of  $x$  in  $I$ , then

$$|f(x) - f(y)| \geq M|x - y|$$

for all values of  $x$  and  $y$  in  $I$ .

- (b) Use the result in part (a) to show that

$$|\tan x - \tan y| \geq |x - y|$$

for all values of  $x$  and  $y$  in the interval  $(-\pi/2, \pi/2)$ .

- (c) Use the result in part (b) to show that

$$|\tan x + \tan y| \geq |x + y|$$

for all values of  $x$  and  $y$  in the interval  $(-\pi/2, \pi/2)$ .

31. (a) Use the Mean-Value Theorem to show that

$$\sqrt{y} - \sqrt{x} < \frac{y-x}{2\sqrt{x}}$$

if  $0 < x < y$ .

- (b) Use the result in part (a) to show that if  $x$  and  $y$  are positive, then  $\sqrt{xy} < \frac{1}{2}(x+y)$ .
32. Show that if  $f$  is differentiable on an open interval  $I$  and  $f'(x) \neq 0$  on  $I$ , the equation  $f(x) = 0$  can have at most one real root in  $I$ .

33. Use the result in Exercise 32 to show the following:

- (a) The equation  $x^3 + 4x - 1 = 0$  has exactly one real root.  
 (b) If  $b^2 - 3ac < 0$  and if  $a \neq 0$ , then the equation

$$ax^3 + bx^2 + cx + d = 0$$

has exactly one real root (possibly repeated).

34. Use the Mean-Value Theorem to prove that

$$1.71 < \sqrt{3} < 1.75$$

[Hint: Let  $f(x) = \sqrt{x}$ ,  $a = 3$ , and  $b = 4$  in the Mean-Value Theorem.]

35. (a) Show that if
- $f$
- and
- $g$
- are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x)$$

for all  $x$ , then  $f^2(x) + g^2(x)$  is a constant.

- (b) Give an example of functions
- $f$
- and
- $g$
- with this property.

36. (a) Show that if
- $f$
- and
- $g$
- are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = f(x)$$

for all  $x$ , then  $f^2(x) - g^2(x)$  is a constant.

- (b) Show that the function  $f(x) = \frac{1}{2}(e^x + e^{-x})$  and the function  $g(x) = \frac{1}{2}(e^x - e^{-x})$  have this property.
37. Let  $g(x) = x^3 - 4x + 6$ . Find  $f(x)$  so that  $f'(x) = g'(x)$  and  $f(1) = 2$ .
38. Let  $f$  and  $g$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Prove: If  $f(a) = g(a)$  and  $f(b) = g(b)$ , then there is a point  $c$  in  $(a, b)$  where  $f'(c) = g'(c)$ .
39. Illustrate the result in Exercise 38 by drawing an appropriate picture.
40. (a) Prove: If  $f''(x) > 0$  for all  $x$  in  $(a, b)$ , then  $f'(x) = 0$  at most once in  $(a, b)$ .  
 (b) Give a geometric interpretation of the result in (a).
41. Prove part (b) of Theorem 5.1.2.
42. Prove part (c) of Theorem 5.1.2.

## SUPPLEMENTARY EXERCISES

1. (a) What inequality must  $f(x)$  satisfy for the function  $f$  to have an absolute maximum on an interval  $I$  at  $x_0$ ?  
 (b) What inequality must  $f(x)$  satisfy for  $f$  to have an absolute minimum on  $I$  at  $x_0$ ?  
 (c) What is the difference between an absolute extremum and a relative extremum?
2. According to the Extreme-Value Theorem, what conditions on a function  $f$  and an interval  $I$  guarantee that  $f$  will have both an absolute maximum and an absolute minimum on  $I$ ?
3. In each part, determine whether the statement is true or false, and justify your answer.
  - (a) If  $f$  is differentiable on the open interval  $(a, b)$ , and if  $f$  has an absolute extremum on that interval, then it must occur at a stationary point of  $f$ .
  - (b) If  $f$  is continuous on the open interval  $(a, b)$ , and if  $f$  is an absolute extremum on that interval, then it must occur at a stationary point of  $f$ .
4. Is it true or false that a particle in rectilinear motion is speeding up when its velocity is increasing and slowing down when its velocity is decreasing? Justify your answer.
5. Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$ , and suppose that  $f(a) = f(b)$ . Is it true or false that  $f$  must have at least one stationary point in  $(a, b)$ ? Justify your answer.
6. Draw an appropriate picture, and describe the basic idea of Newton's Method without using any formulas.
7. In each part, find the absolute minimum  $m$  and the absolute maximum  $M$  of  $f$  on the given interval (if they exist), and state where the absolute extrema occur.
  - (a)  $f(x) = 1/x$ ;  $[-2, -1]$
  - (b)  $f(x) = x^3 - x^4$ ;  $[-1, \frac{3}{2}]$
  - (c)  $f(x) = x^2(x-2)^{1/3}$ ;  $(0, 3]$
  - (d)  $f(x) = e^x/x^2$ ;  $(0, +\infty)$
8. In each part, find the absolute minimum  $m$  and the absolute maximum  $M$  of  $f$  on the given interval (if they exist), and state where the absolute extrema occur.
  - (a)  $f(x) = 2x/(x^2 + 3)$ ;  $(0, 2]$
  - (b)  $f(x) = 2x^5 - 5x^4 + 7$ ;  $(-1, 3)$
  - (c)  $f(x) = -|x^2 - 2x|$ ;  $[1, 3]$
  - (d)  $f(x) = x^x$ ;  $[0, +\infty)$
9. Use Newton's Method to approximate the smallest positive solution of  $\sin x + \cos x = 0$ .
10. Use Newton's Method to approximate all three solutions of  $x^3 - 4x + 1 = 0$ .
11. In each part, determine whether all of the hypotheses of Rolle's Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of  $c$  guaranteed in the conclusion of the theorem.
  - (a)  $f(x) = \sqrt{4 - x^2}$  on  $[-2, 2]$

(b)  $f(x) = x^{2/3} - 1$  on  $[-1, 1]$

(c)  $f(x) = \sin(x^2)$  on  $[0, \sqrt{\pi}]$

12. In each part, determine whether all of the hypotheses of the Mean-Value Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of  $c$  guaranteed in the conclusion of the theorem.

(a)  $f(x) = |x - 1|$  on  $[-2, 2]$

(b)  $f(x) = \frac{x+1}{x-1}$  on  $[2, 3]$

(c)  $f(x) = \begin{cases} 3 - x^2 & \text{if } x \leq 1 \\ 2/x & \text{if } x > 1 \end{cases}$  on  $[0, 2]$

13. A church window consists of a blue semicircular section surmounting a clear rectangular section as shown in the accompanying figure. The blue glass lets through half as much light per unit area as the clear glass. Find the radius  $r$  of the window that admits the most light if the perimeter of the entire window is to be  $P$  feet.

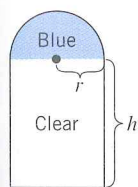


Figure Ex-13

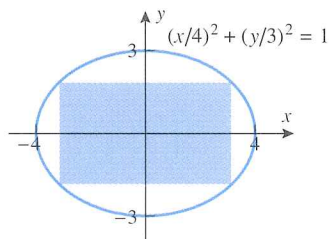


Figure Ex-14

15. (a) Can an object in rectilinear motion reverse direction if its acceleration is constant? Justify your answer using a velocity versus time curve.  
 (b) Can an object in rectilinear motion have increasing speed and decreasing acceleration? Justify your answer using a velocity versus time curve.
16. Suppose that the position function of a particle in rectilinear motion is given by the formula  $s(t) = t/(t^2 + 5)$  for  $t \geq 0$ .
- Use a graphing utility to generate the position, velocity, and acceleration versus time curves.
  - Use the appropriate graph to make a rough estimate of the time when the particle reverses direction, and then find that time exactly.
  - Find the position, velocity, and acceleration at the instant when the particle reverses direction.
  - Use the appropriate graphs to make rough estimates of the time intervals on which the particle is speeding up and the time intervals on which it is slowing down, and then find those time intervals exactly.
  - When does the particle have its maximum and minimum velocities?

17. A basketball player, standing near the basket to grab a rebound, jumps 76.0 cm vertically.

- How much time does the player spend in the top 15.0 cm of the jump and how much time in the bottom 15.0 cm?
- In words, explain why basketball players seem to be suspended in air when they jump.

18. (a) Suppose that an object is released from rest from the top of a high building. Assuming that a free-fall model applies and that time is in seconds and distance is in meters, make a table that shows the distance traveled by the object and its speed to one decimal place at 1-second increments from  $t = 0$  to  $t = 4$ .

- Confirm that doubling the elapsed time doubles the velocity, and explain why this happens.

- Confirm that doubling the elapsed time increases the distance traveled by a factor of 4, and explain why this happens.

19. Let

$$f(x) = \frac{x^3 + 2}{x^4 + 1}$$

- Generate the graph of  $y = f(x)$ , and use the graph to make rough estimates of the coordinates of the absolute extrema.

- Use a CAS to solve the equation  $f'(x) = 0$  and then use it to make more accurate approximations of the coordinates in part (a).

20. As shown in the accompanying figure, suppose that a boat enters the river at the point  $(1, 0)$  and maintains a heading toward the origin. As a result of the strong current, the boat follows the path

$$y = \frac{x^{10/3} - 1}{2x^{2/3}}$$

where  $x$  and  $y$  are in miles.

- Graph the path taken by the boat.
- Can the boat reach the origin? If not, discuss its fate and find how close it comes to the origin.
- What is the velocity of the boat in the  $x$ -direction at the instant when it is closest to the origin if the velocity in the  $y$ -direction is  $-4$  mi/h at this instant?

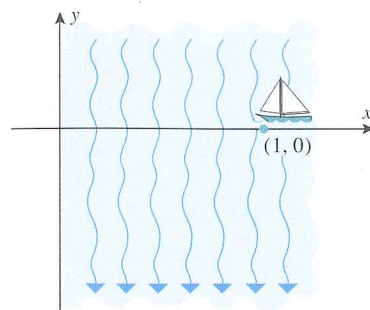


Figure Ex-20

- [c]** 21. Suppose that the position function of a particle in rectilinear motion is given by the formula

$$s(t) = \frac{t^2 + 1}{t^4 + 1}, \quad t \geq 0$$

- Use a CAS to find simplified formulas for the velocity  $v(t)$  and the acceleration  $a(t)$ .
  - Graph the position, velocity, and acceleration versus time curves.
  - Use the appropriate graph to make a rough estimate of the time at which the particle is farthest from the origin and its distance from the origin at that time.
  - Use the appropriate graph to make a rough estimate of the time interval during which the particle is moving in the positive direction.
  - Use the appropriate graphs to make rough estimates of the time intervals during which the particle is speeding up and the time intervals during which it is slowing down.
  - Use the appropriate graph to make a rough estimate of the maximum speed of the particle and the time at which the maximum speed occurs.
- [c]** 22. Suppose that the number of individuals at time  $t$  in a certain wildlife population is given by

$$N(t) = \frac{340}{1 + 9(0.77)^t}, \quad t \geq 0$$

where  $t$  is in years. At approximately what instant of time is the size of the population increasing most rapidly?

23. According to *Kepler's law*, the planets in our solar system move in elliptical orbits around the Sun. If a planet's closest approach to the Sun occurs at time  $t = 0$ , then the distance  $r$  from the center of the planet to the center of the Sun at some later time  $t$  can be determined from the equation

$$r = a(1 - e \cos \phi)$$

where  $a$  is the average distance between centers,  $e$  is a positive constant that measures the "flatness" of the elliptical orbit, and  $\phi$  is the solution of *Kepler's equation*

$$\frac{2\pi t}{T} = \phi - e \sin \phi$$

in which  $T$  is the time it takes for one complete orbit of the planet. Estimate the distance from the Earth to the Sun when  $t = 90$  days. [First find  $\phi$  from Kepler's equation, and then use this value of  $\phi$  to find the distance. Use  $a = 150 \times 10^6$  km,  $e = 0.0167$ , and  $T = 365$  days.]

24. Using the formulas in Exercise 23, find the distance from the planet Mars to the Sun when  $t = 1$  year. For Mars use  $a = 228 \times 10^6$  km,  $e = 0.934$ , and  $T = 1.88$  years.



## EXPANDING THE CALCULUS HORIZON

For additional material relating to this chapter, visit the Anton Website at <http://www.wiley.com/college/anton>