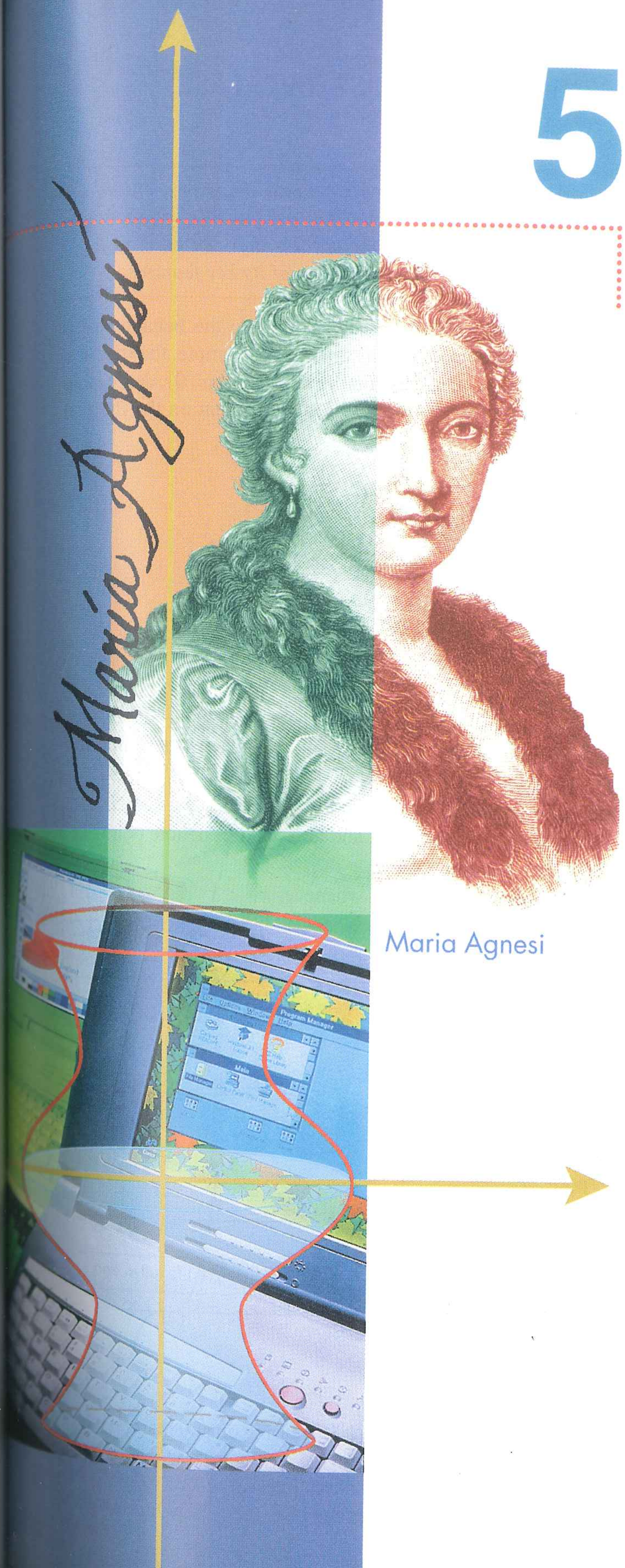
5

ANALYSIS OF FUNCTIONS AND THEIR GRAPHS



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In this chapter we will use methods of calculus to analyze functions and their graphs. We will be concerned here with such matters as identifying where the graph of a function is increasing or decreasing, where its high and low points occur, which way it bends, and what its limiting behavior is at important points.

One of the major goals of this chapter is to show how calculus and graphing utilities, working together, can provide most of the important information about the behavior of functions. Although graphing utilities can give us general information about the shape of a graph, such graphs lack perfect precision, since they are based on numerical approximations that can be affected by compression, distortion, and sampling error—it requires calculus to pin down the *exact* location of the key features and to reveal the nature of the fine detail. On the other hand, graphs produced by graphing utilities often provide information that is useful in pointing the calculus analysis in the right direction.

5.1 ANALYSIS OF FUNCTIONS I: INCREASE, DECREASE, AND CONCAVITY

Although graphing utilities are useful for determining the general shape of a graph, many problems require more precision than graphing utilities are capable of producing. The purpose of this section is to develop mathematical tools that can be used to determine the exact shape of a graph and the precise location of its key features.

INCREASING AND DECREASING FUNCTIONS

The terms *increasing*, *decreasing*, and *constant* are used to describe the behavior of a function over an interval as we travel left to right along its graph. For example, the function graphed in Figure 5.1.1 can be described as increasing on the interval $(-\infty, 0]$, decreasing on the interval $[0, 2]$, increasing again on the interval $[2, 4]$, and constant on the interval $[4, +\infty)$.

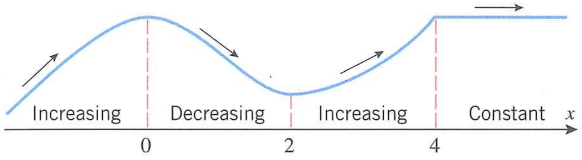


Figure 5.1.1

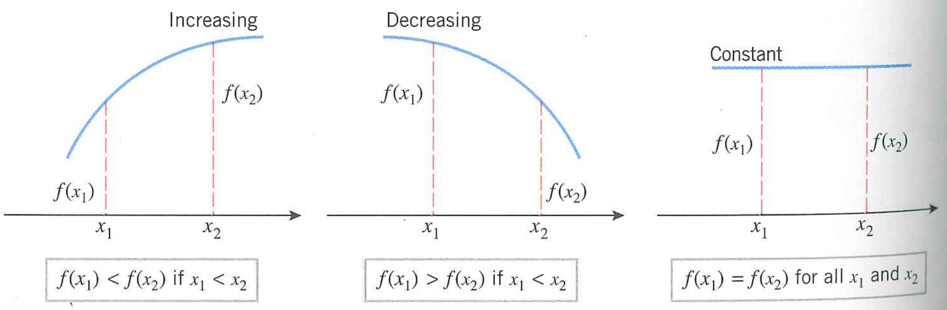
The following definition, which is illustrated in Figure 5.1.2, expresses these intuitive ideas precisely.

5.1.1 DEFINITION. Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(b) f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(c) f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



(*a*) (*b*) (*c*)

Figure 5.1.2

Figure 5.1.3 suggests that a differentiable function f is increasing on any interval where its graph has tangent lines with positive slope, is decreasing on any interval where its graph has tangent lines with negative slope, and is constant on any interval where its graph has tangent lines with zero slope. This intuitive observation suggests the following important theorem that will be proved in Section 6.5.

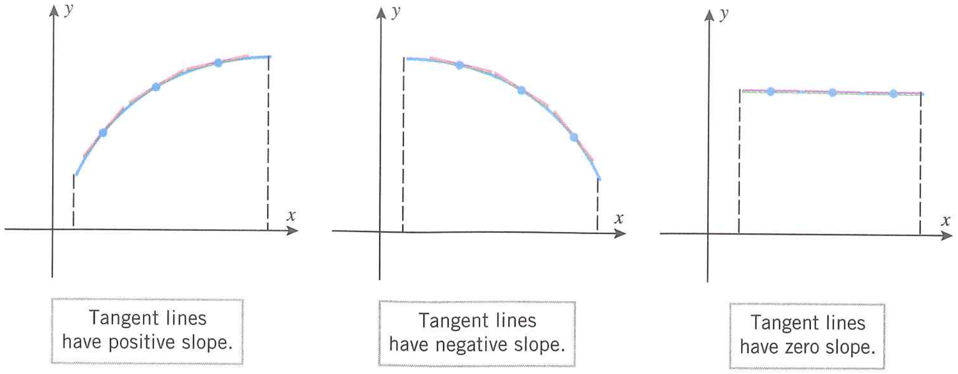


Figure 5.1.3

5.1.2 THEOREM. *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.
(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.
(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

REMARK. Observe that in Theorem 5.1.2 it is only necessary to examine the derivative of f on the open interval (a, b) to determine whether f is increasing, decreasing, or constant on the closed interval $[a, b]$. Moreover, although this theorem was stated for a closed interval $[a, b]$, it is applicable to any interval I on which f is continuous and inside of which f is differentiable. For example, if f is continuous on $(a, +\infty)$ and $f'(x) > 0$ for each x in the interval $(a, +\infty)$, then f is increasing on $[a, +\infty)$; and if $f'(x) < 0$ on $(-\infty, +\infty)$, then f is decreasing on $(-\infty, +\infty)$ [the continuity on $(-\infty, +\infty)$ follows from the differentiability].

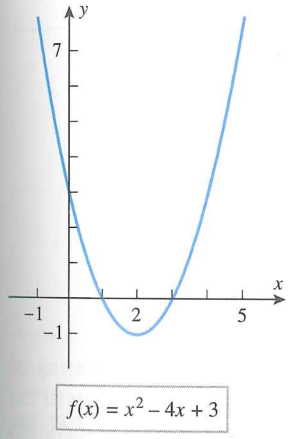


Figure 5.1.4

Example 1

Find the intervals on which the following functions are increasing and the intervals on which they are decreasing.

(a) $f(x) = x^2 - 4x + 3$ (b) $f(x) = x^3$

Solution (a). The graph of f in Figure 5.1.4 suggests that f is decreasing for $x \leq 2$ and increasing for $x \geq 2$. To confirm this, we differentiate f to obtain

$$f'(x) = 2x - 4 = 2(x - 2)$$

It follows that

$$f'(x) < 0 \quad \text{if} \quad -\infty < x < 2$$
$$f'(x) > 0 \quad \text{if} \quad 2 < x < +\infty$$

Since f is continuous at $x = 2$, it follows from Theorem 5.1.2 and the subsequent remark that

f is decreasing on $(-\infty, 2]$
f is increasing on $[2, +\infty)$

These conclusions are consistent with the graph of f in Figure 5.1.4.

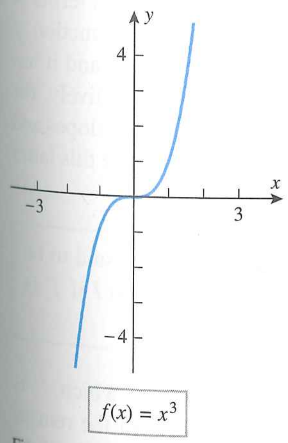


Figure 5.1.5

Solution (b). The graph of f in Figure 5.1.5 suggests that f is increasing over the entire x-axis. To confirm this, we differentiate f to obtain $f'(x) = 3x^2$. Thus,

$$f'(x) > 0 \quad \text{if} \quad -\infty < x < 0$$
$$f'(x) > 0 \quad \text{if} \quad 0 < x < +\infty$$

Since f is continuous at $x = 0$,

$$f \text{ is increasing on } (-\infty, 0]$$
$$f \text{ is increasing on } [0, +\infty)$$

Hence f is increasing over the entire interval $(-\infty, +\infty)$, which is consistent with the graph in Figure 5.1.5 (see Exercise 51). ◀

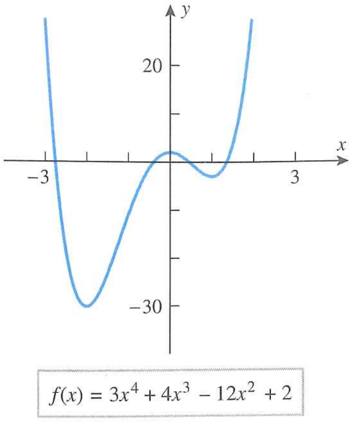


Figure 5.1.6

Example 2

(a) Use the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ in Figure 5.1.6 to make a conjecture about the intervals on which f is increasing or decreasing.

(b) Use Theorem 5.1.2 to determine whether your conjecture is correct.

Solution (a). The graph suggests that f is decreasing if $x \le -2$, increasing if $-2 \le x \le 0$, decreasing if $0 \le x \le 1$, and increasing if $x \ge 1$.

Solution (b). Differentiating f we obtain

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x + 2)(x - 1)$$

The sign analysis of f' in Table 5.1.1 can be obtained using the method of test points discussed in Appendix A. The conclusions in that table confirm the conjecture in part (a). ◀

Table 5.1.1

INTERVAL	$12x$	$x+2$	$x-1$	f'	CONCLUSION
$x < -2$	−	−	−	−	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	−	+	−	+	f is increasing on $[-2, 0]$
$0 < x < 1$	+	+	−	−	f is decreasing on $[0, 1]$
$1 < x$	+	+	+	+	f is increasing on $[1, +\infty)$

CONCAVITY

Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of *curvature*. For example, on both sides of the point in Figure 5.1.7 the graph is increasing, but on the left side it has an upward curvature ("holds water") and on the right side it has a downward curvature ("spills water"). On intervals where the graph of f has upward curvature we say that f is *concave up*, and on intervals where the graph has downward curvature we say that f is *concave down*.

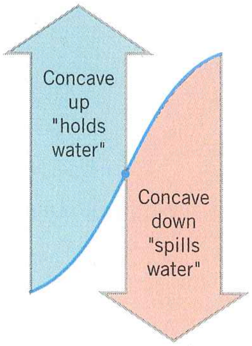


Figure 5.1.7

For differentiable functions, the direction of curvature can be characterized in terms of the tangent lines in two ways: As suggested by Figure 5.1.8, the graph of a function f has upward curvature on intervals where the graph lies above its tangent lines, and it has downward curvature on intervals where it lies below its tangent lines. Alternatively, the graph has upward curvature on intervals where the tangent lines have increasing slopes and downward curvature on intervals where they have decreasing slopes. We will use this latter characterization as our formal definition.

5.1.3 DEFINITION. If f is differentiable on an open interval I, then f is said to be ***concave up*** on I if f' is increasing on I, and f is said to be ***concave down*** on I if f' is decreasing on I.

To apply this definition we need some way to determine the intervals on which f' is increasing or decreasing. One way to do this is to apply Theorem 5.1.2 (and the remark that follows it) to the function f'. It follows from that theorem and remark that f' will be increasing where its derivative f'' is positive and will be decreasing where its derivative f'' is negative. This is the idea behind the following theorem.

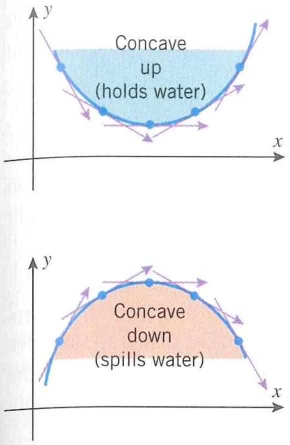


Figure 5.1.8

> **5.1.4 THEOREM.** *Let f be twice differentiable on an open interval I.*
> (a) *If $f''(x) > 0$ on I, then f is concave up on I.*
> (b) *If $f''(x) < 0$ on I, then f is concave down on I.*

Example 3

Find open intervals on which the following functions are concave up and open intervals on which they are concave down.

(a) $f(x) = x^2 - 4x + 3$ (b) $f(x) = x^3$ (c) $f(x) = x^3 - 3x^2 + 1$

Solution (a). Calculating the first two derivatives we obtain

$$f'(x) = 2x - 4 \quad \text{and} \quad f''(x) = 2$$

Since $f''(x) > 0$ for all x, the function f is concave up on $(-\infty, +\infty)$. This is consistent with Figure 5.1.4.

Solution (b). Calculating the first two derivatives we obtain

$$f'(x) = 3x^2 \quad \text{and} \quad f''(x) = 6x$$

Since $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$, the function f is concave down on $(-\infty, 0)$ and concave up on $(0, +\infty)$. This is consistent with Figure 5.1.5.

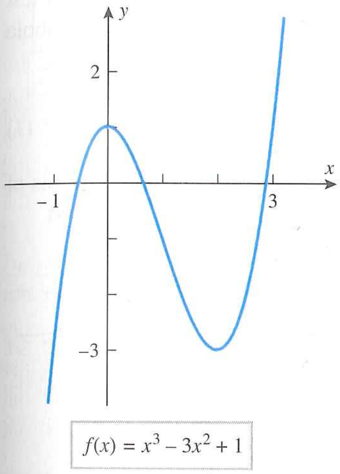


Figure 5.1.9

Solution (c). Calculating the first two derivatives we obtain

$$f'(x) = 3x^2 - 6x \quad \text{and} \quad f''(x) = 6x - 6 = 6(x - 1)$$

Since $f''(x) > 0$ if $x > 1$ and $f''(x) < 0$ if $x < 1$, we conclude that

f is concave up on $(1, +\infty)$
f is concave down on $(-\infty, 1)$

which is consistent with the graph in Figure 5.1.9. ◀

INFLECTION POINTS

Points where a graph changes from concave up to concave down, or vice versa, are of special interest, so there is some terminology associated with them.

> **5.1.5 DEFINITION.** If f is continuous on an open interval containing the point x_0, and if f changes the direction of its concavity at that point, then we say that f has an ***inflection point at*** x_0, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f (Figure 5.1.10).

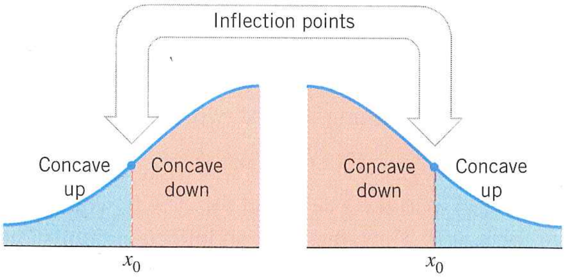


Figure 5.1.10

For example, the function $f(x) = x^3$ has an inflection point at $x = 0$ (Figure 5.1.5), the function $f(x) = x^3 - 3x^2 + 1$ has an inflection point at $x = 1$ (Figure 5.1.9), and the function $f(x) = x^2 - 4x + 3$ has no inflection points (Figure 5.1.4).

Example 4

Use the graph in Figure 5.1.6 to make rough estimates of the locations of the inflection points of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$, and check your estimates by finding the exact location of the inflection points.

Solution. The graph changes from concave up to concave down somewhere between -2 and -1, say roughly at $x = -1.25$; and the graph changes from concave down to concave up somewhere between 0 and 1, say roughly at $x = 0.5$. To find the exact location of the inflection points, we start by calculating the second derivative of f:

$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 - 24x \\ f''(x) &= 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2) \end{aligned}$$

We could analyze the sign of f'' by factoring this function and applying the method of test points (as in Table 5.1.1). However, here is another approach. The graph of f'' is a parabola that opens up, and the quadratic formula shows that the equation $f'' = 0$ has the roots

$$x = \frac{-1-\sqrt{7}}{3} \approx -1.22 \quad \text{and} \quad x = \frac{-1+\sqrt{7}}{3} \approx 0.55 \tag{1}$$

(verify). Thus, from the rough graph of f'' in Figure 5.1.11 we obtain the sign analysis of f'' in Table 5.1.2; this implies that f has inflection points at the points in (1). ◀

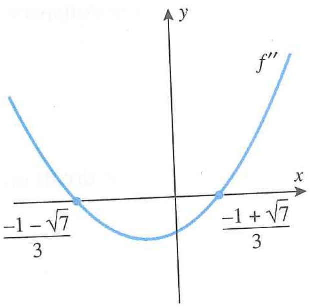


Figure 5.1.11

Table 5.1.2

INTERVAL	SIGN OF f''	CONCLUSION
$x < \frac{-1-\sqrt{7}}{3}$	+	f is concave up
$\frac{-1-\sqrt{7}}{3} < x < \frac{-1+\sqrt{7}}{3}$	−	f is concave down
$x > \frac{-1+\sqrt{7}}{3}$	+	f is concave up

In the preceding example the inflection points of f occurred at points where $f''(x) = 0$. However, inflection points do not always occur at points where $f''(x) = 0$. For example, if the graph of f'' happens to touch the x-axis at a point without crossing over it, then f'' will not change sign at that point, and hence no change in the concavity of f will occur at that point. Here is a specific example.

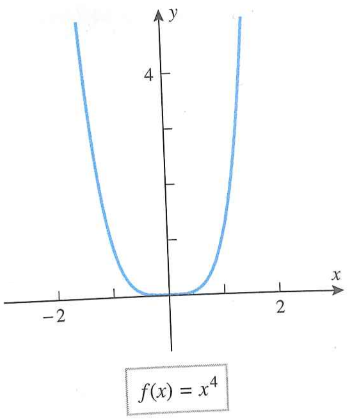


Figure 5.1.12

Example 5

Find the inflection points of $f(x) = x^4$.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 4x^3, \quad f''(x) = 12x^2$$

Here $f''(x) > 0$ for $x < 0$ and for $x > 0$, which implies that f is concave up for $x < 0$ and for $x > 0$. Thus, there are no inflection points; and in particular, there is no inflection point at $x = 0$, even though $f''(0) = 0$ (Figure 5.1.12). ◀

Example 6

Find the inflection points of the following functions, and confirm that your results are consistent with the graphs of the functions.

(a) $f(x) = xe^{-x}$ (b) $f(x) = \sin x, \quad 0 \le x \le 2\pi$ (c) $f(x) = \tan^{-1} x$

Solution (a). Calculating the first two derivatives of f we obtain

$$f'(x) = (1-x)e^{-x}, \qquad f''(x) = (x-2)e^{-x}$$

(verify). Keeping in mind that e^{-x} is always positive, it follows that the sign of f'' is determined by the factor $x - 2$. Thus, $f''(x) < 0$ if $x < 2$, and $f''(x) > 0$ if $x > 2$, which implies that the graph is concave down for $x < 2$ and concave up for $x > 2$. Thus, there is an inflection point at $x = 2$ (Figure 5.1.13*a*).

Solution (b). Calculating the first two derivatives of f we obtain

$$f'(x) = \cos x, \qquad f''(x) = -\sin x$$

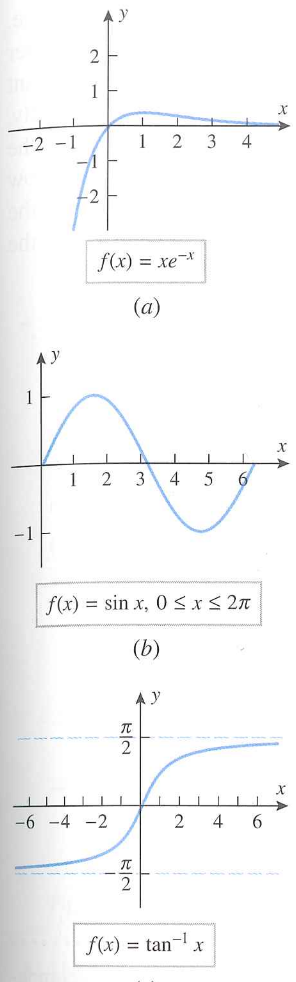
Thus, $f''(x) < 0$ if $0 < x < \pi$, and $f''(x) > 0$ if $\pi < x < 2\pi$, which implies that the graph is concave down for $0 < x < \pi$ and concave up for $\pi < x < 2\pi$. Thus, there is an inflection point at $x = \pi \approx 3.14$ (Figure 5.1.13*b*).

Solution (c). Calculating the first two derivatives of f we obtain

$$f'(x) = \frac{1}{1+x^2}, \qquad f''(x) = -\frac{2x}{\left(1+x^2\right)^2}$$

(verify). Thus, $f''(x) > 0$ if $x < 0$, and $f''(x) < 0$ if $x > 0$, which implies that the graph is concave up for $x < 0$ and concave down for $x > 0$. Thus, there is an inflection point at $x = 0$ (Figure 5.1.13*c*). ◀

FOR THE READER. If you have a CAS, devise a method for using it to find exact values for the inflection points of a function f, and use your method to find the inflection points of $f(x) = x/(x^2+1)$. Verify that your results are consistent with the graph of f.



(*a*)

(*b*)

(*c*)

Figure 5.1.13

INFLECTION POINTS IN APPLICATIONS

Up to now we have viewed the inflection points of a curve $y = f(x)$ as those points where the curve changes the direction of its concavity. However, inflection points also mark the points on the curve where the slopes of the tangent lines change from increasing to decreasing, or vice versa (Figure 5.1.14); stated another way:

> *Inflection points mark the places on the curve $y = f(x)$ where the rate of change of y with respect to x changes from increasing to decreasing, or vice versa.*

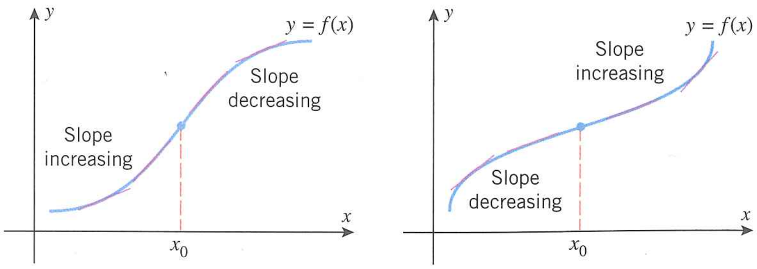


Figure 5.1.14

Note that we are dealing with a rather subtle concept here—a change of a rate of change. However, the following physical example should help to clarify the idea: Suppose that water is added to the flask in Figure 5.1.15 in such a way that the volume increases at a constant rate, and let us examine the rate at which the water level y rises with the time t. Initially, the level y will rise at a slow rate because of the wide base. However, as the diameter of the flask narrows, the rate at which the level y rises will increase until the level is at the narrow point in the neck. From that point on the rate at which the level rises will decrease as the diameter gets wider and wider. Thus, the narrow point in the neck is the point at which the rate of change of y with respect to t changes from increasing to decreasing.

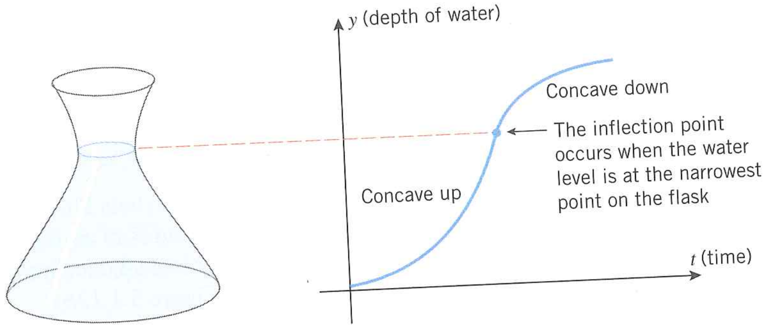


Figure 5.1.15

EXERCISE SET 5.1 Graphing Calculator C CAS

1. In each part, sketch the graph of a function f with the stated properties, and discuss the signs of f' and f''.
   (a) The function f is concave up and increasing on the interval $(-\infty, +\infty)$.
   (b) The function f is concave down and increasing on the interval $(-\infty, +\infty)$.
   (c) The function f is concave up and decreasing on the interval $(-\infty, +\infty)$.
   (d) The function f is concave down and decreasing on the interval $(-\infty, +\infty)$.

2. In each part, sketch the graph of a function f with the stated properties.
   (a) f is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave up on $(0, +\infty)$.
   (b) f is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave down on $(0, +\infty)$.
   (c) f is decreasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave up on $(0, +\infty)$.
   (d) f is decreasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave down on $(0, +\infty)$.

3. Use the graph of the equation $y = f(x)$ in the accompanying figure to find the signs of dy/dx and d^2y/dx^2 at the points A, B, and C.

4. Use the graph of the equation $y = f'(x)$ in the accompanying figure to find the signs of dy/dx and d^2y/dx^2 at the points A, B, and C.

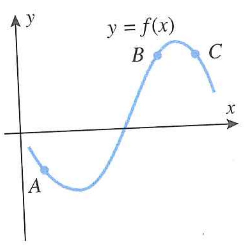


Figure Ex-3

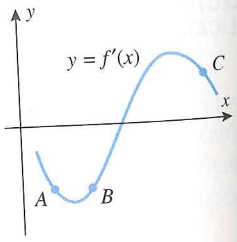


Figure Ex-4

5. Use the graph of $y = f''(x)$ in the accompanying figure to determine the x-coordinates of all inflection points of f. Explain your reasoning.

6. Use the graph of $y = f'(x)$ in the accompanying figure to replace the question mark with $<$, $=$, or $>$, as appropriate. Explain your reasoning.
   (a) $f(0)$? $f(1)$ (b) $f(1)$? $f(2)$ (c) $f'(0)$? 0
   (d) $f'(1)$? 0 (e) $f''(0)$? 0 (f) $f''(2)$? 0

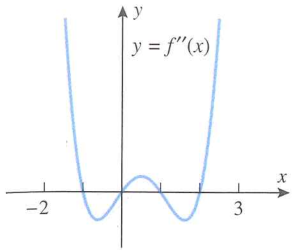


Figure Ex-5

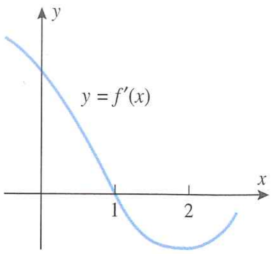


Figure Ex-6

7. In each part, use the graph of $y = f(x)$ in the accompanying figure to find the requested information.
(a) Find the intervals on which f is increasing.
(b) Find the intervals on which f is decreasing.
(c) Find the open intervals on which f is concave up.
(d) Find the open intervals on which f is concave down.
(e) Find all values of x at which f has an inflection point.

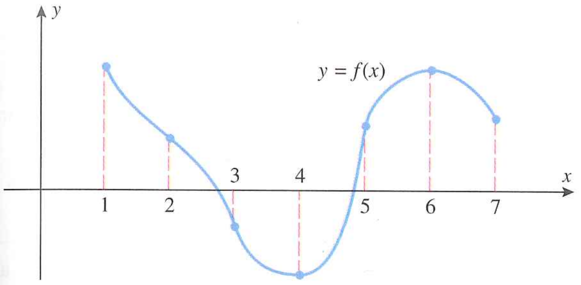


Figure Ex-7

8. Use the graph in Exercise 7 to make a table that shows the signs of f' and f'' over the intervals $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 5)$, $(5, 6)$, and $(6, 7)$.

In Exercises 9–24, find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

9. $f(x) = x^2 - 5x + 6$

10. $f(x) = 4 - 3x - x^2$

11. $f(x) = (x + 2)^3$

12. $f(x) = 5 + 12x - x^3$

13. $f(x) = 3x^4 - 4x^3$

14. $f(x) = x^4 - 8x^2 + 16$

15. $f(x) = \dfrac{x^2}{x^2 + 2}$

16. $f(x) = \dfrac{x}{x^2 + 2}$

17. $f(x) = \sqrt[3]{x + 2}$

18. $f(x) = x^{2/3}$

19. $f(x) = x^{1/3}(x + 4)$

20. $f(x) = x^{4/3} - x^{1/3}$

21. $f(x) = e^{-x^2/2}$

22. $f(x) = xe^{x^2}$

23. $f(x) = \ln(1 + x^2)$

24. $f(x) = x^2 \ln x$

In Exercises 25–30, analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility.

25. $f(x) = \cos x;\ [0, 2\pi]$

26. $f(x) = \sin^2 2x;\ [0, \pi]$

27. $f(x) = \tan x;\ (-\pi/2, \pi/2)$

28. $f(x) = 2x + \cot x;\ (0, \pi)$

29. $f(x) = \sin x \cos x;\ [0, \pi]$

30. $f(x) = \cos^2 x - 2\sin x;\ [0, 2\pi]$

31. In each part sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for all x
(b) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$
(c) $f(2) = 4,\ f''(x) < 0$ for $x \neq 2$ and $\lim_{x \to 2^+} f'(x) = +\infty$, $\lim_{x \to 2^-} f'(x) = -\infty$

32. In each part sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for all x
(b) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for $x < 2$, $f''(x) < 0$ for $x > 2$
(c) $f(2) = 4,\ f''(x) > 0$ for $x \neq 2$ and $\lim_{x \to 2^+} f'(x) = -\infty$, $\lim_{x \to 2^-} f'(x) = +\infty$

33. In each part, assume that a is a constant and find the inflection points, if any.
(a) $f(x) = (x - a)^3$ (b) $f(x) = (x - a)^4$

34. Given that a is a constant and n is a positive integer, what can you say about the existence of inflection points of the function $f(x) = (x - a)^n$? Justify your answer.

If f is increasing on an interval $[0, b)$, then it follows from Definition 5.1.1 that $f(0) < f(x)$ for each x in the interval. Use this result in Exercises 35–38.

35. Show that $\sqrt[3]{1 + x} < 1 + \frac{1}{3}x$ if $x > 0$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = 1 + \frac{1}{3}x - \sqrt[3]{1 + x}$ is increasing on $[0, +\infty)$.]

36. Show that $x < \tan x$ if $0 < x < \pi/2$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = \tan x - x$ is increasing on $[0, \pi/2)$.]

37. Use a graphing utility to make a conjecture about the relative sizes of x and $\sin x$ for $x \geq 0$, and prove your conjecture.

38. (a) Show that $e^x \geq 1 + x$ if $x \geq 0$.
(b) Show that $e^x \geq 1 + x + \frac{1}{2}x^2$ if $x \geq 0$.
(c) Confirm the inequalities in parts (a) and (b) with a graphing utility.

In Exercises 39 and 40, use a graphing utility to generate the graphs of f' and f'' over the stated interval; then use those graphs to estimate the x-coordinates of the inflection points of f, the intervals on which f is concave up or down, and the intervals on which f is increasing or decreasing. Check your estimates by graphing f.

39. $f(x) = x^4 - 24x^2 + 12x, \quad -5 \le x \le 5$

40. $f(x) = \dfrac{1}{1+x^2}, \quad -5 \le x \le 5$

41. For the function $f(x) = e^x/(1+x^2)$, use the method of Example 6 in Section 2.4 to approximate the x-coordinates of the inflection points to two decimal places.

42. For the function f in Exercise 40, use the method of Example 6 in Section 2.4 to approximate the x-coordinates of the inflection points to two decimal places.

In Exercises 43 and 44, use a CAS to find f'', and then use the method of Example 6 in Section 2.4 to approximate the x-coordinates of the inflection points to one decimal place. Confirm that your answer is consistent with the graph of f.

C **43.** $f(x) = \dfrac{10x-3}{3x^2-5x+8}$ C **44.** $f(x) = \dfrac{x^3-8x+7}{\sqrt{x^2+1}}$

45. Use Definition 5.1.1 to prove that $f(x) = x^2$ is increasing on $[0, +\infty)$.

46. Use Definition 5.1.1 to prove that $f(x) = 1/x$ is decreasing on $(0, +\infty)$.

47. In each part, determine whether the statement is true or false. If it is false, find functions for which the statement fails to hold.
(a) If f and g are increasing on an interval, then so is $f+g$.
(b) If f and g are increasing on an interval, then so is $f \cdot g$.

48. In each part, find functions f and g that are increasing on $(-\infty, +\infty)$ and for which $f - g$ has the stated property.
(a) $f - g$ is decreasing on $(-\infty, +\infty)$.
(b) $f - g$ is constant on $(-\infty, +\infty)$.
(c) $f - g$ is increasing on $(-\infty, +\infty)$.

49. (a) Prove that a general cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d \quad (a \ne 0)$$

has exactly one inflection point.
(b) Prove that if a cubic polynomial has three x-intercepts, then the inflection point occurs at the average value of the intercepts.
(c) Use the result in part (b) to find the inflection point of the cubic polynomial $f(x) = x^3 - 3x^2 + 2x$, and check your result by using f'' to determine where f is concave up and concave down.

50. From Exercise 49, the polynomial $f(x) = x^3 + bx^2 + 1$ has one inflection point. Use a graphing utility to reach a conclusion about the effect of the constant b on the location of the inflection point. Use f'' to explain what you have observed graphically.

51. Use Definition 5.1.1 to prove:
(a) If f is increasing on the intervals $(a, c]$ and $[c, b)$, then f is increasing on (a, b).
(b) If f is decreasing on the intervals $(a, c]$ and $[c, b)$, then f is decreasing on (a, b).

52. Use part (a) of Exercise 51 to show that $f(x) = x + \sin x$ is increasing on the interval $(-\infty, +\infty)$.

53. Suppose that the spread of a flu virus on a college campus is modeled by the function

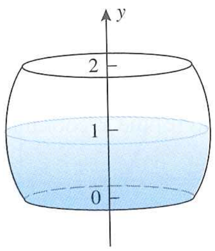
$$y(t) = \frac{1000}{1 + 999e^{-0.9t}}$$

where $y(t)$ is the number of infected students at time t (in days, starting with $t = 0$). Use a graphing utility to estimate the day on which the virus is spreading most rapidly.

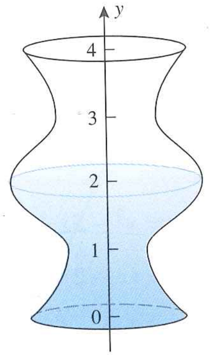
54. Let $y = 1/(1+x^2)$. Find the values of x for which y is increasing and decreasing most rapidly.

In Exercises 55 and 56, suppose that water is flowing at a constant rate into the container shown. Make a rough sketch of the graph of the water level y versus the time t. Make sure that your sketch conveys where the graph is concave up and concave down, and label the y-coordinates of the inflection points.

55.



56.



5.2 ANALYSIS OF FUNCTIONS II: RELATIVE EXTREMA; FIRST AND SECOND DERIVATIVE TESTS

In this section we will discuss methods for finding the high and low points on the graph of a function. The ideas we develop here will have important applications.

RELATIVE MAXIMA AND MINIMA

If we imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys, then the tops of the hills are called *relative maxima*, and the bottoms of the valleys are called *relative minima* (Figure 5.2.1).

The relative maxima are the high points in their *immediate vicinity*, and the relative minima are the low points. Note that a relative maximum need not be the highest point in the entire mountain range, and a relative minimum need not be the lowest point—they are just high and low points *relative* to the nearby terrain. These ideas are captured in the following definition.

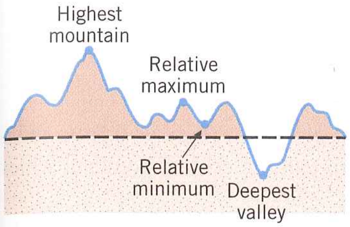


Figure 5.2.1

5.2.1 DEFINITION. A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval. If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.

Example 1

Locate the relative extrema of the four functions graphed in Figure 5.2.2.

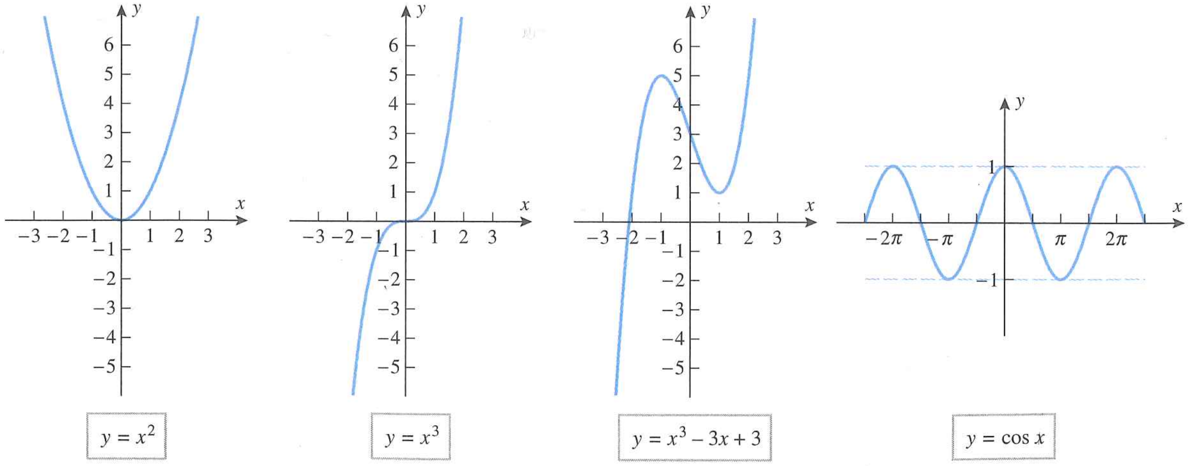
Solution.

(a) The function $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.

(b) The function $f(x) = x^3$ has no relative extrema.

(c) The function $f(x) = x^3 - 3x + 3$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

(d) The function $f(x) = \cos x$ has relative maxima at all even multiples of π and relative minima at all odd multiples of π. ◀



$y = x^2$ $y = x^3$ $y = x^3 - 3x + 3$ $y = \cos x$

Figure 5.2.2

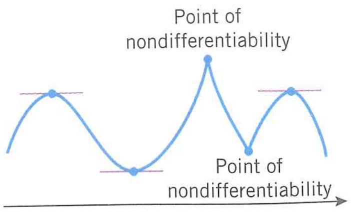


Figure 5.2.3

Relative extrema can be viewed as the transition points that separate the regions where a graph is increasing from those where it is decreasing. As suggested by Figure 5.2.3, the relative extrema of a continuous function f occur either at corners or at points where the graph of f has a horizontal tangent line. This is the content of the following theorem, whose proof is given in Appendix G.

> **5.2.2** THEOREM. *If a function f has any relative extrema, then they occur either at points where $f'(x) = 0$ or at points where f is not differentiable.*

The points at which either $f'(x) = 0$ or f is not differentiable are called the ***critical points*** of f, so that Theorem 5.2.2 can be rephrased as follows:

> *The relative extrema of a function, if any, occur at critical points.*

CRITICAL POINTS

Sometimes we will want to distinguish the critical points at which $f'(x) = 0$ from those points where f is not differentiable, in which case we will call the critical points at which $f'(x) = 0$ the ***stationary points*** of f.

It is important not to read too much into Theorem 5.2.2—the theorem asserts that the relative extrema must occur at critical points, but it does not say that a relative extremum occurs at *every* critical point; that is, there may be critical points at which a relative extremum does not occur. For example, for the eight critical points shown in Figure 5.2.4, relative extrema occur at all of the points in the top row, but not at any of the points in the bottom row.

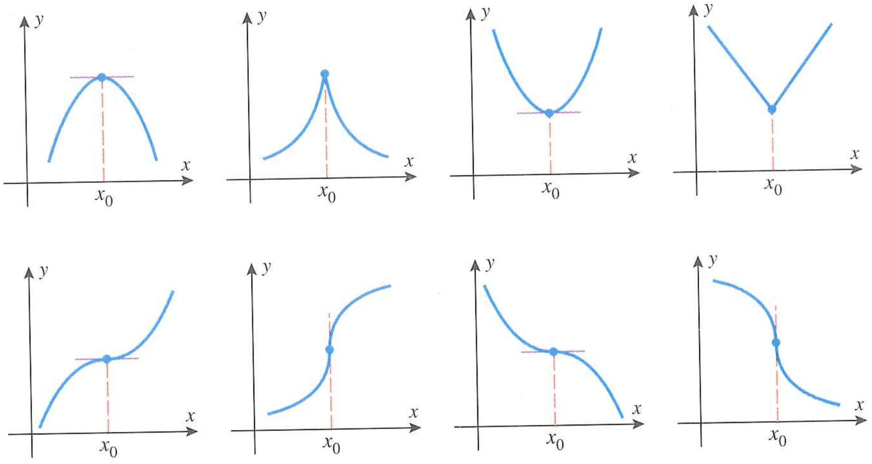


Figure 5.2.4

FIRST DERIVATIVE TEST

To develop an effective method for finding critical points of a function f, we need some criteria that will enable us to distinguish between the critical points where relative extrema occur and those where they do not. One such criterion can be motivated by examining the sign of the first derivative of f on each side of the eight critical points in Figure 5.2.4:

- At the two relative maxima in the top row, f' is positive to the left of x_0 and negative to the right.
- At the two relative minima in the top row, f' is negative to the left of x_0 and positive to the right.
- At the first two critical points in the bottom row, f' is positive on both sides of x_0.
- At the last two critical points in the bottom row, f' is negative on both sides of x_0.

These observations suggest that relative extrema of a function f occur at those critical points, and only those critical points, where f' changes sign. Moreover, if the sign changes from positive to negative, then a relative maximum occurs; and if the sign changes from negative to positive, then a relative minimum occurs. This is the content of the following theorem, whose proof is given at the end of this section.

5.2.3 THEOREM (*First Derivative Test*). *Suppose f is continuous at a critical point x_0.*

(*a*) *If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.*

(*b*) *If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.*

(*c*) *If $f'(x)$ has the same sign [either $f'(x) > 0$ or $f'(x) < 0$] on an open interval extending left from x_0 and on an open interval extending right from x_0, then f does not have a relative extremum at x_0.*

Example 2

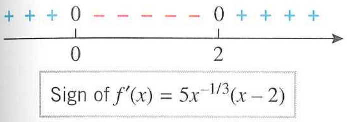
(a) Locate the relative maxima and minima of $f(x) = 3x^{5/3} - 15x^{2/3}$.

(b) Confirm that the results in part (a) agree with the graph of f.

Solution (a). The function f is defined and continuous for all real values of x, and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

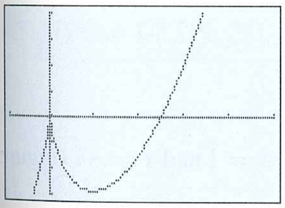
Since $f'(x)$ does not exist if $x = 0$, and since $f'(x) = 0$ if $x = 2$, there are critical points at $x = 0$ and $x = 2$. To apply the first derivative test, we examine the sign of $f'(x)$ on intervals extending to the left and right of the critical points (Figure 5.2.5). Since the sign of the derivative changes from positive to negative at $x = 0$, there is a relative maximum there, and since it changes from negative to positive at $x = 2$, there is a relative minimum there.



Sign of $f'(x) = 5x^{-1/3}(x-2)$

Figure 5.2.5

Solution (b). The result in part (a) agrees with the graph of f shown in Figure 5.2.6. ◀



$[-2, 10] \times [-15, 20]$
$x\text{Scl} = 2, y\text{Scl} = 5$

Figure 5.2.6

FOR THE READER. As discussed in the subsection of Section 1.3 entitled Errors of Omission, many graphing utilities omit portions of the graphs of functions with fractional exponents and must be "tricked" into producing complete graphs; and indeed, for the function in the last example the author's calculator and CAS both failed to produce the portion of the graph over the negative x-axis. To generate the graph in Figure 5.2.6, the author had to apply the techniques discussed in Exercise 29 of Section 1.3 to each term in the formula for f. Use a graphing utility to generate this graph.

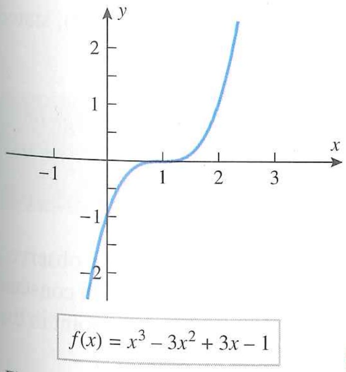
Example 3

Locate the relative extrema of $f(x) = x^3 - 3x^2 + 3x - 1$, if any.

Solution. Since f is differentiable everywhere, the only possible critical points are stationary points. Differentiating f yields

$$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$$

Solving $f'(x) = 0$ yields $x = 1$ as the only stationary point. However, $3(x-1)^2 \geq 0$ for all x, so $f'(x)$ does not change sign at $x = 1$; consequently, f does not have a relative extremum at $x = 1$. Thus, f has no relative extrema (Figure 5.2.7). ◀



$f(x) = x^3 - 3x^2 + 3x - 1$

Figure 5.2.7

FOR THE READER. How many relative extrema can a polynomial of degree n have? Explain your reasoning.

SECOND DERIVATIVE TEST

There is another test for relative extrema that is often easier to apply than the first derivative test. It is based on the geometric observation that a function f has a relative maximum at a stationary point if the graph of f is concave down on an open interval containing the point, and it has a relative minimum if it is concave up (Figure 5.2.8).

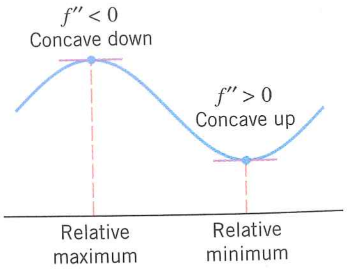


Figure 5.2.8

5.2.4 THEOREM (*Second Derivative Test*). *Suppose that f is twice differentiable at the point x_0.*

(*a*) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(*b*) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(*c*) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.*

REMARK. The proof of parts (a) and (b) is given at the end of this section. For part (c), consider the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$. In all three cases we have $f'(0) = 0$ and $f''(0) = 0$ (verify); but from Figure 1.6.4, $f(x) = x^4$ has a relative minimum at $x = 0$, $f(x) = -x^4$ has a relative maximum at $x = 0$ (why?), and $f(x) = x^3$ has neither a relative maximum nor a relative minimum at $x = 0$.

Example 4

Locate the relative maxima and minima of $f(x) = x^4 - 2x^2$, and confirm that your results are consistent with the graph of f.

Solution.

$$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$$
$$f''(x) = 12x^2 - 4$$

Solving $f'(x) = 0$ yields the stationary points $x = 0$, $x = 1$, and $x = -1$. Evaluating f'' at these points yields

$$f''(0) = -4 < 0$$
$$f''(1) = 8 > 0$$
$$f''(-1) = 8 > 0$$

so there is a relative maximum at $x = 0$ and relative minima at $x = 1$ and $x = -1$ (Figure 5.2.9). ◀

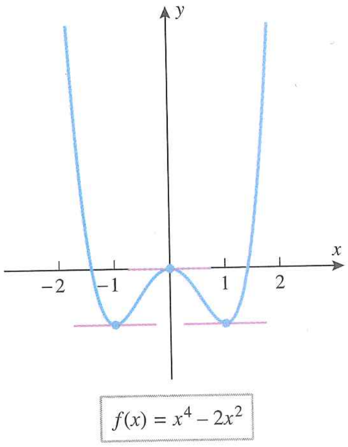


Figure 5.2.9

MORE ON THE SIGNIFICANCE OF INFLECTION POINTS

In Section 5.1 we observed that the inflection points of a curve $y = f(x)$ mark the points where the slopes of the tangent lines change from increasing to decreasing, or vice versa. Thus, in the case where f is twice differentiable, the inflection points mark the places on the curve $y = f(x)$ where $f'(x)$ has a relative maximum or minimum (Figure 5.2.10); stated another way:

Inflection points mark the places on the curve $y = f(x)$ at which the rate of change of y with respect to x has a relative maximum or minimum; that is, they are the places where y is increasing or decreasing most rapidly in the immediate vicinity.

As an illustration of this principle, consider the flask shown in Figure 5.1.15. We observed in Section 5.1 that if water is poured into the flask so that the volume increases at a constant rate, then the graph of y versus t has an inflection point when y is at the narrow point in the neck. However, this is also the place where the water level is rising most rapidly.

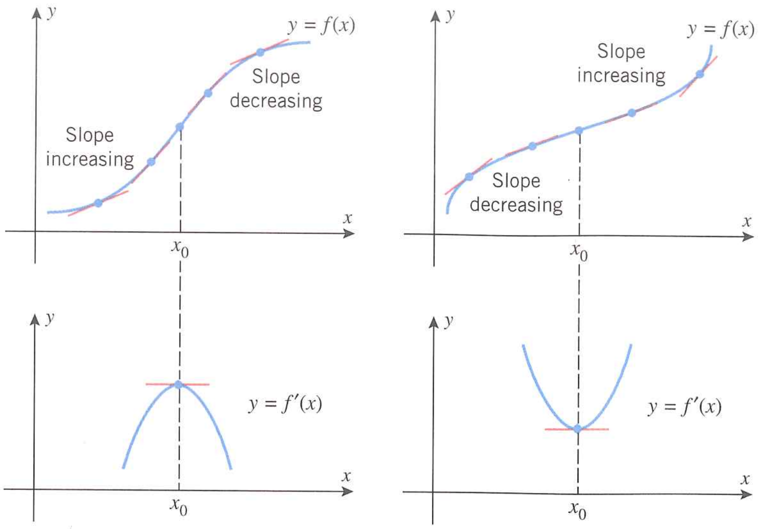


Figure 5.2.10

PROOF OF THE FIRST DERIVATIVE TEST *(Theorem 5.2.3)*

Proof. We will prove part (*a*) and leave parts (*b*) and (*c*) as exercises. We are assuming that $f'(x) > 0$ on the interval (a, x_0) and that $f'(x) < 0$ on the interval (x_0, b), and we want to show that

$$f(x_0) \geq f(x)$$

for all x in the interval (a, b). However, the two hypotheses, together with Theorem 5.1.2 (and its following remark) imply that f is increasing on the interval $(a, x_0]$ and decreasing on the interval $[x_0, b)$. Thus, $f(x_0) \geq f(x)$ for all x in (a, b) with equality only at x_0. ▮

PROOF OF THE SECOND DERIVATIVE TEST *(Theorem 5.2.4)*

Proof. We will prove part (*a*) and leave part (*b*) as an exercise. We want to show that if $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0; that is, there is an open interval (a, b) containing x_0 on which

$$f(x) \geq f(x_0)$$

For simplicity, we will assume that f'' is continuous at x_0. The proof for the case where f is twice differentiable at x_0 is left for more advanced courses. Observe first that the tangent line at x_0 is horizontal [since $f'(x_0) = 0$], and hence its equation is $y = f(x_0)$. Moreover, since $f''(x_0) > 0$, and since f'' is continuous at x_0, there is an open interval (a, b) containing x_0 on which $f''(x) > 0$. This implies that f is concave up on (a, b), and hence its graph lies above the tangent line $y = f(x_0)$ over the interval (a, b). This shows that $f(x) \geq f(x_0)$ on the interval (a, b). ▮

EXERCISE SET 5.2 Graphing Calculator C CAS

1. In each part, sketch the graph of a continuous function f with the stated properties.
  (a) f is concave up on the interval $(-\infty, +\infty)$ and has exactly one relative extremum.
  (b) f is concave up on the interval $(-\infty, +\infty)$ and has no relative extrema.
  (c) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to +\infty$ as $x \to +\infty$.
  (d) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to -\infty$ as $x \to +\infty$.

2. In each part, sketch the graph of a continuous function f with the stated properties.
  (a) f has exactly one relative extremum on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.
  (b) f has exactly two relative extrema on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.
  (c) f has exactly one inflection point and one relative extremum on $(-\infty, +\infty)$.
  (d) f has infinitely many relative extrema, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

3. (a) Use both the first and second derivative tests to show that $f(x) = 3x^2 - 6x + 1$ has a relative minimum at $x = 1$.
  (b) Use both the first and second derivative tests to show that $f(x) = x^3 - 3x + 3$ has a relative minimum at $x = 1$ and a relative maximum at $x = -1$.

4. (a) Use both the first and second derivative tests to show that $f(x) = \sin^2 x$ has a relative minimum at $x = 0$.
  (b) Use both the first and second derivative tests to show that $g(x) = \tan^2 x$ has a relative minimum at $x = 0$.
  (c) Give an informal verbal argument to explain without calculus why the functions in parts (a) and (b) have relative minima at $x = 0$.

5. (a) Show that both of the functions $f(x) = (x-1)^4$ and $g(x) = x^3 - 3x^2 + 3x - 2$ have stationary points at $x = 1$.
  (b) What does the second derivative test tell you about the nature of these stationary points?
  (c) What does the first derivative test tell you about the nature of these stationary points?

6. (a) Show that $f(x) = 1 - x^5$ and $g(x) = 3x^4 - 8x^3$ both have stationary points at $x = 0$.
  (b) What does the second derivative test tell you about the nature of these stationary points?
  (c) What does the first derivative test tell you about the nature of these stationary points?

In Exercises 7–12, locate the critical points, and classify them as stationary points or points of nondifferentiability.

7. (a) $f(x) = x^3 + 3x^2 - 9x + 1$
  (b) $f(x) = x^4 - 6x^2 - 3$

8. (a) $f(x) = 2x^3 - 6x + 7$ (b) $f(x) = 3x^4 - 4x^3$

9. (a) $f(x) = \dfrac{x}{x^2+2}$ (b) $f(x) = x^{2/3}$

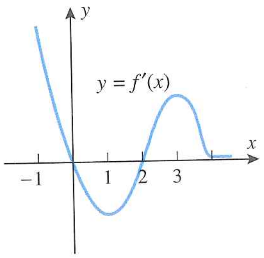
10. (a) $f(x) = \dfrac{x^2-3}{x^2+1}$ (b) $f(x) = \sqrt[3]{x+2}$

11. (a) $f(x) = x^{1/3}(x+4)$ (b) $f(x) = \cos 3x$

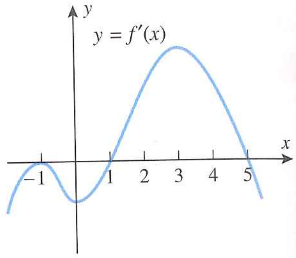
12. (a) $f(x) = x^{4/3} - 6x^{1/3}$ (b) $f(x) = |\sin x|$

In Exercises 13 and 14, use the graph of f' shown in the figure to estimate all values of x at which f has (a) relative minima, (b) relative maxima, and (c) inflection points.

13.



14.



In Exercises 15 and 16, use the given derivative to find the x-coordinates of all critical points of f, and determine whether a relative maximum, relative minimum, or neither occurs there.

15. (a) $f'(x) = x^3(x^2 - 5)$ (b) $f'(x) = xe^{-x}$

16. (a) $f'(x) = x^2(2x+1)(x-1)$
  (b) $f'(x) = \dfrac{9 - 4x^2}{\sqrt[3]{x+1}}$

In Exercises 17–20, find the relative extrema using both the first and second derivative tests.

17. $f(x) = 1 - 4x - x^2$ **18.** $f(x) = 2x^3 - 9x^2 + 12x$

19. $f(x) = \sin^2 x, \quad 0 < x < 2\pi$

20. $f(x) = \frac{1}{2}x - \sin x, \quad 0 < x < 2\pi$

In Exercises 21–34, use any method to find the relative extrema of the function f.

21. $f(x) = x^3 + 5x - 2$ **22.** $f(x) = x^4 - 2x^2 + 7$

23. $f(x) = x(x-1)^2$ **24.** $f(x) = x^4 + 2x^3$

25. $f(x) = 2x^2 - x^4$ **26.** $f(x) = (2x-1)^5$

27. $f(x) = x^{4/5}$ **28.** $f(x) = 2x + x^{2/3}$

29. $f(x) = \dfrac{x^2}{x^2+1}$ **30.** $f(x) = \dfrac{x}{x+2}$

31. $f(x) = \ln(1 + x^2)$

32. $f(x) = x^2 e^x$

33. $f(x) = |x^2 - 4|$

34. $f(x) = \begin{cases} 9 - x, & x \le 3 \\ x^2 - 3, & x > 3 \end{cases}$

In Exercises 35–38, find the relative extrema in the interval $0 < x < 2\pi$, and confirm that your results are consistent with the graph of f generated with a graphing utility.

35. $f(x) = |\sin 2x|$

36. $f(x) = \sqrt{3}x + 2\sin x$

37. $f(x) = \cos^2 x$

38. $f(x) = \dfrac{\sin x}{2 - \cos x}$

In Exercises 39–42, use a graphing utility to make a conjecture about the relative extrema of f, and then check your conjecture using either the first or second derivative test.

39. $f(x) = x \ln x$

40. $f(x) = \dfrac{2}{e^x + e^{-x}}$

41. $f(x) = x^2 e^{-2x}$

42. $f(x) = 10 \ln x - x$

In Exercises 43 and 44, use a graphing utility to generate the graphs of f' and f'' over the stated interval, and then use those graphs to estimate the x-coordinates of the relative extrema of f. Check that your estimates are consistent with the graph of f.

43. $f(x) = x^4 - 24x^2 + 12x, \quad -5 \le x \le 5$

44. $f(x) = \sin \frac{1}{2}x \cos x, \quad -\pi/2 \le x \le \pi/2$

45. For the function f in Exercise 43, use the method of Example 6 in Section 2.4 to approximate the x-coordinates of the relative maxima to two decimal places.

46. For the function f in Exercise 44, use the method of Example 6 in Section 2.4 to approximate the x-coordinates of the relative maxima to two decimal places.

In Exercises 47 and 48, use a CAS to graph f' and f'' over the stated interval. Use those graphs to make a conjecture about the locations and nature of the relative extrema of f, and check your conjecture by graphing f.

C **47.** $f(x) = \dfrac{10x - 3}{3x^2 - 5x + 8}$

C **48.** $f(x) = \dfrac{x^3 - 8x + 7}{\sqrt{x^2 + 1}}$

49. In each part, find k so that f has a relative extremum at the point $x = 3$.

(a) $f(x) = x^2 + \dfrac{k}{x}$

(b) $f(x) = \dfrac{x}{x^2 + k}$

50. Functions of the form

$$f(x) = cx^n e^{-x}, \quad x > 0$$

where n is a positive integer and $c = 1/n!$, arise in the statistical study of traffic flow.

(a) Use a graphing utility to generate the graph of f for $n = 2, 3, 4,$ and 5, and make a conjecture about the number and locations of the relative extrema of f.

(b) Confirm your conjecture using the first derivative test.

51. Functions of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

arise in a wide variety of statistical problems.

(a) Use the first derivative test to show that f has a relative maximum at $x = 0$, and confirm this by using a graphing utility to graph f.

(b) Sketch the graph of

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}$$

where μ is a constant, and label the coordinates of the relative extrema.

C **52.** (a) Use a CAS to graph the function

$$f(x) = \frac{x^4 + 1}{x^2 + 1}$$

and use the graph to estimate the x-coordinates of the relative extrema.

(b) Find the exact x-coordinates by using the CAS to solve the equation $f'(x) = 0$.

53. Find values of a, b, c, and d so that the function

$$f(x) = ax^3 + bx^2 + cx + d$$

has a relative minimum at $(0, 0)$ and a relative maximum at $(1, 1)$.

54. Let h and g have relative maxima at x_0. Prove or disprove:

(a) $h + g$ has a relative maximum at x_0

(b) $h - g$ has a relative maximum at x_0.

55. Sketch some curves that show that the three parts of the first derivative test (Theorem 5.2.3) can be false without the assumption that f is continuous at x_0.