Pierre de Fermat

4

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this chapter we will study logarithms and exponents from the function point of view. These functions have applications in the study of population growth, sound, heating and cooling, earthquakes, and carbon dating, to name a few. We will review the algebraic aspects of logarithms and exponents, but we will focus mainly on those aspects of logarithmic and exponential functions that relate to calculus. The heart of this chapter is Section 4.1 on inverse functions, in which we develop fundamental ideas that link logarithmic and exponential functions together numerically, algebraically, and graphically. We also apply inverse functions to the study of inverse trigonometric functions (Section 4.5) and to the problem of differentiating functions whose formulas cannot be expressed in the form $y = f(x)$ (Section 4.3). We show how these methods of differentiation can be applied to problems involving rates of change (Section 4.6); and finally, we develop a powerful tool for evaluating limits, especially limits involving logarithmic and exponential functions.

4.1 INVERSE FUNCTIONS

In everyday language the term "inversion" conveys the idea of a reversal. For example, in meteorology a temperature inversion is a reversal in the usual temperature properties of air layers; in music an inversion is a recurring theme that uses the same notes in reverse order; and in grammar an inversion is a reversal of the normal order of words. In mathematics the term ***inverse*** *is used to describe functions that are reverses of one another in the sense that each undoes the effect of the other. The purpose of this section is to discuss this fundamental mathematical idea.*

INVERSE FUNCTIONS

The idea of solving an equation $y = f(x)$ for x as a function of y, say $x = g(y)$, is one of the most important ideas in mathematics. Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

$$y = x^3 + 1 \qquad \boxed{y = f(x)}$$

can be solved for x as a function of y:

$$x = \sqrt[3]{y - 1} \qquad \boxed{x = g(y)}$$

The first equation is better for computing y if x is known, and the second is better for computing x if y is known (Figure 4.1.1).

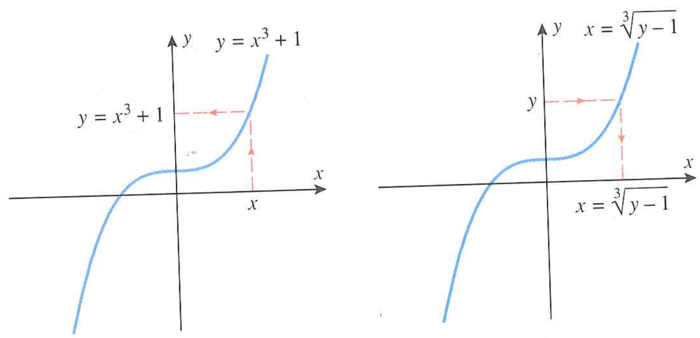


Figure 4.1.1

Our primary interest in this section is to identify relationships that may exist between the functions f and g when an equation $y = f(x)$ is expressed as $x = g(y)$, or conversely. For example, consider the functions $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y - 1}$ discussed above. When these functions are composed in either order they cancel out the effect of one another in the sense that

$$g(f(x)) = \sqrt[3]{f(x) - 1} = \sqrt[3]{(x^3 + 1) - 1} = x \tag{1}$$
$$f(g(y)) = [g(y)]^3 + 1 = (\sqrt[3]{y - 1})^3 + 1 = y$$

The first of these equations states that each output of the composition $g(f(x))$ is the same as the input, and the second states that each output of the composition $f(g(y))$ is the same as the input. Pairs of functions with these two properties are so important that there is some terminology for them.

4.1.1 DEFINITION. If the functions f and g satisfy the two conditions

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$
$$f(g(y)) = y \text{ for every } y \text{ in the domain of } g$$

then we say that f and g are ***inverse functions***. Moreover, we call ***f an inverse of g*** and ***g an inverse of f***.

Example 1

It follows from (1) that $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y-1}$ are inverse functions. ◀

It can be shown that a function cannot have two different inverses. Thus, if a function f has an inverse, then the inverse is unique, and we are entitled to talk about *the* inverse of f. The inverse of a function f is commonly denoted by f^{-1} (read "f inverse"). Thus, instead of using g in Example 1, the inverse of $f(x) = x^3$ could have been expressed as $f^{-1}(y) = \sqrt[3]{y-1}$.

WARNING. The symbol f^{-1} should always be interpreted as the inverse of f and *never* as the reciprocal $1/f$.

It is important to understand that a function is determined by the relationship that it establishes between its inputs and outputs and not by the letter used for the independent variable. Thus, even though the formulas $f(x) = 3x$ and $f(y) = 3y$ use different independent variables, they define the *same* function f, since the two formulas have the same "form" and hence assign the same value to each input; for example, in either notation $f(2) = 6$. As we progress through this text, there will be certain occasions on which we will want the independent variables for f and f^{-1} to be the same, and other occasions on which we will want them to be different. Thus, in Example 1 we could have expressed the inverse of $f(x) = x^3 + 1$ as $f^{-1}(x) = \sqrt[3]{x-1}$ had we wanted f and f^{-1} to have the same independent variable.

If we use the notation f^{-1} (rather than g) in Definition 4.1.1, and if we use x as the independent variable in the formulas for both f and f^{-1}, then the defining equations relating these functions are

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in the domain of } f^{-1} \end{aligned} \tag{2}$$

Example 2

Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$
$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$
$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x$$ ◀

REMARK. The results in Example 2 should make sense to you intuitively, since the operations of multiplying by 2 and multiplying by $\frac{1}{2}$ in either order cancel the effect of one another, as do the operations of cubing and taking a cube root.

DOMAIN AND RANGE OF INVERSE FUNCTIONS

The equations in (2) imply certain relationships between the domains and ranges of f and f^{-1}. For example, in the first equation the quantity $f(x)$ is an input of f^{-1}, so points in the range of f lie in the domain of f^{-1}; and in the second equation the quantity $f^{-1}(x)$ is an input of f, so points in the range of f^{-1} lie in the domain of f. All of this suggests the following relationships, which we state without formal proof:

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f \tag{3}$$

At the beginning of this section we solved the equation $y = f(x) = x^3 + 1$ for x as a function of y to obtain $x = g(y) = \sqrt[3]{y-1}$, and we observed in Example 1 that g is the inverse of f. This was not accidental—whenever an equation $y = f(x)$ is solved for x as a function of y, say $x = g(y)$, then f and g will be inverses. We can see why this is so by making two substitutions:

- Substitute $y = f(x)$ into $x = g(y)$. This yields $x = g(f(x))$, which is the first equation in Definition 4.1.1.
- Substitute $x = g(y)$ into $y = f(x)$. This yields $y = f(g(y))$, which is the second equation in Definition 4.1.1.

Since f and g satisfy the two conditions in Definition 4.1.1, we conclude that they are inverses. Thus, we have the following result.

4.1.2 THEOREM. *If an equation $y = f(x)$ can be solved for x as a function of y, then f has an inverse and the resulting equation is $x = f^{-1}(y)$.*

A METHOD FOR FINDING INVERSES

Example 3

Find the inverse of $f(x) = \sqrt{3x-2}$.

Solution. From Theorem 4.1.2 we can find a formula for $f^{-1}(y)$ by solving the equation

$$y = \sqrt{3x-2}$$

for x as a function of y. The computations are

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2 + 2)$$

from which it follows that

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2)$$

At this point we have successfully produced a formula for f^{-1}; however, we are not quite done, since there is no guarantee that the natural domain associated with this formula is the correct domain for f^{-1}. To determine whether this is so, we will examine the range of $y = f(x) = \sqrt{3x-2}$. The range consists of all y in the interval $[0, +\infty)$, so from (3) this interval is also the domain of $f^{-1}(y)$; thus, the inverse of f is given by the formula

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2), \quad y \geq 0$$ ◀

REMARK. When a formula for f^{-1} is obtained by solving the equation $y = f(x)$ for x as a function of y, the resulting formula has y as the independent variable. If it is preferable to have x as the independent variable for f^{-1}, then there are two ways to proceed: you can solve $y = f(x)$ for x as a function of y, and then replace y by x in the *final* formula for f^{-1}, or you can interchange x and y in the *original* equation and solve the equation $x = f(y)$ for y in terms of x, in which case the final equation will be $y = f^{-1}(x)$. In Example 3, either of these procedures will produce $f^{-1}(x) = \tfrac{1}{3}(x^2 + 2)$, $x \geq 0$.

Theorem 4.1.2 not only provides a method for finding the inverse of a function f, but it also provides an interpretation of what the values of f^{-1} represent. The theorem tells us that for a given y, the quantity $f^{-1}(y)$ is that number x with the property that $f(x) = y$. For example, if $f^{-1}(1) = 4$, then you know that $f(4) = 1$; and similarly, if $f(3) = 7$, then you know that $f^{-1}(7) = 3$.

EXISTENCE OF INVERSE FUNCTIONS

Not every function has an inverse. In general, in order for a function f to have an inverse it must assign distinct outputs to distinct inputs. To see why this is so, consider the function $f(x) = x^2$. Since $f(2) = f(-2) = 4$, the function f assigns the same output to two distinct inputs. If f were to have an inverse, then the equation $f(2) = 4$ would imply that $f^{-1}(4) = 2$, and the equation $f(-2) = 4$ would imply that $f^{-1}(4) = -2$. This is obviously impossible, since we cannot have two different values for $f^{-1}(4)$. Thus, $f(x) = x^2$ has no inverse. Another way to see that $f(x) = x^2$ has no inverse is to attempt to find the inverse by solving the equation $y = x^2$ for x in terms of y. We run into trouble immediately because the resulting equation, $x = \pm\sqrt{y}$, does not express x as a *single* function of y.

Functions that assign distinct outputs to distinct inputs are sufficiently important that there is a name for them—they are said to be ***one-to-one*** or ***invertible***. Stated algebraically, a function f is one-to-one if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$; and stated geometrically, a function f is one-to-one if the graph of $y = f(x)$ is cut at most once by any horizontal line (Figure 4.1.2).

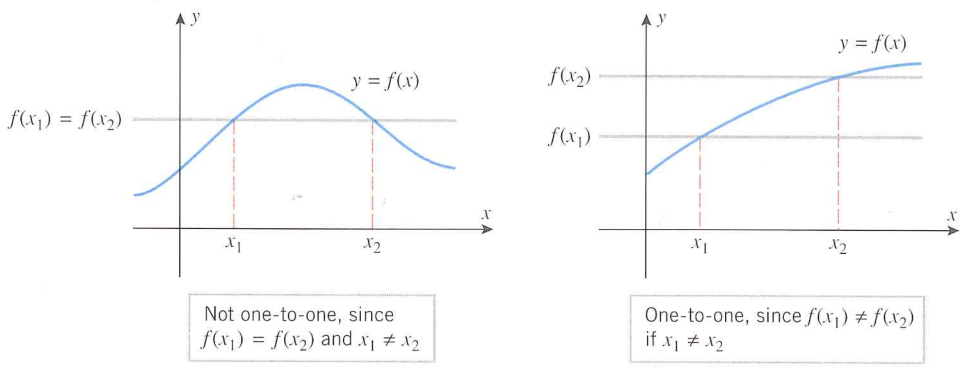


Figure 4.1.2

One can prove that a function f has an inverse if and only if it is one-to-one, and this provides us with the following geometric test for determining whether a function has an inverse.

4.1.3 THEOREM (*The Horizontal Line Test*). *A function f has an inverse if and only if its graph is cut at most once by any horizontal line.*

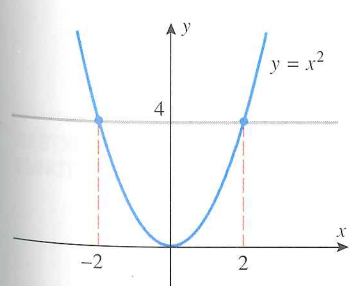


Figure 4.1.3

Example 4

We observed above that the function $f(x) = x^2$ does not have an inverse. This is confirmed by the horizontal line test, since the graph of $y = x^2$ is cut more than once by certain horizontal lines (Figure 4.1.3). ◀

figure 4.1.3

Example 5

We saw in Example 2(b) that the function $f(x) = x^3$ has an inverse [namely, $f^{-1}(x) = x^{1/3}$]. The existence of an inverse is confirmed by the horizontal line test, since the graph of $y = x^3$ is cut at most once by any horizontal line (Figure 4.1.4). ◀

Example 6

Explain why the function f that is graphed in Figure 4.1.5 has an inverse, and find $f^{-1}(3)$.

Solution. The function f has an inverse since its graph passes the horizontal line test. To evaluate $f^{-1}(3)$, we view $f^{-1}(3)$ as that number x for which $f(x) = 3$. From the graph we see that $f(2) = 3$, so $f^{-1}(3) = 2$. ◀

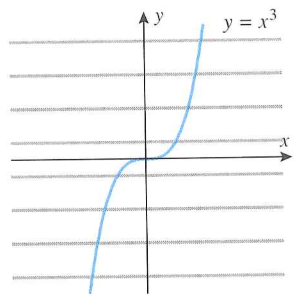


Figure 4.1.4

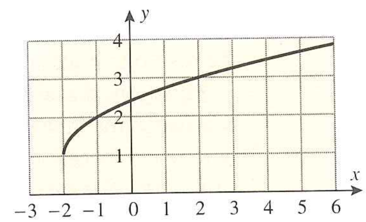


Figure 4.1.5

GRAPHS OF INVERSE FUNCTIONS

Our next objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions, which means that we will be comparing the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$. In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. Similarly, reversing the coordinates of a point on the graph of f^{-1} produces a point on the graph of f (verify). However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure 4.1.6), and hence the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line (Figure 4.1.7). In summary, we have the following result.

4.1.4 THEOREM. *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each is the mirror image of the other with respect to that line.*

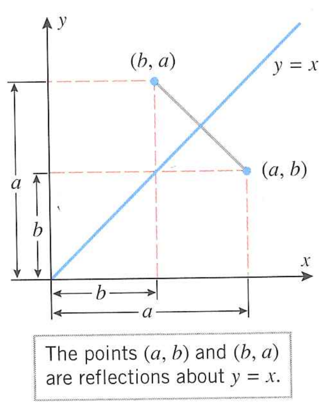


Figure 4.1.6

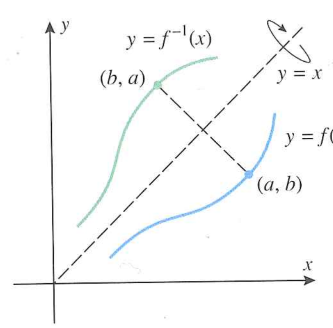


Figure 4.1.7

Example 7

Figure 4.1.8 shows the graphs of the inverse functions discussed in Examples 2 and 3. ◀

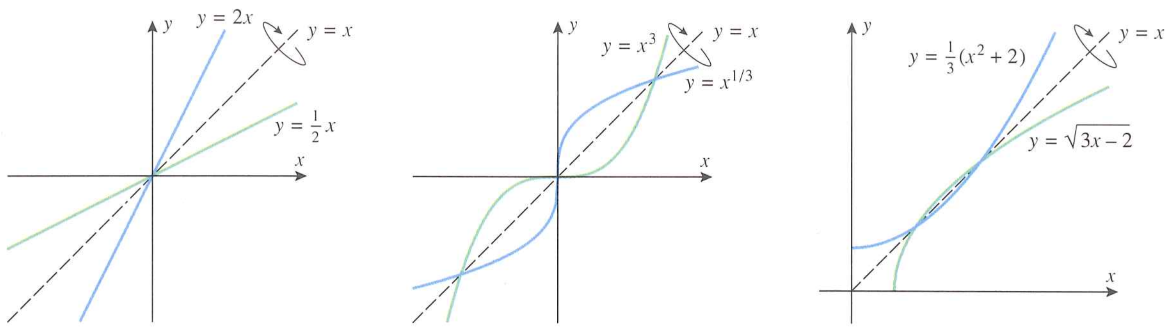


Figure 4.1.8

INCREASING OR DECREASING FUNCTIONS HAVE INVERSES

If the graph of a function f is always increasing or always decreasing over the domain of f, then the graph of f can be cut at most once by any horizontal line and consequently the function f must have an inverse. One way to tell whether the graph of a function is increasing or decreasing over an interval is by examining the slopes of its tangent lines. We will prove in the next chapter that the graph of f must be increasing on any interval where $f'(x) > 0$ (since the tangent lines have positive slope) and must be decreasing on any interval where $f'(x) < 0$ (since the tangent lines have negative slope) (Figure 4.1.9). These intuitive observations suggest the following theorem, which we state without formal proof.

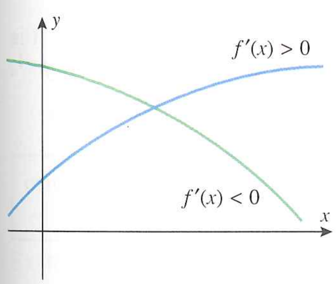


Figure 4.1.9

> **4.1.5 THEOREM.** *If the domain of f is an interval on which $f'(x) > 0$ or on which $f'(x) < 0$, then the function f has an inverse.*

Example 8

The graph of $f(x) = x^5 + x + 1$ is always increasing on $(-\infty, +\infty)$, since

$$f'(x) = 5x^4 + 1 > 0$$

for all x. However, there is no easy way to solve the equation $y = x^5 + x + 1$ for x in terms of y (try it), so even though we know that f has an inverse, we cannot produce a formula for it. ◀

REMARK. What is important to understand here is that our inability to find a formula for the inverse does not negate the existence of the inverse; indeed, one of our goals in later sections will be to develop ways of finding properties of functions in which there are no explicit formulas for the functions to work with.

RESTRICTING DOMAINS TO MAKE FUNCTIONS INVERTIBLE

Sometimes a function that is not one-to-one can be made one-to-one by restricting its domain. For example, although the function $f(x) = x^2$ is not one-to-one, the functions

$$g(x) = x^2, \quad x \geq 0$$
$$h(x) = x^2, \quad x \leq 0$$

which result from restricting the domain of f, are one-to-one since their graphs pass the horizontal line test [the graph of g is the right half of the parabola $y = x^2$ and the graph of h is the left half (Figure 4.1.10)]. The inverses of g and h can be found by solving each of the equations $y = g(x)$ and $y = h(x)$ for x as a function of y. For example, to find the inverse of g we solve

$$y = x^2, \quad x \geq 0$$

for x, which yields $x = \sqrt{y}$; hence, $g^{-1}(y) = \sqrt{y}$. Similarly, $h^{-1}(y) = -\sqrt{y}$. Geometrically, the graphs of $g(x) = x^2, x \geq 0$ and $g^{-1}(x) = \sqrt{x}$ are reflections of one another about the line $y = x$ (Figure 4.1.11), which reveals that the graph of $y = \sqrt{x}$ is a portion of a reflected parabola.

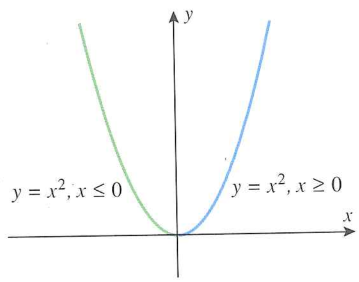


Figure 4.1.10

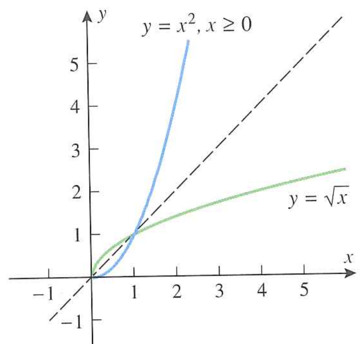


Figure 4.1.11

CONTINUITY OF INVERSE FUNCTIONS

Because the graphs of f and f^{-1} are reflections of one another about the line $y = x$, it is intuitively obvious that if the graph of f has no breaks, then neither will the graph of f^{-1}. This suggests the following result, which we state without proof.

> **4.1.6 THEOREM.** *If a function f is continuous and has an inverse, then f^{-1} is also continuous.*

For example, even though we cannot find a formula for f^{-1} in Example 8, the continuity of the polynomial f guarantees that f^{-1} is a continuous function.

DIFFERENTIABILITY OF INVERSE FUNCTIONS

Suppose that f is a continuous one-to-one function. Speaking informally, the points of nondifferentiability of f^{-1} occur most commonly at corners or points of vertical tangency in the graph of $y = f^{-1}(x)$. However, the graph of $y = f^{-1}(x)$ is the reflection about $y = x$ of the graph of $y = f(x)$; hence, corners in the graph of f^{-1} are reflections of corners in the graph of f, and points of vertical tangency in the graph of f^{-1} are reflections of points of horizontal tangency in the graph of f. This suggests that if f is a differentiable function whose derivative is nonzero, then f^{-1} will be a differentiable function. The following theorem, which we state without proof, makes this idea precise.

> **4.1.7 THEOREM (*Differentiability of Inverse Functions*).** *Suppose that the function f is invertible and differentiable on an interval I. Then f^{-1} is differentiable at any point x where $f'(f^{-1}(x)) \neq 0$.*

Example 9

We showed in Example 8 that the function $f(x) = x^5 + x + 1$ has an inverse. Use Theorem 4.1.7 to show that f^{-1} is differentiable on the interval $(-\infty, +\infty)$.

Solution. Let I denote the interval $(-\infty, +\infty)$. We must show that for each x in I, the function f has a nonzero derivative at the point $f^{-1}(x)$. But this is so because the derivative of f is

$$f'(x) = 5x^4 + 1$$

which is nonzero for all x. ◀

GRAPHING INVERSE FUNCTIONS WITH GRAPHING UTILITIES

Most graphing utilities cannot graph inverse functions directly. However, there is a way of graphing inverse functions by expressing the graph parametrically. To see how this can be done, suppose that we are interested in graphing the inverse of a one-to-one function f. We observed in Section 1.7 that the equation $y = f(x)$ can be expressed parametrically as

$$x = t, \quad y = f(t) \tag{4}$$

Moreover, we know that the graph of f^{-1} can be obtained by interchanging x and y, since this reflects the graph of f about the line $y = x$. Thus, from (4) the graph of f^{-1} can be represented parametrically as

$$x = f(t), \quad y = t \tag{5}$$

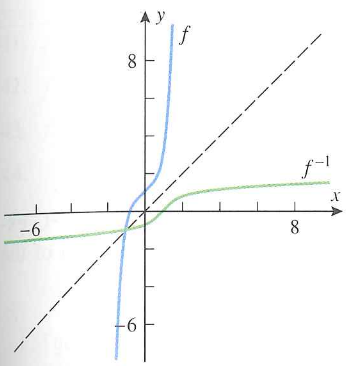


Figure 4.1.12

For example, Figure 4.1.12 shows the graph of $f(x) = x^5 + x + 1$ and its inverse generated with a graphing utility. The graph of f was generated from the parametric equations

$$x = t, \quad y = t^5 + t + 1$$

and the graph of f^{-1} was generated from the parametric equations

$$x = t^5 + t + 1, \quad y = t$$

EXERCISE SET 4.1 Graphing Calculator

1. In (a)–(d), determine whether f and g are inverse functions.
(a) $f(x) = 4x, \; g(x) = \frac{1}{4}x$
(b) $f(x) = 3x + 1, \; g(x) = 3x - 1$
(c) $f(x) = \sqrt[3]{x - 2}, \; g(x) = x^3 + 2$
(d) $f(x) = x^4, \; g(x) = \sqrt[4]{x}$

2. Check your answers to Exercise 1 with a graphing utility by determining whether the graphs of f and g are reflections of one another about the line $y = x$.

3. In each part, determine whether the function f defined by the table is one-to-one.

(a)

x	1	2	3	4	5	6
$f(x)$	–2	–1	0	1	2	3

(b)

x	1	2	3	4	5	6
$f(x)$	4	–7	6	–3	1	4

4. In each part, determine whether the function f is one-to-one, and justify your answer.
(a) $f(t)$ is the number of people in line at a movie theater at time t.
(b) $f(x)$ is your weight on your xth birthday.
(c) $f(v)$ is the weight of v cubic inches of lead.

5. In each part, use the horizontal line test to determine whether the function f is one-to-one.
(a) $f(x) = 3x + 2$ (b) $f(x) = \sqrt{x - 1}$
(c) $f(x) = |x|$ (d) $f(x) = x^3$
(e) $f(x) = x^2 - 2x + 2$ (f) $f(x) = \sin x$

6. In each part, generate the graph of the function f with a graphing utility, and determine whether f is one-to-one.
(a) $f(x) = x^3 - 3x + 2$ (b) $f(x) = x^3 - 3x^2 + 3x - 1$

7. In each part, determine whether f is one-to-one.
(a) $f(x) = \tan x$
(b) $f(x) = \tan x, \quad -\pi < x < \pi$
(c) $f(x) = \tan x, \quad -\pi/2 < x < \pi/2$

8. In each part, determine whether f is one-to-one.
(a) $f(x) = \cos x$
(b) $f(x) = \cos x, \quad -\pi/2 \le x \le \pi/2$
(c) $f(x) = \cos x, \quad 0 \le x \le \pi$

9. (a) The accompanying figure shows the graph of a function f over its domain $-8 \le x \le 8$. Explain why f has an inverse, and use the graph to find $f^{-1}(2)$, $f^{-1}(-1)$, and $f^{-1}(0)$.
(b) Find the domain and range of f^{-1}.
(c) Sketch the graph of f^{-1}.

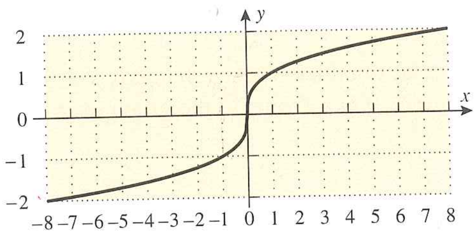


Figure Ex-9

10. (a) Explain why the function f graphed in the accompanying figure has no inverse on its domain $-3 \le x \le 4$.
(b) Subdivide the domain into three adjacent intervals on each of which the function f has an inverse.

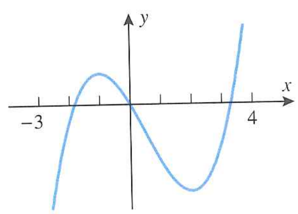


Figure Ex-10

In Exercises 11 and 12, determine whether the function f is one-to-one by examining the sign of $f'(x)$.

11. (a) $f(x) = x^2 + 8x + 1$
(b) $f(x) = 2x^5 + x^3 + 3x + 2$
(c) $f(x) = 2x + \sin x$

12. (a) $f(x) = x^3 + 3x^2 - 8$
(b) $f(x) = x^5 + 8x^3 + 2x - 1$
(c) $f(x) = \dfrac{x}{x+1}$

In Exercises 13–23, find a formula for $f^{-1}(x)$.

13. $f(x) = x^5$
14. $f(x) = 6x$
15. $f(x) = 7x - 6$
16. $f(x) = \dfrac{x+1}{x-1}$
17. $f(x) = 3x^3 - 5$
18. $f(x) = \sqrt[5]{4x+2}$
19. $f(x) = \sqrt[3]{2x-1}$
20. $f(x) = 5/(x^2+1), \quad x \ge 0$
21. $f(x) = 3/x^2, \quad x < 0$
22. $f(x) = \begin{cases} 2x, & x \le 0 \\ x^2, & x > 0 \end{cases}$
23. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \ge 2 \end{cases}$

24. Find a formula for $p^{-1}(x)$, given that
$$p(x) = x^3 - 3x^2 + 3x - 1$$

In Exercises 25–29, find a formula for $f^{-1}(x)$, and state the domain of f^{-1}.

25. $f(x) = (x+2)^4, \quad x \ge 0$
26. $f(x) = \sqrt{x+3}$
27. $f(x) = -\sqrt{3-2x}$
28. $f(x) = 3x^2 + 5x - 2, \quad x \ge 0$
29. $f(x) = x - 5x^2, \quad x \ge 1$

30. The formula $F = \frac{9}{5}C + 32$, where $C \ge -273.15°$C expresses the Fahrenheit temperature F as a function of the Celsius temperature C.
(a) Find a formula for the inverse function.
(b) In words, what does the inverse function tell you?
(c) Find the domain and range of the inverse function.

31. (a) One meter is about 6.214×10^{-4} miles. Find a formula $y = f(x)$ that expresses a length x in meters as a function of the same length y in miles.
(b) Find a formula for the inverse of f.
(c) In practical terms, what does the formula $x = f^{-1}(y)$ tell you?

32. Suppose that f is a one-to-one, continuous function such that $\lim_{x\to 3} f(x) = 7$. Find $\lim_{x\to 7} f^{-1}(x)$, and justify your reasoning.

33. Let $f(x) = x^2, x > 1$, and $g(x) = \sqrt{x}$.
(a) Show that $f(g(x)) = x, x > 1$, and $g(f(x)) = x$, $x > 1$.
(b) Show that f and g are *not* inverses of one another by showing that the graphs of $y = f(x)$ and $y = g(x)$ are not reflections of one another about $y = x$.
(c) Do parts (a) and (b) contradict one another? Explain.

34. Let $f(x) = ax^2 + bx + c$, $a > 0$. Find f^{-1} if the domain of f is restricted to
(a) $x \ge -b/(2a)$
(b) $x \le -b/(2a)$.

35. (a) Show that $f(x) = (3-x)/(1-x)$ is its own inverse.
(b) What does the result in part (a) tell you about the graph of f?

36. Suppose that a line of nonzero slope m intersects the x-axis at $(x_0, 0)$. Find an equation for the reflection of this line about $y = x$.

37. (a) Show that $f(x) = x^3 - 3x^2 + 2x$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the largest value of k such that f is one-to-one on the interval $(-k, k)$.

38. (a) Show that the function $f(x) = x^4 - 2x^3$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the smallest value of k such that f is one-to-one on the interval $[k, +\infty)$.

39. Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

40. Let $f(x) = \dfrac{x^3}{x^2+1}$. Find x if $f^{-1}(x) = 2$.

In Exercises 41–44, use a graphing utility and parametric equations to display the graphs of f and f^{-1} on the same screen.

41. $f(x) = x^3 + 0.2x - 1, \quad -1 \le x \le 2$

42. $f(x) = \sqrt{x^2+2} + x, \quad -5 \le x \le 5$

43. $f(x) = \cos(\cos 0.5x), \quad 0 \le x \le 3$

44. $f(x) = x + \sin x, \quad 0 \le x \le 6$

45. Prove that if $a^2 + bc \ne 0$, then the graph of

$$f(x) = \frac{ax+b}{cx-a}$$

is symmetric about the line $y = x$.

46. (a) Prove: If f and g are one-to-one, then so is the composition $f \circ g$.
(b) Prove: If f and g are one-to-one, then

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

47. Sketch the graph of a function that is one-to-one on $(-\infty, +\infty)$, yet not increasing on $(-\infty, +\infty)$ and not decreasing on $(-\infty, +\infty)$.

48. Prove: A one-to-one function f cannot have two different inverses.

49. Let $F(x) = f(2g(x))$ where $f(x) = x^4 + x^3 + 1$ for $0 \le x \le 2$, and $g(x) = f^{-1}(x)$. Find $F(3)$.

4.2 LOGARITHMIC AND EXPONENTIAL FUNCTIONS

When logarithms were introduced in the seventeenth century as a computational tool, they provided scientists of that period computing power that was previously unimaginable. Although computers and calculators have largely replaced logarithms for numerical calculations, the logarithmic functions and their relatives have wide-ranging applications in mathematics and science. Some of these will be introduced in this section.

IRRATIONAL EXPONENTS

In algebra, integer and rational powers of a number b are defined by

$$b^n = b \times b \times \cdots \times b \quad (n \text{ factors}), \qquad b^{-n} = \frac{1}{b^n}, \qquad b^0 = 1,$$

$$b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p, \qquad b^{-p/q} = \frac{1}{b^{p/q}}$$

If b is negative, then some of the fractional powers of b will have imaginary values; for example, $(-2)^{1/2} = \sqrt{-2}$. To avoid this complication we will assume throughout this section that $b \ge 0$, even if it is not stated explicitly.

Observe that the preceding definitions do not include *irrational* powers of b such as

$$2^\pi, \quad 3^{\sqrt{2}}, \quad \text{and} \quad \pi^{-\sqrt{7}}$$

There are various methods for defining irrational powers. One approach is to define irrational powers of b as limits of rational powers of b. For example, to define 2^π we can start with the decimal representation of π, namely,

$$3.1415926\ldots$$

From this decimal we can form a sequence of rational numbers that gets closer and closer to π, namely,

$$3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159$$

and from these we can form a sequence of *rational* powers of 2:

$$2^{3.1}, \quad 2^{3.14}, \quad 2^{3.141}, \quad 2^{3.1415}, \quad 2^{3.14159}$$

Table 4.2.1

x	2^x
3	8.000000
3.1	8.574188
3.14	8.815241
3.141	8.821353
3.1415	8.824411
3.14159	8.824962
3.141592	8.824974

Since the exponents of the terms in this sequence approach a limit of π, it seems plausible that the terms themselves approach a limit, and it would seem reasonable to *define* 2^π to be this limit. Table 4.2.1 provides numerical evidence that the sequence does, in fact, have a limit and that to four decimal places the value of this limit is $2^\pi \approx 8.8250$. More generally, for any irrational exponent p and positive number b, we can define b^p as the limit of the rational powers of b created from the decimal expansion of p.

FOR THE READER. Confirm the approximation $2^\pi \approx 8.8250$ by computing 2^π directly using your calculating utility.

Although our definition of b^p for irrational p certainly seems reasonable, there is a lot of tedious mathematical detail required to make the definition precise. We will not be concerned with such matters here and will accept without proof that the following familiar laws hold for all real exponents:

$$b^p b^q = b^{p+q}, \quad \frac{b^p}{b^q} = b^{p-q}, \quad (b^p)^q = b^{pq}$$

THE FAMILY OF EXPONENTIAL FUNCTIONS

A function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, is called an ***exponential function with base b***. Some examples are

$$f(x) = 2^x, \quad f(x) = \left(\tfrac{1}{2}\right)^x, \quad f(x) = \pi^x$$

Note that an exponential function has a constant base and variable exponent. Thus, functions such as $f(x) = x^2$ and $f(x) = x^\pi$ would not be classified as exponential functions, since they have a variable base and a constant exponent. Functions of this type, which are called ***power functions***, will be studied later.

It can be shown that exponential functions are continuous and have one of the basic two shapes shown in Figure 4.2.1*a*, depending on whether $0 < b < 1$ or $b > 1$. Figure 4.2.1*b* shows the graphs of some specific exponential functions.

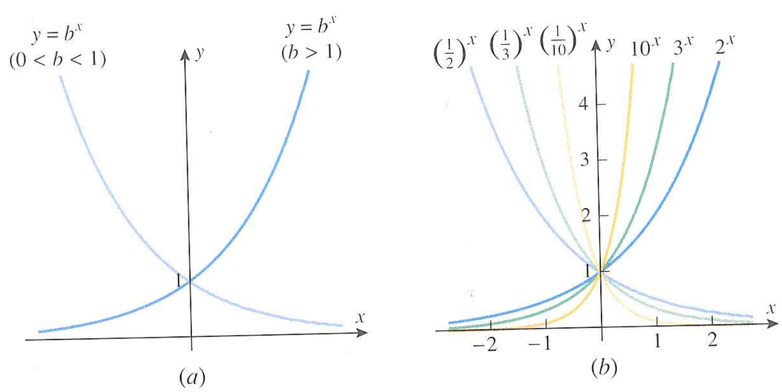


Figure 4.2.1

REMARK. If $b = 1$, then the function b^x is constant, since $b^x = 1^x = 1$. This case is of no interest to us here, so we have excluded it from the family of exponential functions.

FOR THE READER. Use your graphing utility to confirm that the graphs $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$ agree with Figure 4.2.1*b*, and explain why the two graphs are reflections of one another about the y-axis.

Since it is not our objective in this section to develop the properties of exponential functions in rigorous mathematical detail, we will simply observe without proof that the following properties of exponential functions are consistent with the graphs shown in Figure 4.2.1.

> **4.2.1 THEOREM.** *If $b > 0$ and $b \neq 1$, then:*
>
> (*a*) *The function $f(x) = b^x$ is defined for all real values of x, so its natural domain is $(-\infty, +\infty)$.*
>
> (*b*) *The function $f(x) = b^x$ is continuous on the interval $(-\infty, +\infty)$, and its range is $(0, +\infty)$.*

LOGARITHMS

Recall from algebra that a logarithm is an exponent. More precisely, if $b > 0$ and $b \neq 1$, then for positive values of x the ***logarithm to the base b of x*** is denoted by

$$\log_b x$$

and is defined to be that exponent to which b must be raised to produce x. For example,

$$\log_{10} 100 = 2, \quad \log_{10}(1/1000) = -3, \quad \log_2 16 = 4, \quad \log_b 1 = 0, \quad \log_b b = 1$$

$10^2 = 100$ $\quad$ $10^{-3} = 1/1000$ $\quad$ $2^4 = 16$ $\quad$ $b^0 = 1$ $\quad$ $b^1 = b$

Historically, the first logarithms ever studied were the logarithms with base 10, called ***common logarithms***. For such logarithms it is usual to suppress explicit reference to the base and write $\log x$ rather than $\log_{10} x$. More recently, logarithms with base 2 have played a role in computer science, since they arise naturally in the binary number system. However, the most widely used logarithms in applications are the ***natural logarithms***, which have an irrational base denoted by the letter e in honor of the Swiss mathematician Leonard Euler (p. 19), who first suggested its application to logarithms in an unpublished paper written in 1728. This constant, whose value to six decimal places is

$$e \approx 2.718282 \tag{1}$$

arises as the horizontal asymptote of the graph of the equation

$$y = \left(1 + \frac{1}{x}\right)^x \tag{2}$$

(Figure 4.2.2).

THE VALUES OF $(1 + 1/x)^x$ APPROACH e

x	$1 + \frac{1}{x}$	$\left(1 + \frac{1}{x}\right)^x$
1	2	≈ 2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

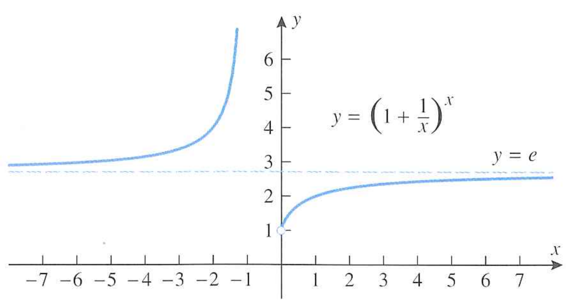


Figure 4.2.2

The fact that $y = e$ is a horizontal asymptote of (2) as $x \to +\infty$ and as $x \to -\infty$ is expressed by the limits

$$e = \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x \quad \text{and} \quad e = \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x \tag{3–4}$$

Later, we will show that these limits can be derived from the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x} \tag{5}$$

which is sometimes taken as the definition of the number e.

It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x"), rather than $\log_e x$. Thus, $\ln x$ can be viewed as that power to which e must be raised to produce x. For example,

$$\ln 1 = 0, \qquad \ln e = 1, \qquad \ln 1/e = -1, \qquad \ln(e^2) = 2$$

Since $e^0 = 1$ | Since $e^1 = e$ | Since $e^{-1} = 1/e$ | Since $e^2 = e^2$

In general, the statements

$$y = \ln x \quad \text{and} \quad x = e^y$$

are equivalent.

The exponential function $f(x) = e^x$ is called the ***natural exponential function***. To simplify typography, this function is sometimes written as $\exp x$. Thus, for example, you might see the relationship $e^{x_1 + x_2} = e^{x_1} e^{x_2}$ expressed as

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$

This notation is also used by graphing and calculating utilities, and it is typical to access the function e^x with some variation of the command EXP.

FOR THE READER. Most scientific calculating utilities provide some way of evaluating common logarithms, natural logarithms, and powers of e. Check your documentation to see how this is done, and then confirm the approximation $e \approx 2.718282$ and the values that appear in the table in Figure 4.2.2.

LOGARITHMIC FUNCTIONS

Figure 4.2.1a suggests that if $b > 0$ and $b \neq 1$, then the graph of $y = b^x$ passes the horizontal line test, and this implies that the function $f(x) = b^x$ has an inverse. To find a formula for this inverse (with x as the independent variable), we can solve the equation $x = b^y$ for y as a function of x. This can be done by taking the logarithm to the base b of both sides of this equation. This yields

$$\log_b x = \log_b(b^y) \tag{6}$$

However, if we think of $\log_b(b^y)$ as that exponent to which b must be raised to produce b^y, then it becomes evident that $\log_b(b^y) = y$. Thus, (6) can be rewritten as

$$y = \log_b x$$

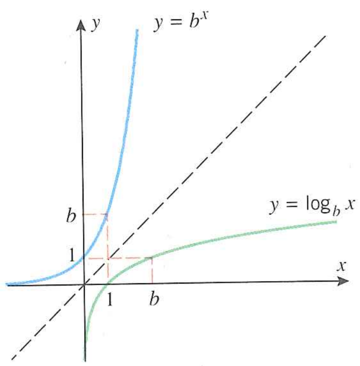


Figure 4.2.3

from which we conclude that the inverse of $f(x) = b^x$ is $f^{-1}(x) = \log_b x$. This implies that the graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$ (Figure 4.2.3). We call $\log_b x$ the ***logarithmic function with base b***.

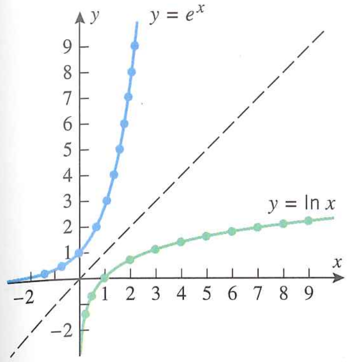
Recall from Section 4.1 that a one-to-one function f and its inverse satisfy the equations

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in the domain of } f^{-1}$$

In particular, if we take $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, and if we keep in mind that the domain of f^{-1} is the same as the range of f, then we obtain

$$\begin{aligned} \log_b(b^x) &= x \quad \text{for all real values of } x \\ b^{\log_b x} &= x \quad \text{for } x > 0 \end{aligned} \tag{7}$$



x	$y = \ln x$
0.25	−1.39
0.50	−0.69
1	0
2	0.69
3	1.10
4	1.39
5	1.61
6	1.79
7	1.95
8	2.08
9	2.20

x	$y = e^x$
−1.39	0.25
−0.69	0.50
0	1
0.69	2
1.10	3
1.39	4
1.61	5
1.79	6
1.95	7
2.08	8
2.20	9

Figure 4.2.4

In the special case where $b = e$, these equations become

$$\begin{aligned} \ln(e^x) &= x \quad \text{for all real values of } x \\ e^{\ln x} &= x \quad \text{for } x > 0 \end{aligned} \tag{8}$$

In words, the equations in (7) tell us that the functions b^x and $\log_b x$ cancel out the effect of one another when composed in either order; for example,

$$\log 10^x = x, \quad 10^{\log x} = x, \quad \ln e^x = x, \quad e^{\ln x} = x, \quad \ln e^5 = 5, \quad e^{\ln \pi} = \pi$$

FOR THE READER. Figure 4.2.4 shows computer-generated tables and graphs of $y = e^x$ and $y = \ln x$. Use your calculating and graphing utilities to generate the graphs and table values.

The inverse relationship between b^x and $\log_b x$ allows us to translate properties of exponential functions into properties of logarithmic functions, and vice versa.

4.2.2 THEOREM (***Comparison of Exponential and Logarithmic Functions for b > 1***).

$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
range $b^x = (0, +\infty)$	domain $\log_b x = (0, +\infty)$
domain $b^x = (-\infty, +\infty)$	range $\log_b x = (-\infty, +\infty)$
$0 < b^x < 1$ *if* $x < 0$	$\log_b x < 0$ *if* $0 < x < 1$

It follows from Theorem 4.1.2 that the equation $y = e^x$ can be solved for x in terms of y as $x = \ln y$, provided (of course) that y is in the domain of the natural logarithm function and x is in the domain of the natural exponential function; that is, $y > 0$ and x is any real number. Thus,

$y = e^x$ is equivalent to $x = \ln y$ if $y > 0$ and x is any real number

More generally, if $b > 0$ and $b \neq 1$, then

$y = b^x$ is equivalent to $x = \log_b y$ if $y > 0$ and x is any real number

You should recall the following algebraic properties of logarithms from your earlier studies.

4.2.3 THEOREM (***Algebraic Properties of Logarithms***).

$\log_b(ac) = \log_b a + \log_b c$	Product property
$\log_b(a/c) = \log_b a - \log_b c$	Quotient property
$\log_b(a^r) = r \log_b a$	Power property
$\log_b(1/c) = -\log_b c$	Reciprocal property

These properties are often used to expand a single logarithm into sums, differences, and multiples of other logarithms and, conversely, to condense sums, differences, and multiples of logarithms into a single logarithm. For example,

$$\log \frac{xy^5}{\sqrt{z}} = \log xy^5 - \log \sqrt{z} = \log x + \log y^5 - \log z^{1/2} = \log x + 5\log y - \tfrac{1}{2}\log z$$

$$5\log 2 + \log 3 - \log 8 = \log 32 + \log 3 - \log 8 = \log \frac{32 \cdot 3}{8} = \log 12$$

$$\tfrac{1}{3}\ln x - \ln(x^2-1) + 2\ln(x+3) = \ln x^{1/3} - \ln(x^2-1) + \ln(x+3)^2 = \ln \frac{\sqrt[3]{x}(x+3)^2}{x^2-1}$$

REMARK. Expressions of the form $\log_b(u+v)$ and $\log_b(u-v)$ have no useful simplifications in terms of $\log_b u$ and $\log_b v$. In particular,

$$\log_b(u+v) \neq \log_b u + \log_b v$$

$$\log_b(u-v) \neq \log_b u - \log_b v$$

SOLVING EQUATIONS INVOLVING EXPONENTIALS AND LOGARITHMS

Equations of the form $\log_b x = k$ can be solved by converting them to the exponential form $x = b^k$, and equations of the form $b^x = k$ can be solved by taking a logarithm of both sides (usually log or ln).

Example 1

Find x such that

(a) $\log x = \sqrt{2}$ (b) $\ln(x+1) = 5$ (c) $5^x = 7$

Solution (a). Converting the equation to exponential form yields

$$x = 10^{\sqrt{2}} \approx 25.95$$

Solution (b). Converting the equation to exponential form yields

$$x + 1 = e^5 \quad \text{or} \quad x = e^5 - 1 \approx 147.41$$

Solution (c). Taking the natural logarithm of both sides and using the power property of logarithms yields

$$x \ln 5 = \ln 7 \quad \text{or} \quad x = \frac{\ln 7}{\ln 5} \approx 1.21$$

◀

Example 2

A satellite that requires 7 watts of power to operate at full capacity is equipped with a radioisotope power supply whose power output in watts is given by the equation

$$P = 75e^{-t/125}$$

where t is the time in days that the supply is used. How long can the satellite operate at full capacity?

Solution. The power P will fall to 7 watts when

$$7 = 75e^{-t/125}$$

The solution for t is as follows:

$$7/75 = e^{-t/125}$$

$$\ln(7/75) = \ln(e^{-t/125})$$

$$\ln(7/75) = -t/125$$

$$t = -125\ln(7/75) \approx 296.4$$

so the satellite can operate at full capacity for about 296 days. ◀

Here is a more complicated example.

Example 3

Solve $\dfrac{e^x - e^{-x}}{2} = 1$ for x.

Solution. Multiplying both sides of the given equation by 2 yields

$$e^x - e^{-x} = 2$$

or equivalently,

$$e^x - \frac{1}{e^x} = 2$$

Multiplying through by e^x yields

$$e^{2x} - 1 = 2e^x \quad \text{or} \quad e^{2x} - 2e^x - 1 = 0$$

This is really a quadratic equation in disguise, as can be seen by rewriting it in the form

$$\left(e^x\right)^2 - 2e^x - 1 = 0$$

and letting $u = e^x$ to obtain

$$u^2 - 2u - 1 = 0$$

Solving for u by the quadratic formula yields

$$u = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

or, since $u = e^x$,

$$e^x = 1 \pm \sqrt{2}$$

But e^x cannot be negative, so we discard the negative value $1 - \sqrt{2}$; thus,

$$\begin{aligned} e^x &= 1 + \sqrt{2} \\ \ln e^x &= \ln(1 + \sqrt{2}) \\ x &= \ln(1 + \sqrt{2}) \approx 0.881 \end{aligned}$$

◀

CHANGE OF BASE FORMULA FOR LOGARITHMS

Scientific calculators generally provide keys for evaluating common logarithms and natural logarithms but have no keys for evaluating logarithms with other bases. However, this is not a serious deficiency because it is possible to express a logarithm with any base in terms of logarithms with any other base (see Exercise 40). For example, the following formula expresses a logarithm with base b in terms of natural logarithms:

$$\log_b x = \frac{\ln x}{\ln b} \tag{9}$$

We can derive this result by letting $y = \log_b x$, from which it follows that $b^y = x$. Taking the natural logarithm of both sides of this equation we obtain $y \ln b = \ln x$, from which (9) follows.

Example 4

Use a calculating utility to evaluate $\log_2 5$ by expressing this logarithm in terms of natural logarithms.

Solution. From (9) we obtain

$$\log_2 5 = \frac{\ln 5}{\ln 2} \approx 2.321928$$

◀

LOGARITHMIC SCALES IN SCIENCE AND ENGINEERING

Logarithms are used in science and engineering to deal with quantities whose units vary over an excessively wide range of values. For example, the "loudness" of a sound can be measured by its ***intensity*** I (in watts per square meter), which is related to the energy transmitted by the sound wave—the greater the intensity, the greater the transmitted energy, and the louder the sound is perceived by the human ear. However, intensity units are unwieldy because they vary over an enormous range. For example, a sound at the threshold of human hearing has an intensity of about 10^{-12} W/m^2, a close whisper has an intensity that is about 100 times the hearing threshold, and a jet engine at 50 meters has an intensity that is about $1{,}000{,}000{,}000{,}000 = 10^{12}$ times the hearing threshold. To see how logarithms can be used to reduce this wide spread, observe that if

$$y = \log x$$

then increasing x by a *factor* of 10 *adds* 1 unit to y since

$$\log 10x = \log 10 + \log x = 1 + y$$

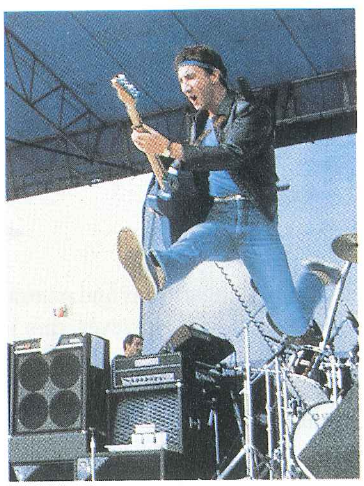
Physicists and engineers take advantage of this property by measuring loudness in terms of the ***sound level*** β, which is defined by

$$\beta = 10\log(I/I_0)$$

where $I_0 = 10^{-12}$ W/m^2 is a reference intensity close to the threshold of human hearing. The units of β are ***decibels*** (dB), named in honor of the telephone inventor Alexander Graham Bell. With this scale of measurement, *multiplying* the intensity I by a factor of 10 *adds* 10 dB to the sound level β (verify). This results in a more tractable scale than intensity for measuring sound loudness (Table 4.2.2). Some other familiar logarithmic scales are the ***Richter scale*** used to measure earthquake intensity and the **pH *scale*** used to measure acidity in chemistry, both of which are discussed in the exercises.

Table 4.2.2

β (dB)	I/I_0
0	$10^0 = 1$
10	$10^1 = 10$
20	$10^2 = 100$
30	$10^3 = 1{,}000$
40	$10^4 = 10{,}000$
50	$10^5 = 100{,}000$
⋮	⋮
120	$10^{12} = 1{,}000{,}000{,}000{,}000$



Peter Townsend of the Who sustained permanent hearing reduction due to the high decibel level of his band's music.

Example 5

In 1976 the rock group The Who set the record for the loudest concert: 120 dB. By comparison, a jackhammer positioned at the same spot as The Who would have produced a sound level of 92 dB. What is the ratio of the sound intensity of The Who to the sound intensity of a jackhammer?

Solution. Let I_1 and $\beta_1 (= 120$ dB$)$ denote the intensity and sound level of The Who, and let I_2 and β_2 $(= 92$ dB$)$ denote the intensity and sound level of the jackhammer. Then

$$I_1/I_2 = (I_1/I_0)/(I_2/I_0)$$
$$\log(I_1/I_2) = \log(I_1/I_0) - \log(I_2/I_0)$$
$$10\log(I_1/I_2) = 10\log(I_1/I_0) - 10\log(I_2/I_0)$$
$$10\log(I_1/I_2) = \beta_1 - \beta_2 = 120 - 92 = 28$$
$$\log(I_1/I_2) = 2.8$$

Thus, $I_1/I_2 = 10^{2.8} \approx 631$, which tells us that the sound intensity of The Who was 631 times greater than a jackhammer! ◀

EXPONENTIAL AND LOGARITHMIC GROWTH

The growth patterns of e^x and $\ln x$ illustrated by Table 4.2.3 are worth noting. Both functions increase as x increases, but they increase in dramatically different ways—e^x increases extremely rapidly and $\ln x$ increases extremely slowly. For example, at $x = 10$ the value of e^x is over 22,000, but at $x = 1000$ the value of $\ln x$ has not even reached 7.

The table strongly suggests that $e^x \to +\infty$ as $x \to +\infty$. However, the growth of $\ln x$ is so slow that its limiting behavior as $x \to +\infty$ is not clear from the table. However, in spite of its slow growth, it is still true that $\ln x \to +\infty$ as $x \to +\infty$. To see that this is so, choose any positive number M (as large as you like). The value of $\ln x$ will reach M when $x = e^M$, since

$$\ln x = \ln(e^M) = M$$

Table 4.2.3

x	e^x	$\ln x$
1	2.72	0.00
2	7.39	0.69
3	20.09	1.10
4	54.60	1.39
5	148.41	1.61
6	403.43	1.79
7	1096.63	1.95
8	2980.96	2.08
9	8103.08	2.20
10	22026.47	2.30
100	2.69×10^{43}	4.61
1000	1.97×10^{434}	6.91

Since $\ln x$ increases as x increases, we can conclude that $\ln x > M$ for $x > e^M$; hence, $\ln x \to +\infty$ as $x \to +\infty$ since the values of $\ln x$ eventually exceed any positive number M (Figure 4.2.5).

In summary,

$$\lim_{x\to+\infty} e^x = +\infty \qquad \lim_{x\to+\infty} \ln x = +\infty \tag{10–11}$$

The following limits, which are consistent with Figure 4.2.5, can be deduced numerically by constructing appropriate tables of values (verify):

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to 0^+} \ln x = -\infty \tag{12–13}$$

The following limits can be deduced numerically, but they can be seen more readily by noting that the graph of $y = e^{-x}$ is the reflection about the y-axis of the graph of $y = e^x$ (Figure 4.2.6):

$$\lim_{x\to+\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = +\infty \tag{14–15}$$

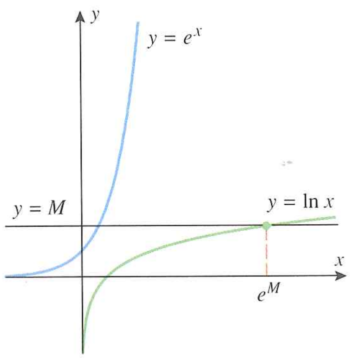


Figure 4.2.5

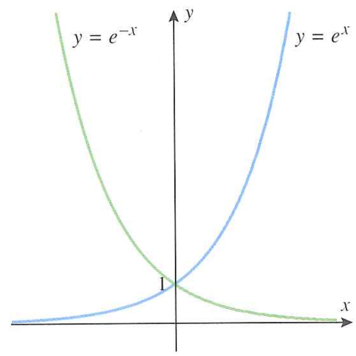


Figure 4.2.6

EXERCISE SET 4.2 Graphing Calculator

In Exercises 1 and 2, simplify the expression without using a calculating utility.

1. (a) $-8^{2/3}$ (b) $(-8)^{2/3}$ (c) $8^{-2/3}$

2. (a) 2^{-4} (b) $4^{1.5}$ (c) $9^{-0.5}$

In Exercises 3 and 4, use a calculating utility to approximate the expression. Round your answer to four decimal places.

3. (a) $2^{1.57}$ (b) $5^{-2.1}$

4. (a) $\sqrt[5]{24}$ (b) $\sqrt[8]{0.6}$

In Exercises 5 and 6, find the exact value of the expression without using a calculating utility.

5. (a) $\log_2 16$ (b) $\log_2\left(\frac{1}{32}\right)$
(c) $\log_4 4$ (d) $\log_9 3$

6. (a) $\log_{10}(0.001)$ (b) $\log_{10}(10^4)$
(c) $\ln(e^3)$ (d) $\ln(\sqrt{e})$

In Exercises 7 and 8, use a calculating utility to approximate the expression. Round your answer to four decimal places.

7. (a) $\log 23.2$ (b) $\ln 0.74$

8. (a) $\log 0.3$ (b) $\ln \pi$

In Exercises 9 and 10 use the logarithm properties in Theorem 4.2.3 to rewrite the expression in terms of r, s, and t, where $r = \ln a$, $s = \ln b$, and $t = \ln c$.

9. (a) $\ln a^2\sqrt{bc}$ (b) $\ln \dfrac{b}{a^3c}$

10. (a) $\ln \dfrac{\sqrt[3]{c}}{ab}$ (b) $\ln \sqrt{\dfrac{ab^3}{c^2}}$

In Exercises 11 and 12, expand the logarithm in terms of sums, differences, and multiples of simpler logarithms.

11. (a) $\log(10x\sqrt{x-3})$ (b) $\ln \dfrac{x^2 \sin^3 x}{\sqrt{x^2+1}}$

12. (a) $\log \dfrac{\sqrt[3]{x+2}}{\cos 5x}$ (b) $\ln \sqrt{\dfrac{x^2+1}{x^3+5}}$

In Exercises 13–15, rewrite the expression as a single logarithm.

13. $4\log 2 - \log 3 + \log 16$

14. $\frac{1}{2}\log x - 3\log(\sin 2x) + 2$

15. $2\ln(x+1) + \frac{1}{3}\ln x - \ln(\cos x)$

In Exercises 16–25, solve for x without using a calculating utility.

16. $\log_{10}(1+x) = 3$

17. $\log_{10}(\sqrt{x}) = -1$

18. $\ln(x^2) = 4$

19. $\ln(1/x) = -2$

20. $\log_3(3^x) = 7$

21. $\log_5(5^{2x}) = 8$

22. $\log_{10} x^2 + \log_{10} x = 30$

23. $\log_{10} x^{3/2} - \log_{10}\sqrt{x} = 5$

24. $\ln 4x - 3\ln(x^2) = \ln 2$

25. $\ln(1/x) + \ln(2x^3) = \ln 3$

In Exercises 26–31, solve for x without using a calculating utility. Use the natural logarithm anywhere that logarithms are needed.

26. $3^x = 2$

27. $5^{-2x} = 3$

28. $3e^{-2x} = 5$

29. $2e^{3x} = 7$

30. $e^x - 2xe^x = 0$

31. $xe^{-x} + 2e^{-x} = 0$

In Exercises 32 and 33, rewrite the given equation as a quadratic equation in u, where $u = e^x$; then solve for x.

32. $e^{2x} - e^x = 6$

33. $e^{-2x} - 3e^{-x} = -2$

In Exercises 34–36, sketch the graph of the equation without using a graphing utility.

34. (a) $y = 1 + \ln(x-2)$ (b) $y = 3 + e^{x-2}$

35. (a) $y = \left(\frac{1}{2}\right)^{x-1} - 1$ (b) $y = \ln|x|$

36. (a) $y = 1 - e^{-x+1}$ (b) $y = 3\ln\sqrt[3]{x-1}$

37. Use a calculating utility and the change of base formula (9) to find the values of $\log_2 7.35$ and $\log_5 0.6$, rounded to four decimal places.

In Exercises 38 and 39, graph the functions on the same screen of a graphing utility. [Use the change of base formula (9), where needed].

38. $y = \ln x,\ y = e^x,\ \log x,\ 10^x$

39. $y = \log_2 x,\ \ln x,\ \log_5 x,\ \log x$

40. (a) Derive the general change of base formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

(b) Use the result in part (a) to find the exact value of $(\log_2 81)(\log_3 32)$ without using a calculating utility. [*Hint:* Take $x = a$.]

41. Use a graphing utility to estimate where the graphs of $y = x^{0.2}$ and $y = \ln x$ intersect.

42. The United States public debt D, in billions of dollars, has been modeled as $D = 0.051517(1.1306727)^x$, where x is the number of years since 1900. Based on this model, when did the debt first reach one trillion dollars?

43. (a) Is the curve in the accompanying figure the graph of an exponential function? Explain your reasoning.
(b) Find the equation of an exponential function that passes through the point (4, 2).
(c) Find the equation of an exponential function that passes through the point $\left(2, \frac{1}{4}\right)$.
(d) Use a graphing utility to generate the graph of an exponential function that passes through the point (2, 5).

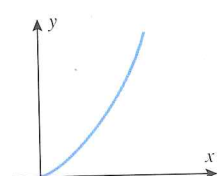


Figure Ex-43

44. (a) Make a conjecture about the general shape of the graph of $y = \log(\log x)$, and sketch the graph of this equation and $y = \log x$ in the same coordinate system.
(b) Check your work in part (a) with a graphing utility.

45. Find the fallacy in the following "proof" that $\frac{1}{8} > \frac{1}{4}$. Multiply both sides of the inequality $3 > 2$ by $\log \frac{1}{2}$ to get

$$\begin{aligned} 3\log\tfrac{1}{2} &> 2\log\tfrac{1}{2} \\ \log\left(\tfrac{1}{2}\right)^3 &> \log\left(\tfrac{1}{2}\right)^2 \\ \log\tfrac{1}{8} &> \log\tfrac{1}{4} \\ \tfrac{1}{8} &> \tfrac{1}{4} \end{aligned}$$

46. Prove the four algebraic properties of logarithms in Theorem 4.2.3.

47. If equipment in the satellite of Example 2 requires 15 watts to operate correctly, what is the operational lifetime of the power supply?

48. The equation $Q = 12e^{-0.055t}$ gives the mass Q in grams of radioactive potassium-42 that will remain from some initial quantity after t hours of radioactive decay.
(a) How many grams were there initially?
(b) How many grams remain after 4 hours?
(c) How long will it take to reduce the amount of radioactive potassium-42 to half of the initial amount?

49. The acidity of a substance is measured by its pH value, which is defined by the formula

$$\text{pH} = -\log[H^+]$$

where the symbol $[H^+]$ denotes the concentration of hydrogen ions measured in moles per liter. Distilled water has a pH of 7; a substance is called *acidic* if it has pH < 7 and *basic* if it has pH > 7. Find the pH of each of the following substances and state whether it is acidic or basic.

	SUBSTANCE	$[H^+]$
(a)	Arterial blood	3.9×10^{-8} mol/L
(b)	Tomatoes	6.3×10^{-5} mol/L
(c)	Milk	4.0×10^{-7} mol/L
(d)	Coffee	1.2×10^{-6} mol/L

50. Use the definition of pH in Exercise 49 to find $[H^+]$ in a solution having a pH equal to
(a) 2.44 (b) 8.06

51. The perceived loudness β of a sound in decibels (dB) is related to its intensity I in watts/square meter (W/m^2) by the equation

$$\beta = 10\log(I/I_0)$$

where $I_0 = 10^{-12}$ W/m^2. Damage to the average ear occurs at 90 dB or greater. Find the decibel level of each of the following sounds and state whether it will cause ear damage.

	SOUND	I
(a)	Jet aircraft (from 500 ft)	1.0×10^2 W/m^2
(b)	Amplified rock music	1.0 W/m^2
(c)	Garbage disposal	1.0×10^{-4} W/m^2
(d)	TV (mid volume from 10 ft)	3.2×10^{-5} W/m^2

In Exercises 52–54, use the definition of the decibel level of a sound (see Exercise 51).

52. If one sound is three times as intense as another, how much greater is its decibel level?

53. According to one source, the noise inside a moving automobile is about 70 dB, while an electric blender generates 93 dB. Find the ratio of the intensity of the noise of the blender to that of the automobile.

54. Suppose that the decibel level of an echo is $\frac{2}{3}$ the decibel level of the original sound. If each echo results in another echo, how many echoes will be heard from a 120-dB sound given that the average human ear can hear a sound as low as 10 dB?

55. On the ***Richter scale***, the magnitude M of an earthquake is related to the released energy E in joules (J) by the equation

$$\log E = 4.4 + 1.5M$$

(a) Find the energy E of the 1906 San Francisco earthquake that registered $M = 8.2$ on the Richter scale.
(b) If the released energy of one earthquake is 10 times that of another, how much greater is its magnitude on the Richter scale?

56. Suppose that the magnitudes of two earthquakes differ by 1 on the Richter scale. Find the ratio of the released energy of the larger earthquake to that of the smaller earthquake. [*Note:* See Exercise 55 for terminology.]

In Exercises 57 and 58, use Formula (3) or (5), as appropriate, to find the limit.

57. Find $\lim_{x\to 0}(1-2x)^{1/x}$. [*Hint:* Let $t = -2x$.]

58. Find $\lim_{x\to +\infty}(1+3/x)^x$. [*Hint:* Let $t = 3/x$.]

4.3 IMPLICIT DIFFERENTIATION

In earlier sections we were concerned with differentiating functions that were given by equations of the form $y = f(x)$. In this section we will consider methods for differentiating functions for which it is inconvenient or impossible to express them in this form.

FUNCTIONS DEFINED EXPLICITLY AND IMPLICITLY

Up to now, we have been concerned with differentiating functions that are expressed in the form $y = f(x)$. An equation of this form is said to define y ***explicitly*** as a function of x, because the variable y appears alone on one side of the equation. However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$yx + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$. However, this equation still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1}$$

Thus, we say that (1) defines y ***implicitly*** as a function of x, the function being

$$f(x) = \frac{x-1}{x+1}$$

An equation in x and y can implicitly define more than one function of x; for example, if we solve the equation

$$x^2 + y^2 = 1 \tag{2}$$

for y in terms of x, we obtain $y = \pm\sqrt{1-x^2}$, so we have found two functions that are defined implicitly by (2), namely

$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2} \tag{3}$$

The graphs of these functions are the upper and lower semicircles of the circle $x^2 + y^2 = 1$ (Figure 4.3.1).

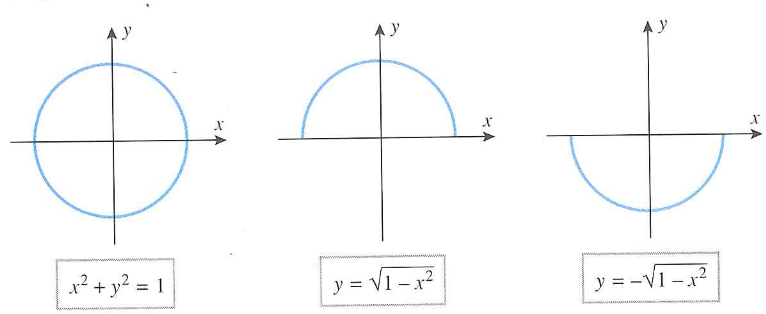


Figure 4.3.1

Observe that the complete circle $x^2 + y^2 = 1$ does not pass the vertical line test, and hence is not itself the graph of a function of x. However, the upper and lower semicircles (which are only portions of the entire circle) do pass the vertical line test, and hence are graphs of functions. In general, if we have an equation in x and y, then any segment of its graph that passes the vertical line test can be viewed as the graph of a function defined by the equation. Thus, we make the following definition.

> **4.3.1 DEFINITION.** We will say that a given equation in x and y defines the function f ***implicitly*** if the graph of $y = f(x)$ coincides with some segment of the graph of the equation.

Thus, for example, the equation $x^2 + y^2 = 1$ defines the functions $f_1(x) = \sqrt{1-x^2}$ and $f_2(x) = -\sqrt{1-x^2}$ implicitly, since the graphs of these functions are segments of the circle $x^2 + y^2 = 1$.

Sometimes it may be difficult or impossible to solve an equation in x and y for y in terms of x. For example, with persistence the equation

$$x^3 + y^3 = 3xy \tag{4}$$

can be solved for y in terms of x, but the algebra is tedious and the resulting formulas are complicated. On the other hand, the equation

$$\sin(xy) = y$$

cannot be solved for y in terms of x by any elementary method. Thus, even though an equation in x and y may define one or more functions of x, it may not be practical or possible to find explicit formulas for those functions.

GRAPHS OF EQUATIONS IN x AND y

When an equation in x and y cannot be solved for y in terms of x (or x in terms of y), it may be difficult or time-consuming to obtain even a rough sketch of the graph, so the graphing of such equations is usually best left for graphing utilities. In particular, the CAS programs *Mathematica* and *Maple* both have "implicit plot" capabilities for graphing such equations. For example, Figure 4.3.2 shows the graph of Equation (4), which is called the ***Folium of Descartes***.

FOR THE READER. Figure 4.3.3 shows the graphs of two functions (in solid color) that are defined implicitly by (4). Sketch some more.

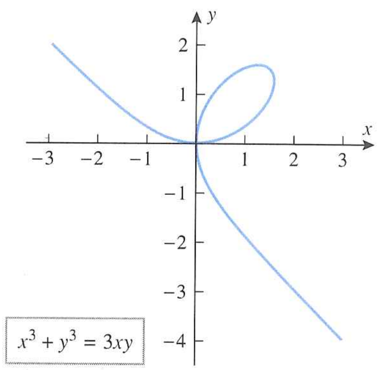


Figure 4.3.2

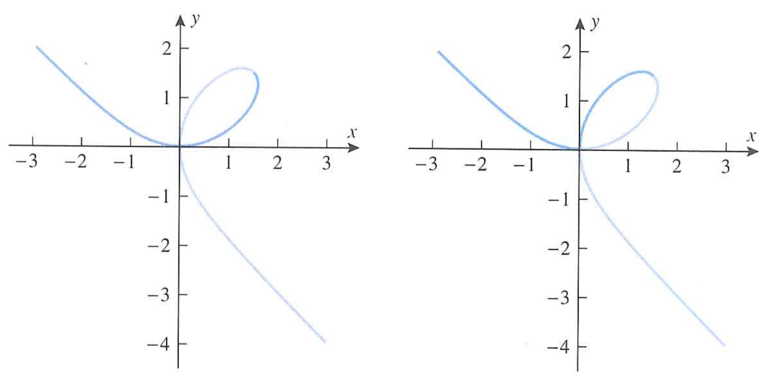


Figure 4.3.3

IMPLICIT DIFFERENTIATION

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. To illustrate this, let us consider the simple equation

$$xy = 1 \tag{5}$$

One way to find dy/dx is to rewrite this equation as

$$y = \frac{1}{x} \tag{6}$$

from which it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2} \tag{7}$$

However, there is another way to obtain this derivative. We can differentiate both sides of

(5) *before* solving for y in terms of x, treating y as a (temporarily unspecified) differentiable function of x. With this approach we obtain

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

$$x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

If we now substitute (6) into the last expression, we obtain

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

which agrees with (7). This method of obtaining derivatives is called ***implicit differentiation***.

Example 1

Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$

The chain rule was used here because y is a function of x.

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

Solving for dy/dx we obtain

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y} \tag{8}$$

Note that this formula involves both x and y. In order to obtain a formula for dy/dx that involves x alone, we would have to solve the original equation for y in terms of x and then substitute in (8). However, it is impossible to do this, so we are forced to leave the formula for dy/dx in terms of x and y. ◀

Example 2

Use implicit differentiation to find d^2y/dx^2 if $4x^2 - 2y^2 = 9$.

Solution. Differentiating both sides of $4x^2 - 2y^2 = 9$ implicitly yields

$$8x - 4y\frac{dy}{dx} = 0$$

from which we obtain

$$\frac{dy}{dx} = \frac{2x}{y} \tag{9}$$

Differentiating both sides of (9) implicitly yields

$$\frac{d^2y}{dx^2} = \frac{(y)(2) - (2x)(dy/dx)}{y^2} \tag{10}$$

Substituting (9) into (10) and simplifying using the original equation, we obtain

$$\frac{d^2y}{dx^2} = \frac{2y - 2x(2x/y)}{y^2} = \frac{2y^2 - 4x^2}{y^3} = -\frac{9}{y^3}$$ ◀

In Examples 1 and 2, the resulting formulas for dy/dx involved both x and y. Although it is usually more desirable to have the formula for dy/dx expressed in terms of x alone, having the formula in terms of x and y is not an impediment to finding slopes and equations of tangent lines provided the x- and y-coordinates of the point of tangency are known. This is illustrated in the following example.

Example 3

Find the slopes of the tangent lines at $(2, -1)$ and $(2, 1)$ to $y^2 - x + 1 = 0$.

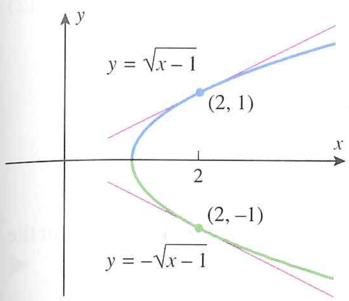


Figure 4.3.4

Solution. We could proceed by solving the equation for y in terms of x, and then evaluating the derivative of $y = \sqrt{x-1}$ at $(2, 1)$ and the derivative of $y = -\sqrt{x-1}$ at $(2, -1)$ (Figure 4.3.4). However, implicit differentiation is more efficient since it gives the slopes of *both* functions. Differentiating implicitly yields

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$

$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the tangent lines at those points are

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2}$$ ◀

Example 4

(a) Use implicit differentiation to find dy/dx for the Folium of Descartes $x^3 + y^3 = 3xy$.

(b) Find an equation for the tangent line to the Folium of Descartes at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(c) At what points is the tangent line to the Folium of Descartes horizontal?

Solution (a). Differentiating both sides of the given equation implicitly yields

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \tag{11}$$

Solution (b). At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, we have $x = \frac{3}{2}$ and $y = \frac{3}{2}$, so from (11) the slope $m_{\tan}$ of the tangent line at this point is

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=3/2\\y=3/2}} = \frac{(3/2) - (3/2)^2}{(3/2)^2 - (3/2)} = -1$$

Thus, the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$y - \tfrac{3}{2} = -1\left(x - \tfrac{3}{2}\right) \quad \text{or} \quad x + y = 3$$

which is consistent with Figure 4.3.5.

Solution (c). The tangent line is horizontal at the points where $dy/dx = 0$, and from (11) this occurs where $y - x^2 = 0$ or

$$y = x^2 \tag{12}$$

Substituting this expression for y in the equation $x^3 + y^3 = 3xy$ for the curve yields

$$x^3 + (x^2)^3 = 3x^3$$
$$x^6 - 2x^3 = 0$$
$$x^3(x^3 - 2) = 0$$

whose solutions are $x = 0$ and $x = 2^{1/3}$. Thus, from (12), the tangent line is horizontal at the points $(0, 0)$ and $(2^{1/3}, 2^{2/3}) \approx (1.26, 1.59)$, which is consistent with Figure 4.3.6. ◀

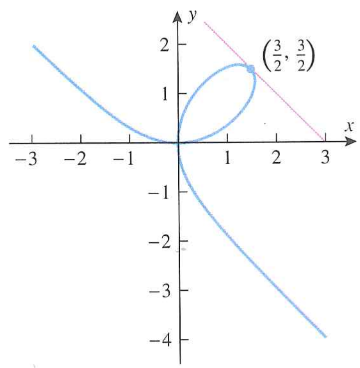


Figure 4.3.5

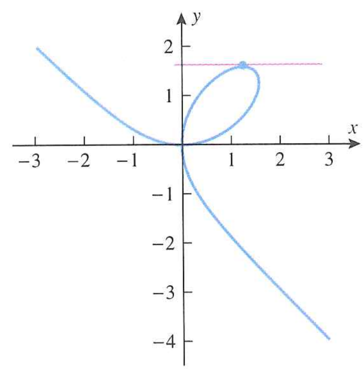


Figure 4.3.6

DIFFERENTIABILITY OF FUNCTIONS DEFINED IMPLICITLY

When differentiating implicitly, it is assumed that y represents a differentiable function of x. If this is not so, then the resulting calculations may be nonsense. For example, if we differentiate the equation

$$x^2 + y^2 + 1 = 0 \tag{13}$$

we obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}$$

However, this derivative is meaningless because (13) does not define a function of x. (The left side of the equation is greater than zero.)

Sometimes it is possible to identify points of nondifferentiability graphically. For example, the first function in Figure 4.3.3 is differentiable at each point of its domain because there are no corners, discontinuities, or points of vertical tangency; however, the second function is not differentiable at the origin.

In general, it can be difficult to determine analytically whether functions defined implicitly are differentiable, so we will leave such matters for more advanced courses.

DERIVATIVES OF RATIONAL POWERS OF x

In Theorem 3.3.8 and the discussion immediately following it, we showed that the formula

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{14}$$

holds for integer values of n and for $n = \frac{1}{2}$. We will now use implicit differentiation to show that this formula holds for any rational exponent. More precisely, we will show that if r is a rational number, then

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{15}$$

wherever x^r and x^{r-1} are defined. For now, we will assume without proof that x^r is differentiable; the justification for this will be considered later.

Let $y = x^r$. Since r is a rational number, it can be expressed as a ratio of integers $r = m/n$. Thus, $y = x^r = x^{m/n}$ can be written as

$$y^n = x^m \quad \text{so that} \quad \frac{d}{dx}[y^n] = \frac{d}{dx}[x^m]$$

By differentiating implicitly with respect to x and using (14), we obtain

$$ny^{n-1}\frac{dy}{dx} = mx^{m-1} \tag{16}$$

But

$$y^{n-1} = \left[x^{m/n}\right]^{n-1} = x^{m-(m/n)}$$

Thus, (16) can be written as

$$nx^{m-(m/n)}\frac{dy}{dx} = mx^{m-1}$$

so that

$$\frac{dy}{dx} = \frac{m}{n}x^{(m/n)-1} = rx^{r-1}$$

which establishes (15).

Example 5

From (15)

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$

$$\frac{d}{dx}[x^{-7/8}] = -\frac{7}{8}x^{(-7/8)-1} = -\frac{7}{8}x^{-15/8}$$

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

◀

If u is a differentiable function of x, and r is a rational number, then the chain rule yields the following generalization of (15):

$$\frac{d}{dx}[u^r] = ru^{r-1} \cdot \frac{du}{dx} \tag{17}$$

Example 6

$$\frac{d}{dx}\left[x^2 - x + 2\right]^{3/4} = \frac{3}{4}\left(x^2 - x + 2\right)^{-1/4} \cdot \frac{d}{dx}[x^2 - x + 2]$$

$$= \frac{3}{4}\left(x^2 - x + 2\right)^{-1/4}(2x - 1)$$

$$\frac{d}{dx}[(\sec \pi x)^{-4/5}] = -\frac{4}{5}(\sec \pi x)^{-9/5} \cdot \frac{d}{dx}[\sec \pi x]$$

$$= -\frac{4}{5}(\sec \pi x)^{-9/5} \cdot \sec \pi x \tan \pi x \cdot \pi$$

$$= -\frac{4\pi}{5}(\sec \pi x)^{-4/5} \tan \pi x$$

◀

DERIVATIVES OF INVERSE FUNCTIONS

We conclude this section with a brief discussion of the general relationship between the derivatives of f and f^{-1}. For this purpose, suppose that both functions are differentiable, and let

$$y = f^{-1}(x) \tag{18}$$

Rewriting this equation as

$$x = f(y) \tag{19}$$

and differentiating implicitly with respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[f(y)]$$

$$1 = f'(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)} \tag{20}$$

Thus, from (18) we obtain the following formula that relates the derivative of f^{-1} to the derivative of f.

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} \tag{21}$$

For example, if $f^{-1}(x) = \sqrt{x}$, then $f(x) = x^2$, so $f'(x) = 2x$; this formula implies that

$$\frac{d}{dx}[\sqrt{x}\,] = \frac{1}{2(f^{-1}(x))} = \frac{1}{2\sqrt{x}}$$

which is consistent with the known derivative formula for $\sqrt{x}$.

An alternative version of Formula (21) that uses dependent variables can be obtained by using (19) to rewrite $f'(y)$ as dx/dy, in which case (21) becomes

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{22}$$

For example, if $y = \sqrt{x}$, then $x = y^2$. Thus, $dx/dy = 2y$, and (22) implies that

$$\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

which again is consistent with the known derivative formula for $y = \sqrt{x}$.

If an explicit formula can be obtained for the inverse of a function, then the differentiability and the derivative of the inverse can usually be deduced from that formula. However, if no explicit formula for the inverse can be obtained, then Theorem 4.1.7 is the primary mathematical tool for establishing differentiability of the inverse. Once differentiability has been established, a formula for the derivative of the inverse can be obtained either by differentiating implicitly or by using Formulas (21) or (22). The following example illustrates this.

Example 7

We showed in Example 9 of Section 4.1 that the inverse of the function $f(x) = x^5 + x + 1$ is differentiable on the interval $(-\infty, +\infty)$. However, there is no way to obtain an explicit formula for f^{-1}, so we must resort to indirect methods to differentiate this function.

(a) Find the derivative of f^{-1} by using Formula (22).

(b) Find the derivative of f^{-1} by differentiating implicitly.

Solution (a). If we let $y = f^{-1}(x)$, then

$$x = f(y) = y^5 + y + 1 \tag{23}$$

from which it follows that

$$\frac{dx}{dy} = 5y^4 + 1$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{5y^4 + 1} \tag{24}$$

Although it would be preferable to have dy/dx expressed as a function of x, we are forced to leave it in terms of y, since we cannot solve (23) for y in terms of x.

Solution (b). Differentiating (23) implicitly with respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = 5y^4\frac{dy}{dx} + \frac{dy}{dx}$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1}$$

which agrees with (24). ◀

EXERCISE SET 4.3 [C] CAS

In Exercises 1–8, find dy/dx.

1. $y = \sqrt[3]{2x - 5}$
2. $y = \sqrt[3]{2 + \tan(x^2)}$
3. $y = \left(\frac{x-1}{x+2}\right)^{3/2}$
4. $y = \sqrt{\frac{x^2+1}{x^2-5}}$
5. $y = x^3\left(5x^2 + 1\right)^{-2/3}$
6. $y = \frac{(3-2x)^{4/3}}{x^2}$
7. $y = [\sin(3/x)]^{5/2}$
8. $y = \left[\cos(x^3)\right]^{-1/2}$

In Exercises 9 and 10: (a) Find dy/dx by differentiating implicitly. (b) Solve the equation for y as a function of x, and find dy/dx from that equation. (c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone.

9. $x^3 + xy - 2x = 1$
10. $\sqrt{y} - e^x = 2$

In Exercises 11–20, find dy/dx by implicit differentiation.

11. $x^2 + y^2 = 100$
12. $x^3 - y^3 = 6xy$
13. $x^2y + 3xy^3 - x = 3$
14. $x^3y^2 - 5x^2y + x = 1$
15. $\frac{1}{y} + \frac{1}{x} = 1$
16. $x^2 = \frac{x+y}{x-y}$
17. $\sin(x^2y^2) = x$
18. $x^2 = \frac{\cot y}{1 + \csc y}$
19. $\tan^3(xy^2 + y) = x$
20. $\frac{xy^3}{1 + \sec y} = 1 + y^4$

In Exercises 21–26, find d^2y/dx^2 by implicit differentiation.

21. $3x^2 - 4y^2 = 7$
22. $x^3 + y^3 = 1$
23. $x^3y^3 - 4 = 0$
24. $2xy - y^2 = 3$
25. $y + \sin y = x$
26. $x\cos y = y$

In Exercises 27 and 28, find the slope of the tangent line to the curve at the given points in two ways: first by solving for y in terms of x and differentiating and then by implicit differentiation.

27. $x^2 + y^2 = 1$; $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$
28. $y^2 - x + 1 = 0$; $(10, 3)$, $(10, -3)$

In Exercises 29–32, use implicit differentiation to find the slope of the tangent line to the curve at the specified point, and check that your answer is consistent with the accompanying graph.

29. $x^4 + y^4 = 16$; $(1, \sqrt[4]{15})$ [***Lamé's special quartic***]

30. $y^3 + yx^2 + x^2 - 3y^2 = 0$; $(0, 3)$ [***trisectrix***]

31. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$; $(3, 1)$ [***lemniscate***]

32. $x^{2/3} + y^{2/3} = 4$; $(-1, 3\sqrt{3})$ [***four-cusped hypocycloid***]

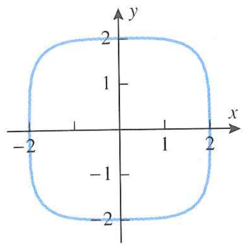


Figure Ex-29

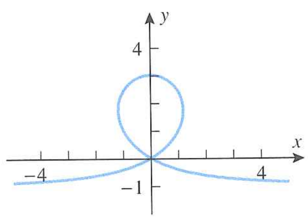


Figure Ex-30

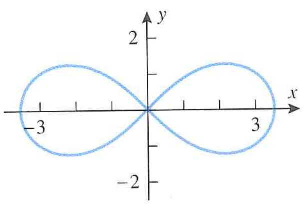


Figure Ex-31

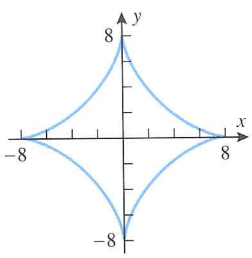


Figure Ex-32

C **33.** If you have a CAS, read the documentation on "implicit plotting," and then generate the four curves in Exercises 29–32.

C **34.** Curves with equations of the form $y^2 = x(x - a)(x - b)$, where $a < b$ are called ***bipartite cubics***.
(a) Use the implicit plotting capability of a CAS to graph the bipartite cubic $y^2 = x(x - 1)(x - 2)$.
(b) At what points does the curve in part (a) have a horizontal tangent line?
(c) Solve the equation in part (a) for y in terms of x, and use the result to explain why the graph consists of two separate parts (i.e., is *bipartite*).
(d) Graph the equation in part (a) without using the implicit plotting capability of the CAS.

C **35.** (a) Use the implicit plotting capability of a CAS to graph the rotated ellipse $x^2 - xy + y^2 = 4$.
(b) Use the graph to estimate the x-coordinates of all horizontal tangent lines.
(c) Find the exact values for the x-coordinates in part (b).

In Exercises 36–39, use implicit differentiation to find the specified derivative.

36. $\sqrt{u} + \sqrt{v} = 5$; du/dv

37. $a^4 - t^4 = 6a^2 r$; da/dt

38. $y = \sin x$; dx/dy.

39. $a^2\omega^2 + b^2\lambda^2 = 1$ (a, b constants); $d\omega/d\lambda$

40. At what point(s) is the tangent line to the curve $y^2 = 2x^3$ perpendicular to the line $4x - 3y + 1 = 0$?

41. Find the values of a and b for the curve $x^2y + ay^2 = b$ if the point $(1, 1)$ is on its graph and the tangent line at $(1, 1)$ has the equation $4x + 3y = 7$.

42. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the x-axis.

43. Find equations for two lines through the origin that are tangent to the curve $x^2 - 4x + y^2 + 3 = 0$.

44. Use implicit differentiation to show that the equation of the tangent line to the curve $y^2 = kx$ at (x_0, y_0) is

$$y_0 y = \tfrac{1}{2}k(x + x_0)$$

45. Find dy/dx if

$$2y^3 t + t^3 y = 1 \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{\cos t}$$

In Exercises 46 and 47, find dy/dt in terms of x, y, and dx/dt, assuming that x and y are differentiable functions of the variable t. [*Hint:* Differentiate both sides of the given equation with respect to t.]

46. $x^3y^2 + y = 3$

47. $xy^2 = \sin 3x$

48. (a) Show that $f(x) = x^{4/3}$ is differentiable at 0, but not twice differentiable at 0.
(b) Show that $f(x) = x^{7/3}$ is twice differentiable at 0, but not three times differentiable at 0.
(c) Find an exponent k such that $f(x) = x^k$ is $(n-1)$ times differentiable at 0, but not n times differentiable at 0.

In Exercises 49 and 50, find all rational values of r such that $y = x^r$ satisfies the given equation.

49. $3x^2y'' + 4xy' - 2y = 0$

50. $16x^2y'' + 24xy' + y = 0$

Two curves are said to be ***orthogonal*** if their tangent lines are perpendicular at each point of intersection, and two families of curves are said to be ***orthogonal trajectories*** of one another if each member of one family is orthogonal to each member of the other family. This terminology is used in Exercises 51 and 52.

51. The accompanying figure shows some typical members of the families of circles $x^2 + (y - c)^2 = c^2$ (black curves) and $(x - k)^2 + y^2 = k^2$ (gray curves). Show that these families are orthogonal trajectories of one another. [*Hint:* For the tangent lines to be perpendicular at a point of intersection, the slopes of those tangent lines must be negative reciprocals of one another.]

52. The accompanying figure shows some typical members of the families of hyperbolas $xy = c$ (black curves) and $x^2 - y^2 = k$ (gray curves), where $c \neq 0$ and $k \neq 0$. Use the hint in Exercise 51 to show that these families are orthogonal trajectories of one another.

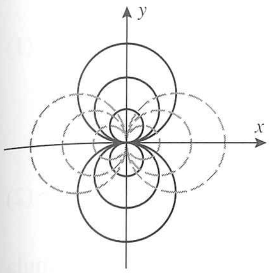


Figure Ex-51

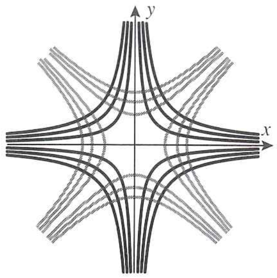


Figure Ex-52

In Exercises 53–56, find the derivative of f^{-1} by using Formula (22), and check your result by differentiating implicitly.

53. $f(x) = 5x^3 + x - 7$

54. $f(x) = 1/x^2,\ x > 0$

55. $f(x) = 2x^5 + x^3 + 1$

56. $f(x) = 5x - \sin 2x$

4.4 DERIVATIVES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this section we will obtain derivative formulas for logarithmic and exponential functions, and we will discuss the general relationship between the derivative of a one-to-one function and its inverse.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

The natural logarithm plays a special role in calculus that can be motivated by differentiating $\log_b x$, where b is an arbitrary base. For this purpose, we will *assume* that $\log_b x$ is differentiable, and hence continuous, for $x > 0$. We will also need the limit

$$\lim_{v \to 0} (1 + v)^{1/v} = e$$

that was given in Formula (5) of Section 4.2 (with x rather than v as the variable).

Using the definition of a derivative, we obtain

$$\frac{d}{dx}[\log_b x] = \lim_{h \to 0} \frac{\log_b(x+h) - \log_b x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \log_b\left(\frac{x+h}{x}\right)$$ The quotient property of logarithms in Theorem 4.2.3

$$= \lim_{h \to 0} \frac{1}{h} \log_b\left(1 + \frac{h}{x}\right)$$

$$= \lim_{v \to 0} \frac{1}{vx} \log_b(1 + v)$$ Let $v = h/x$ and note that $v \to 0$ as $h \to 0$.

$$= \frac{1}{x} \lim_{v \to 0} \frac{1}{v} \log_b(1 + v)$$ $1/x$ does not vary with v, so it can be moved through the limit sign.

$$= \frac{1}{x} \lim_{v \to 0} \log_b(1 + v)^{1/v}$$ The power property of logarithms in Theorem 4.2.3

$$= \frac{1}{x} \log_b\left[\lim_{v \to 0} (1 + v)^{1/v}\right]$$ $\log_b x$ is continuous, so we can move the limit through the function symbol.

$$= \frac{1}{x} \log_b e$$

Thus,

$$\frac{d}{dx}[\log_b x] = \frac{1}{x} \log_b e, \quad x > 0$$

But from Formula (9) of Section 4.2 we have $\log_b e = 1/\ln b$, so we can rewrite this derivative formula as

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \quad x > 0 \tag{1}$$

In the special case where $b = e$, we have $\log_b e = \ln e = 1$, so this formula becomes

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0 \tag{2}$$

Thus, among all possible bases, the base $b = e$ produces the simplest derivative formula for $\log_b x$. This is one of the reasons why the natural logarithm function is preferred over other logarithms in calculus.

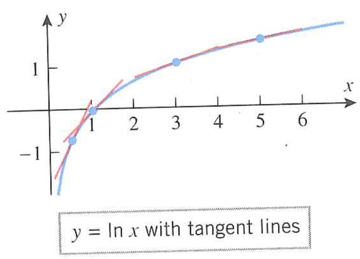


Figure 4.4.1

Example 1

(a) Figure 4.4.1 shows the graph of $y = \ln x$ and its tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5. Find the slopes of those tangent lines.

(b) Do you think that the graph of $y = \ln x$ has any horizontal tangent lines? Use the derivative of $\ln x$ to justify your answer.

Solution (a). From (2), the slopes of the tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5 are $1/x = 2, 1, \frac{1}{3}$, and $\frac{1}{5}$, which is consistent with Figure 4.4.1.

Solution (b). From the graph of $y = \ln x$, it does not appear that there are any horizontal tangent lines. This is confirmed by the fact that $dy/dx = 1/x$ is not equal to zero for any real value of x. ◀

If u is a differentiable function of x, and if $u(x) > 0$, then applying the chain rule to (1) and (2) produces the following generalized derivative formulas:

$$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \tag{3–4}$$

Example 2

Find $\dfrac{d}{dx}[\ln(x^2 + 1)]$.

Solution. From (4) with $u = x^2 + 1$,

$$\frac{d}{dx}[\ln(x^2 + 1)] = \frac{1}{x^2 + 1} \cdot \frac{d}{dx}[x^2 + 1] = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}$$ ◀

When possible, the properties of logarithms in Theorem 4.2.3 should be used to convert products, quotients, and exponents into sums, differences, and constant multiples *before* differentiating a function involving logarithms.

Example 3

$$\frac{d}{dx}\left[\ln\left(\frac{x^2 \sin x}{\sqrt{1 + x}}\right)\right] = \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1 + x)\right]$$
$$= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1 + x)}$$
$$= \frac{2}{x} + \cot x - \frac{1}{2 + 2x}$$ ◀

Example 4

Find $\dfrac{d}{dx}[\ln |x|]$.

Solution. The function $\ln|x|$ is defined for all x, except $x = 0$; we will consider the cases $x > 0$ and $x < 0$ separately.

If $x > 0$, then $|x| = x$, so

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

If $x < 0$, then $|x| = -x$, so from (4) we have

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)} \cdot \frac{d}{dx}[-x] = \frac{1}{x}$$

Since the same formula results in both cases, we have shown that

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{if } x \neq 0 \tag{5}$$

Example 5

From (5) and the chain rule,

$$\frac{d}{dx}[\ln|\sin x|] = \frac{1}{\sin x} \cdot \frac{d}{dx}[\sin x] = \frac{\cos x}{\sin x} = \cot x$$ ◀

LOGARITHMIC DIFFERENTIATION

We now consider a technique called ***logarithmic differentiation*** that is useful for differentiating functions that are composed of products, quotients, and powers.

Example 6

The derivative of

$$y = \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4} \tag{6}$$

is messy to calculate directly. However, if we first take the natural logarithm of both sides and then use its properties, we can write

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x - 14) - 4\ln(1 + x^2)$$

Differentiating both sides with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x - 14} - \frac{8x}{1 + x^2} \tag{7}$$

Thus, on solving for dy/dx and using (6) we obtain

$$\frac{dy}{dx} = \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4}\left[\frac{2}{x} + \frac{1}{3x - 6} - \frac{8x}{1 + x^2}\right] \tag{8}$$

REMARK. Since $\ln y$ is defined only for $y > 0$, logarithmic differentiation of $y = f(x)$ is valid only on intervals where $f(x)$ is positive. Thus, the derivative obtained in the preceding example is valid on the interval $(2, +\infty)$, since the given function is positive for $x > 2$. However, the formula is actually valid on the interval $(-\infty, 2)$ as well. This can be seen by taking absolute values before proceeding with the logarithmic differentiation and noting that $\ln|y|$ is defined for all y except $y = 0$. If we do this and simplify using properties of logarithms and absolute values, we obtain

$$\ln|y| = 2\ln|x| + \tfrac{1}{3}\ln|7x - 14| - 4\ln|1 + x^2|$$

Differentiating both sides with respect to x yields (7), and hence results in (8).

In general, if the derivative of $y = f(x)$ is to be obtained by logarithmic differentiation, then the same formula for dy/dx will result regardless of whether one first takes absolute values or not. Thus, a derivative formula obtained by logarithmic differentiation will be valid except perhaps at points where $f(x)$ is zero. The formula may, in fact, be valid at those points as well, but it is not guaranteed.

DERIVATIVES OF IRRATIONAL POWERS OF x

We know from Formula (15) of Section 4.3 that the differentiation formula

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{9}$$

holds for rational values of r. We will now use logarithmic differentiation to show that this formula holds if r is *any* real number (rational or irrational). In our computations we will assume that x^r is a differentiable function and that the familiar laws of exponents hold for real exponents.

Let $y = x^r$, where r is a real number. The derivative dy/dx can be obtained by logarithmic differentiation as follows:

$$\ln y = \ln x^r = r \ln x$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[r \ln x]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x}y = \frac{r}{x}x^r = rx^{r-1}$$

which establishes (9) for real values of r. Thus, for example,

$$\frac{d}{dx}[x^\pi] = \pi x^{\pi-1} \quad \text{and} \quad \frac{d}{dx}\left[x^{\sqrt{2}}\right] = \sqrt{2}x^{\sqrt{2}-1} \tag{10}$$

DERIVATIVES OF EXPONENTIAL FUNCTIONS

To obtain a derivative formula for the exponential function

$$y = b^x \tag{11}$$

we rewrite this equation as

$$x = \log_b y$$

and differentiate implicitly using (3) to obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

which we can rewrite using (11) as

$$\frac{dy}{dx} = y \ln b = b^x \ln b$$

Thus, we have shown that if b^x is a differentiable function, then its derivative with respect to x is

$$\frac{d}{dx}[b^x] = b^x \ln b \tag{12}$$

In the special case where $b = e$ we have $\ln e = 1$, so that (12) becomes

$$\frac{d}{dx}[e^x] = e^x \tag{13}$$

Moreover, if u is a differentiable function of x, then it follows from (12) and (13) that

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx} \tag{14–15}$$

REMARK. It is important to distinguish between differentiating b^x (variable exponent and constant base) and x^b (variable base and constant exponent). For example, compare the derivative of x^π in (10) to the following derivative of π^x, which is obtained from (12):

$$\frac{d}{dx}[\pi^x] = \pi^x \ln \pi$$

Example 7

The following computations use (14) and (15).

$$\frac{d}{dx}[2^{\sin x}] = (2^{\sin x})(\ln 2) \cdot \frac{d}{dx}[\sin x] = (2^{\sin x})(\ln 2)(\cos x)$$

$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$

$$\frac{d}{dx}\left[e^{x^3}\right] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2e^{x^3}$$

$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x)e^{\cos x}$$ ◀

Example 8

A glass of lemonade with a temperature of 40°F sits in a room whose temperature is a constant 70°F. Using a principle of physics, called ***Newton's Law of Cooling***, one can show that if the temperature of the lemonade reaches 52°F in 1 hour, then the temperature T of the lemonade as a function of the elapsed time t is modeled approximately by the equation

$$T = 70 - 30e^{-0.5t}$$

where T is in °F and t is in hours. The graph of this equation, shown in Figure 4.4.2, confirms our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room.

(a) In words, what happens to the *rate* of temperature rise over time?

(b) Use a derivative to confirm your conclusion.

Solution (a). The rate of change of temperature with respect to time is the slope of the tangent line to the graph of T versus t. As t increases, these slopes decrease, so the temperature rises at an ever-decreasing rate.

Solution (b). The rate of change of temperature with respect to time is

$$\frac{dT}{dt} = \frac{d}{dt}[70 - 30e^{-0.5t}] = -30(-0.5)e^{-0.5t} = 15e^{-0.5t}$$

As t increases, this derivative decreases, which confirms the conclusion in part (a). ◀

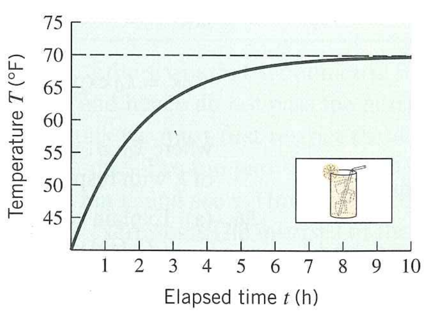


Figure 4.4.2

Exercise Set 4.4 Graphing Calculator

In Exercises 1–30, find dy/dx.

1. $y = \ln 2x$
2. $y = \ln(x^3)$
3. $y = (\ln x)^2$
4. $y = \ln(\sin x)$
5. $y = \ln|\tan x|$
6. $y = \ln(2 + \sqrt{x})$
7. $y = \ln\left(\dfrac{x}{1+x^2}\right)$
8. $y = \ln(\ln x)$
9. $y = \ln|x^3 - 7x^2 - 3|$
10. $y = x^3 \ln x$
11. $y = \sqrt{\ln x}$
12. $y = \sqrt{1 + \ln^2 x}$
13. $y = \cos(\ln x)$
14. $y = \sin^2(\ln x)$
15. $y = x^3 \log_2(3 - 2x)$
16. $y = x\left[\log_2(x^2 - 2x)\right]^3$
17. $y = \dfrac{x^2}{1 + \log x}$
18. $y = \dfrac{\log x}{1 + \log x}$
19. $y = e^{7x}$
20. $y = e^{-5x^2}$
21. $y = x^3 e^x$
22. $y = e^{1/x}$
23. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
24. $y = \sin(e^x)$
25. $y = e^{x \tan x}$
26. $y = \dfrac{e^x}{\ln x}$
27. $y = e^{(x - e^{3x})}$
28. $y = \exp(\sqrt{1 + 5x^3})$
29. $y = \ln(1 - xe^{-x})$
30. $y = \ln(\cos e^x)$

In Exercises 31 and 32, find dy/dx by implicit differentiation.

31. $y + \ln xy = 1$
32. $y = \ln(x \tan y)$

In Exercises 33 and 34, use the method of Example 3 to help perform the indicated differentiation.

33. $\dfrac{d}{dx}\left[\ln \dfrac{\cos x}{\sqrt{4 - 3x^2}}\right]$
34. $\dfrac{d}{dx}\left[\ln \sqrt{\dfrac{x-1}{x+1}}\right]$

In Exercises 35–38, find dy/dx using the method of logarithmic differentiation.

35. $y = x\sqrt[3]{1 + x^2}$
36. $y = \sqrt[5]{\dfrac{x-1}{x+1}}$
37. $y = \dfrac{(x^2 - 8)^{1/3}\sqrt{x^3 + 1}}{x^6 - 7x + 5}$
38. $y = \dfrac{\sin x \cos x \tan^3 x}{\sqrt{x}}$

In Exercises 39–42, find $f'(x)$ by Formula (14) and then by logarithmic differentiation.

39. $f(x) = 2^x$
40. $f(x) = 3^{-x}$
41. $f(x) = \pi^{\sin x}$
42. $f(x) = \pi^{x \tan x}$

In Exercises 43–46, find dy/dx using the method of logarithmic differentiation.

43. $y = (x^3 - 2x)^{\ln x}$
44. $y = x^{\sin x}$
45. $y = (\ln x)^{\tan x}$
46. $y = (x^2 + 3)^{\ln x}$

47. Show that for any constants A and B, the function

$$y = Ae^{2x} + Be^{-4x}$$

satisfies the equation

$$y'' + 2y' - 8y = 0$$

48. Show that for any constants A and k, the function $y = Ae^{kt}$ satisfies the equation $dy/dt = ky$.

49. Let $f(x) = e^{kx}$ and $g(x) = e^{-kx}$. Find
(a) $f^{(n)}(x)$ (b) $g^{(n)}(x)$.

50. Find dy/dt if $y = e^{-\lambda t}(A \sin \omega t + B \cos \omega t)$, where A, B, λ, and ω are constants.

51. Find $f'(x)$ if

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right]$$

where μ and σ are constants and $\sigma \neq 0$.

52. Show that
(a) $y = xe^{-x}$ satisfies the equation $xy' = (1 - x)y$
(b) $y = xe^{-x^2/2}$ satisfies the equation $xy' = (1 - x^2)y$.

53. Find
(a) $\dfrac{d}{dx}[\log_x e]$ (b) $\dfrac{d}{dx}[\log_x 2]$.

54. Recall from Section 4.2 that the loudness β of a sound in decibels (db) is given by $\beta = 10 \log(I/I_0)$, where I is the intensity of the sound in watts per square meter (W/m^2) and I_0 is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of β with respect to I at the point where
(a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$

55. The equilibrium constant k of a balanced chemical reaction changes with the absolute temperature T according to the law

$$k = k_0 \exp\left(-\frac{q(T - T_0)}{2T_0 T}\right)$$

where k_0, q, and T_0 are constants. Find the rate of change of k with respect to T.

56. (a) Explain why Formula (12) cannot be used to find $(d/dx)[x^x]$.
(b) Find this derivative by logarithmic differentiation.

57. Find $f'(x)$ if $f(x) = x^e$.

58. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

In Exercises 59 and 60, find the limit by interpreting the expression as an appropriate derivative.

59. (a) $\lim_{h\to 0} \dfrac{\ln(1+h)}{h}$ (b) $\lim_{h\to 0} \dfrac{10^h - 1}{h}$

60. (a) $\lim_{h\to 0} \dfrac{\ln(e^2+h) - 2}{h}$ (b) $\lim_{x\to 1} \dfrac{2^x - 2}{x - 1}$

61. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
(b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
(c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
(d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain your reasoning.
(e) Find the exact x-coordinates of all horizontal tangent lines to the curve.

62. (a) Use a graphing utility to graph the function

$$f(x) = 2x^3 + x^2 - 20x + 4$$

over the interval $-5 < x < 5$.
(b) Working with the graph in part (a), make a rough sketch of the graph of $f'(x)$ over the interval $-5 < x < 5$.
(c) Check your work in part (b) by generating the graph of $f'(x)$ with a graphing utility.
(d) Find the exact locations of the horizontal tangent lines to the graph of f over the interval $-5 < x < 5$.
(e) Confirm that the result in part (d) is consistent with the graph of $f'(x)$ in part (c).

63. (a) Sketch the curves $y = e^x$ and $y = -e^x$ in the same coordinate system; then make a conjecture about the general shape of the equation $y = e^x \cos \pi x$ for $x \geq 0$, and sketch its graph in the same coordinate system as the two exponential functions.
(b) Check your conjecture in part (a) by using a graphing utility to generate the graphs of $y = e^x$, $y = -e^x$, and $y = e^x \cos \pi x$ in the same window for $0 \leq x \leq 3$.

64. Suppose that the population of oxygen-dependent bacteria in a pond is modeled by the equation

$$P(t) = \frac{60}{5 + 7e^{-t}}$$

where $P(t)$ is the population (in billions) t days after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time? Check your conclusion by finding $\lim_{t\to +\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

65. Suppose that the population of deer on an island is modeled by the equation

$$P(t) = \frac{95}{5 - 4e^{-t/4}}$$

where $P(t)$ is the number of deer t weeks after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to +\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

4.5 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle whose trigonometric functions are known. As you may recall, problems of this type involve the computation of "arc functions" such as $\arcsin x$, $\arccos x$, $\arctan x$, *and so forth. In this section we will consider this idea from the viewpoint of inverse functions, with the goal of developing derivative formulas for the inverse trigonometric functions.*

INVERSE TRIGONOMETRIC FUNCTIONS

None of the six basic trigonometric functions is one-to-one because they all repeat periodically and hence do not pass the horizontal line test. Thus, to define inverse trigonometric functions we must first restrict the domains of the trigonometric functions to make them one-to-one. The top part of Figure 4.5.1 shows how these restrictions are made for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. (Inverses of $\cot x$ and $\csc x$ are of lesser importance and will be left for the exercises.) The inverses of these restricted functions are denoted by

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \sec^{-1} x$$

(or alternatively by $\arcsin x$, $\arccos x$, $\arctan x$, $\arcsec x$) and are defined as follows:

4.5.1 DEFINITION. The ***inverse sine function***, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\pi/2 \le x \le \pi/2$$

4.5.2 DEFINITION. The ***inverse cosine function***, denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$

4.5.3 DEFINITION. The ***inverse tangent function***, denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\pi/2 < x < \pi/2$$

4.5.4 DEFINITION.* The ***inverse secant function***, denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \text{ with } x \ne \pi/2$$

REMARK. The notations $\sin^{-1} x$, $\cos^{-1} x, \ldots$ are reserved exclusively for the inverse trigonometric functions and are not used for reciprocals of the trigonometric functions. For example, to denote the reciprocal $1/\sin x$ in exponent form, we would write $(\sin x)^{-1}$ and *never* $\sin^{-1} x$.

The graphs of the inverse trigonometric functions, which are shown in the bottom part of Figure 4.5.1, are obtained by reflecting the graphs in the top part of the figure about the line $y = x$. If you have trouble visualizing these relationships, then look at Figure 4.5.2

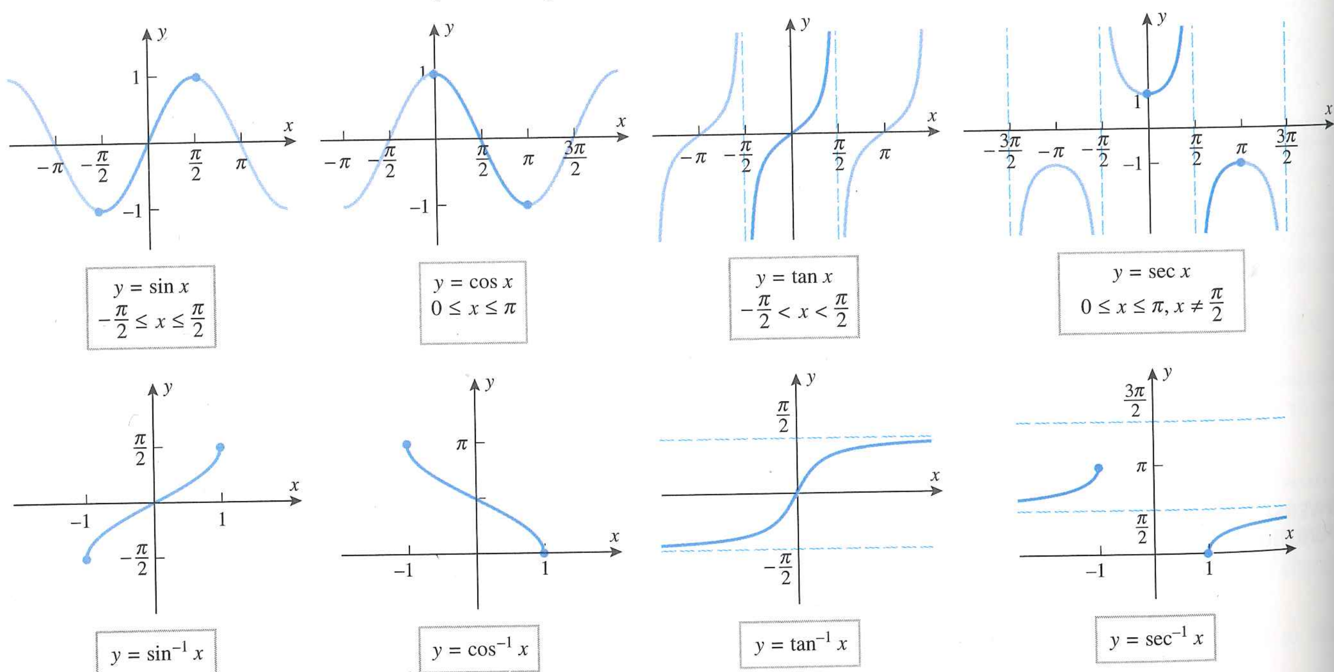


Figure 4.5.1

*There is no universal agreement on the definition of $\sec^{-1} x$, and some mathematicians prefer to restrict the domain of $\sec x$ so that $0 \le x < \pi/2$ or $\pi \le x < 3\pi/2$, which was the definition used in earlier editions of this text. Each definition has advantages and disadvantages, but we have changed to the current definition to conform with the conventions used by the CAS programs *Mathematica*, *Maple*, and *Derive*.

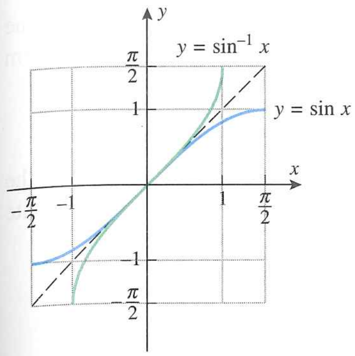


Figure 4.5.2

for a more detailed illustration for the inverse sine. It may also help to keep in mind that reflection about $y = x$ converts vertical lines to horizontal lines, and vice versa, and that x-intercepts reflect into y-intercepts, and vice versa.

Table 4.5.1 summarizes the basic properties of the inverse sine, cosine, tangent, and secant functions. You should confirm that the domains and ranges listed in this table are consistent with the graphs in the bottom part of Figure 4.5.1.

Table 4.5.1

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

EVALUATING INVERSE TRIGONOMETRIC FUNCTIONS

It follows from Theorem 4.1.2 that the equations $y = \sin^{-1} x$ and $x = \sin y$ are equivalent provided (of course) that y is in the domain of the restricted sine function and x is in the domain of the inverse sine function; that is, $-\pi/2 \le y \le \pi/2$ and $-1 \le x \le 1$. Thus,

$$y = \sin^{-1} x \text{ is equivalent to } \sin y = x \text{ if } \begin{cases} -1 \le x \le 1 \\ -\pi/2 \le y \le \pi/2 \end{cases}$$

Similarly,

$$y = \cos^{-1} x \text{ is equivalent to } \cos y = x \text{ if } \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \pi \end{cases}$$

$$y = \tan^{-1} x \text{ is equivalent to } \tan y = x \text{ if } \begin{cases} -\infty < x < +\infty \\ -\pi/2 < y < \pi/2 \end{cases}$$

$$y = \sec^{-1} x \text{ is equivalent to } \sec y = x \text{ if } \begin{cases} x \ge 1 \\ 0 \le y < \pi/2 \end{cases} \text{ or } \begin{cases} x \le -1 \\ \pi/2 < y \le \pi \end{cases}$$

A common problem in trigonometry is to find an angle whose sine is known. For example, you might want to find an angle θ in radian measure such that

$$\sin\theta = \tfrac{1}{2} \tag{1}$$

and, more generally, for a given value of y in the interval $-1 \le y \le 1$ you might want to solve the equation

$$\sin\theta = y \tag{2}$$

Because $\sin\theta$ repeats periodically, such equations have infinitely many solutions for θ; however, if we solve this equation as

$$\theta = \sin^{-1} y$$

then we isolate the specific solution that lies in the interval $[-\pi/2, \pi/2]$, since this is the range of the inverse sine. For example, Figure 4.5.3 shows four solutions of Equation (1), namely, $-11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. Of these, $\pi/6$ is the solution in the interval $[-\pi/2, \pi/2]$, so

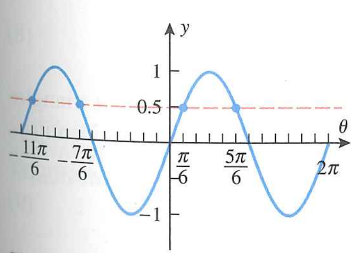


Figure 4.5.3

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \pi/6 \tag{3}$$

FOR THE READER. Refer to the documentation for your calculating utility to determine how to calculate inverse sines, inverse cosines, and inverse tangents; and then confirm Equation (3) numerically by showing that

$$\sin^{-1}(0.5) \approx 0.523598775598\ldots \approx \pi/6$$

In general, if we view $\theta = \sin^{-1} y$ as an angle in radian measure whose sine is y, then the restriction $-\pi/2 \leq \theta \leq \pi/2$ imposes the geometric requirement that the angle θ terminate in either the first or fourth quadrant or on an axis adjacent to those quadrants.

Example 1

Find exact values of

(a) $\sin^{-1}(1/\sqrt{2})$ (b) $\sin^{-1}(-1)$

by inspection, and confirm your results numerically using a calculating utility.

Solution (a). Because $\sin^{-1}(1/\sqrt{2}) > 0$, we can view $\theta = \sin^{-1}(1/\sqrt{2})$ as that angle in the first quadrant such that $\sin\theta = 1/\sqrt{2}$. Thus, $\sin^{-1}(1/\sqrt{2}) = \pi/4$. You can confirm this with your calculating utility by showing that $\sin^{-1}(1/\sqrt{2}) \approx 0.785 \approx \pi/4$.

Solution (b). Because $\sin^{-1}(-1) < 0$, we can view $\theta = \sin^{-1}(-1)$ as an angle in the fourth quadrant (or an adjacent axis) such that $\sin\theta = -1$. Thus, $\sin^{-1}(-1) = -\pi/2$. You can confirm this with your calculating utility by showing that $\sin^{-1}(-1) \approx -1.57 \approx -\pi/2$. ◀

FOR THE READER. If $\theta = \cos^{-1} y$ is viewed as an angle in radian measure whose cosine is y, in what possible quadrants can θ lie? Answer the same question for $\theta = \tan^{-1} y$ and $\theta = \sec^{-1} y$.

FOR THE READER. Most calculators do not provide a direct method for calculating inverse secants. In such situations the identity

$$\sec^{-1} x = \cos^{-1}(1/x) \tag{4}$$

is useful (Exercise 16). Use this formula to show that

$$\sec^{-1}(2.25) \approx 1.11 \quad \text{and} \quad \sec^{-1}(-2.25) \approx 2.03$$

If you have a calculating utility (such as a CAS) that can find $\sec^{-1} x$ directly, use it to check these values.

IDENTITIES FOR INVERSE TRIGONOMETRIC FUNCTIONS

If we interpret $\sin^{-1} x$ as an angle in radian measure whose sine is x, and if that angle is *nonnegative*, then we can represent $\sin^{-1} x$ geometrically as an angle in a right triangle in which the hypotenuse has length 1 and the side opposite to the angle $\sin^{-1} x$ has length x (Figure 4.5.4*a*). By the Theorem of Pythagoras the side adjacent to the angle $\sin^{-1} x$ has length $\sqrt{1-x^2}$. Moreover, the angle opposite to $\sin^{-1} x$ is $\cos^{-1} x$, since the cosine of that angle is x (Figure 4.5.4*b*). This triangle motivates a number of useful identities involving inverse trigonometric functions that are valid for $-1 \leq x \leq 1$; for example,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \tag{5}$$

$$\cos(\sin^{-1} x) = \sqrt{1-x^2} \tag{6}$$

$$\sin(\cos^{-1} x) = \sqrt{1-x^2} \tag{7}$$

$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}} \tag{8}$$

In a similar manner, $\tan^{-1} x$ and $\sec^{-1} x$ can be represented as angles in the right triangles shown in Figures 4.5.4*c* and 4.5.4*d* (verify). Those triangles reveal more useful identities; for example,

$$\sec(\tan^{-1} x) = \sqrt{1+x^2} \tag{9}$$

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{x} \quad (x \geq 1) \tag{10a}$$

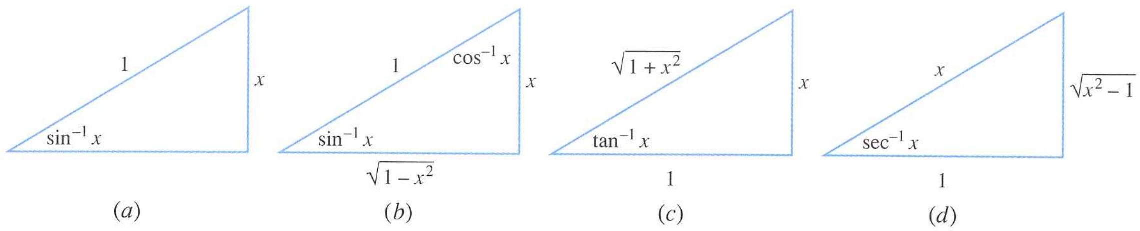


Figure 4.5.4

REMARK. We leave it as an exercise to use (4) and (7) to obtain the following identity that is valid for $x \geq 1$ and $x \leq 1$ (Exercise 48):

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2 - 1}}{|x|} \qquad (|x| \geq 1) \tag{10b}$$

REMARK. There is nothing to be gained by memorizing these identities; what is important to understand is the *method* that was used to obtain them.

Referring to Figure 4.5.1, observe that the inverse sine and inverse tangent are odd functions; that is,

$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad \text{and} \quad \tan^{-1}(-x) = -\tan^{-1}(x) \tag{11–12}$$

Example 2

Figure 4.5.5 shows a computer-generated graph of $y = \sin^{-1}(\sin x)$. One might think that this graph should be the line $y = x$, since $\sin^{-1}(\sin x) = x$. Why isn't it?

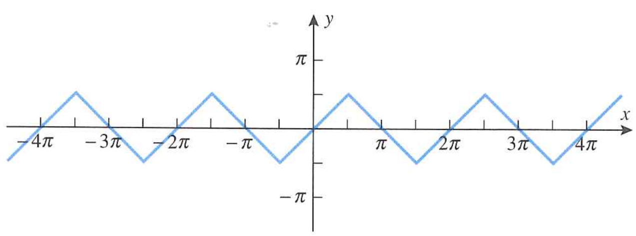


Figure 4.5.5

Solution. The relationship $\sin^{-1}(\sin x) = x$ is valid on the interval $-\pi/2 \leq x \leq \pi/2$, so we can say with certainty that the graphs of $y = \sin^{-1}(\sin x)$ and $y = x$ coincide on this interval (which is confirmed by Figure 4.5.5). However, outside of this interval the relationship $\sin^{-1}(\sin x) = x$ need not hold. For example, if x lies in the interval $\pi/2 \leq x \leq 3\pi/2$, then the quantity $x - \pi$ lies in the interval $-\pi/2 \leq x \leq \pi/2$, so

$$\sin^{-1}[\sin(x - \pi)] = x - \pi$$

Thus, by using the identity $\sin(x - \pi) = -\sin x$ and the fact that $\sin^{-1}$ is an odd function, we can express $\sin^{-1}(\sin x)$ as

$$\sin^{-1}(\sin x) = \sin^{-1}[-\sin(x - \pi)] = -\sin^{-1}[\sin(x - \pi)] = -(x - \pi)$$

This shows that on the interval $\pi/2 \leq x \leq 3\pi/2$ the graph of $y = \sin^{-1}(\sin x)$ coincides with the line $y = -(x - \pi)$, which has slope -1 and an x-intercept at $x = \pi$. This agrees with Figure 4.5.5. ◀

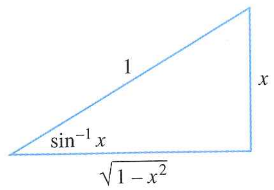
DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

Recall that if f is a one-to-one function whose derivative is known, then there are two basic ways to obtain a derivative formula for $f^{-1}(x)$—we can rewrite the equation $y = f^{-1}(x)$ as $x = f(y)$, and differentiate implicitly, or we can apply Formula (21) or (22) of Section 4.3. Here we will use implicit differentiation to obtain the derivative formula for $y = \sin^{-1} x$. Rewriting this equation as $x = \sin y$ and differentiating implicitly, we obtain

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$



$\cos(\sin^{-1} x) = \sqrt{1-x^2}$

Figure 4.5.6

At this point we have succeeded in obtaining the derivative; however, this derivative formula can be simplified by applying Formula (6), which is derived from the triangle in Figure 4.5.6. This yields

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Thus, we have shown that

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \tag{13}$$

If u is a differentiable function of x, then (13) and the chain rule produce the following generalized derivative formula:

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{14}$$

The method used to obtain this formula can also be used to obtain generalized derivative formulas for the other inverse trigonometric functions. These formulas, which are valid for $-1 < u < 1$, are

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad \frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{15–16}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx}, \qquad \frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx} \tag{17–18}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad \frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \tag{19–20}$$

DIFFERENTIABILITY OF THE INVERSE TRIGONOMETRIC FUNCTIONS

In the derivation of (13) we *assumed* that $\sin^{-1} x$ is differentiable. However, we can establish the differentiability with the help of Theorem 4.1.7. Since $f(x) = \sin x$ and $f'(x) = \cos x$, it follows from that theorem that the function $f^{-1}(x) = \sin^{-1} x$ will be differentiable at any point x where $\cos(\sin^{-1} x) \neq 0$ or from (6) where $\sqrt{1-x^2} \neq 0$. Thus, $\sin^{-1} x$ is differentiable on the interval $(-1, 1)$. The differentiability of the remaining inverse trigonometric functions can be deduced similarly.

REMARK. Observe that $\sin^{-1} x$ is only differentiable on the interval $(-1, 1)$, even though its domain is $[-1, 1]$. However, it can be seen geometrically that $\sin^{-1}$ cannot be differentiable at $x = \pm 1$. Just observe that the graph of $y = \sin x$ has horizontal tangent lines at $(\pi/2, 1)$ and $(-\pi/2, -1)$ and that these become points of vertical tangency for $y = \sin^{-1} x$ when reflected around the line $y = x$.

Example 3

Find dy/dx if

(a) $y = \sin^{-1}(x^3)$ (b) $y = \sec^{-1}(e^x)$

Solution (a). From (14)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2) = \frac{3x^2}{\sqrt{1-x^6}}$$

Solution (b). From (19)

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}}$$

EXERCISE SET 4.5 Graphing Calculator

1. Find the exact value of
(a) $\sin^{-1}(-1)$ (b) $\cos^{-1}(-1)$
(c) $\tan^{-1}(-1)$ (d) $\sec^{-1}(1)$.

2. Find the exact value of
(a) $\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right)$ (b) $\cos^{-1}\left(\frac{1}{2}\right)$
(c) $\tan^{-1}(1)$ (d) $\sec^{-1}(-2)$.

3. Given that $\theta = \sin^{-1}\left(-\frac{1}{2}\sqrt{3}\right)$, find the exact values of $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

4. Given that $\theta = \cos^{-1}\left(\frac{1}{2}\right)$, find the exact values of $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

5. Given that $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

6. Make a table that lists the six inverse trigonometric functions together with their domains and ranges.

7. Find the exact value of
(a) $\sin^{-1}(\sin\pi/7)$ (b) $\sin^{-1}(\sin\pi)$
(c) $\sin^{-1}(\sin 5\pi/7)$ (d) $\sin^{-1}(\sin 630)$.

8. Find the exact value of
(a) $\cos^{-1}(\cos\pi/7)$ (b) $\cos^{-1}(\cos\pi)$
(c) $\cos^{-1}(\cos 12\pi/7)$ (d) $\cos^{-1}(\cos 200)$.

9. For which values of x is it true that
(a) $\cos^{-1}(\cos x) = x$ (b) $\cos(\cos^{-1} x) = x$
(c) $\tan^{-1}(\tan x) = x$ (d) $\tan(\tan^{-1} x) = x$

In Exercises 10 and 11, find the exact value of the given quantity.

10. $\sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right]$ **11.** $\sin\left[2\cos^{-1}\left(\frac{3}{5}\right)\right]$

In Exercises 12 and 13, complete the identities using the triangle method (Figure 4.5.4).

12. (a) $\sin(\cos^{-1} x) = ?$ (b) $\tan(\cos^{-1} x) = ?$
(c) $\csc(\tan^{-1} x) = ?$ (d) $\sin(\tan^{-1} x) = ?$

13. (a) $\cos(\tan^{-1} x) = ?$ (b) $\tan(\cot^{-1} x) = ?$
(c) $\sin(\sec^{-1} x) = ?$ (d) $\cot(\csc^{-1} x) = ?$

14. (a) Use a calculating utility set to radian measure to make tables of values of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for $x = -1, -0.8, -0.6, \ldots, 0, 0.2, \ldots, 1$. Round your answers to two decimal places.
(b) Plot the points obtained in part (a), and use the points to sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$. Confirm that your sketches agree with those in Figure 4.5.1.
(c) Use your graphing utility to graph $y = \sin^{-1} x$ and $y = \cos^{-1} x$; confirm that the graphs agree with those in Figure 4.5.1.

The function $\cot^{-1} x$ is defined to be the inverse of the restricted cotangent function

$$\cot x, \quad 0 < x < \pi$$

and the function $\csc^{-1} x$ is defined to be the inverse of the restricted cosecant function

$$\csc x, \quad -\pi/2 < x < \pi/2, \quad x \neq 0$$

Use these definitions in Exercises 15 and 16 and in all subsequent exercises that involve these functions.

15. (a) Sketch the graphs of $\cot^{-1} x$ and $\csc^{-1} x$.
(b) Find the domain and range of $\cot^{-1} x$ and $\csc^{-1} x$.

16. Show that

(a) $\cot^{-1} x = \tan^{-1} \frac{1}{x}$, if $x > 0$

(b) $\sec^{-1} x = \cos^{-1} \frac{1}{x}$, if $|x| \geq 1$

(c) $\csc^{-1} x = \sin^{-1} \frac{1}{x}$, if $|x| \geq 1$.

17. Most scientific calculators have keys for the values of only $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$. The formulas in Exercise 16 show how a calculator can be used to obtain values of $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ for positive values of x. Use these formulas and a calculator to find numerical values for each of the following inverse trigonometric functions. Express your answers in degrees, rounded to the nearest tenth of a degree.
(a) $\cot^{-1} 0.7$ (b) $\sec^{-1} 1.2$ (c) $\csc^{-1} 2.3$

In Exercises 18–20, use a calculating utility to approximate the solution of the equation. Where radians are used, express your answer to four decimal places, and where degrees are used, express it to the nearest tenth of a degree. [*Note:* In each part, the solution is not in the range of the relevant inverse trigonometric function.]

18. (a) $\sin x = 0.37$, $\pi/2 < x < \pi$
(b) $\sin\theta = -0.61$, $180^\circ < \theta < 270^\circ$

19. (a) $\cos x = -0.85$, $\pi < x < 3\pi/2$
(b) $\cos\theta = 0.23$, $-90^\circ < \theta < 0^\circ$

20. (a) $\tan x = 3.16$, $-\pi < x < -\pi/2$
(b) $\tan\theta = -0.45$, $90^\circ < \theta < 180^\circ$

In Exercises 21–28, find dy/dx.

21. (a) $y = \sin^{-1}\left(\frac{1}{3}x\right)$ (b) $y = \cos^{-1}(2x+1)$

22. (a) $y = \tan^{-1}(x^2)$ (b) $y = \cot^{-1}(\sqrt{x})$

23. (a) $y = \sec^{-1}(x^7)$ (b) $y = \csc^{-1}(e^x)$

24. (a) $y = (\tan x)^{-1}$ (b) $y = \dfrac{1}{\tan^{-1} x}$

25. (a) $y = \sin^{-1}(1/x)$ (b) $y = \cos^{-1}(\cos x)$

26. (a) $y = \ln(\cos^{-1} x)$ (b) $y = \sqrt{\cot^{-1} x}$

27. (a) $y = e^x \sec^{-1} x$ (b) $y = x^2\left(\sin^{-1} x\right)^3$

28. (a) $y = \sin^{-1} x + \cos^{-1} x$ (b) $y = \sec^{-1} x + \csc^{-1} x$

In Exercises 29 and 30, find dy/dx by implicit differentiation.

29. $x^3 + x\tan^{-1} y = e^y$

30. $\sin^{-1}(xy) = \cos^{-1}(x-y)$

31. (a) Referring to the graph of $y = \sin^{-1} x$ in Figure 4.5.1, make a rough sketch of the graph of dy/dx.
(b) Check your work in part (a) using a graphing utility to generate the graph of dy/dx.

32. (a) Referring to the graph of $y = \tan^{-1} x$ in Figure 4.5.1, make a rough sketch of the graph of dy/dx.
(b) Check your work in part (a) using a graphing utility to generate the graph of dy/dx.

33. (a) Make a conjecture about the shape of the graph of

$$y = \cos^{-1}(\cos x)$$

and sketch the graph for $-4\pi \leq x \leq 4\pi$.
(b) Check your work in part (a) using a graphing utility to generate the graph.

34. (a) Use a calculating utility to evaluate $\sin^{-1}(\sin^{-1} 0.25)$ and $\sin^{-1}(\sin^{-1} 0.9)$, and explain what you think is happening in the second calculation.
(b) For what values of x in the interval $-1 \leq x \leq 1$ will your calculating utility produce a real value for the function $\sin^{-1}(\sin^{-1} x)$?

35. In each part, sketch the graph and check your work with a graphing utility.
(a) $y = \sin^{-1} 2x$ (b) $y = \tan^{-1} \frac{1}{2}x$

36. In each part, express x in terms of k and an appropriate inverse trigonometric function. [*Note:* x may not be in the range of the inverse trigonometric function.]
(a) $\cos x = k$, if $0 < k < 1$ and $3\pi/2 < x < 2\pi$
(b) $\tan x = k$, if $k < 0$ and $\pi/2 < x < \pi$
(c) $\sin 2x = k$, if $0 < k < 1$ and $0 < x < \pi/2$.
[*Hint:* Consider the following cases: $0 < 2x < \pi/2$ and $\pi/2 < 2x < \pi$.]

37. An Earth-observing satellite has horizon sensors that can measure the angle θ shown in the accompanying figure. Let R be the radius of the Earth (assumed spherical) and h the distance between the satellite and the Earth's surface.
(a) Show that $\sin\theta = \dfrac{R}{R+h}$.
(b) Find θ, to the nearest degree, for a satellite that is 10,000 km from the Earth's surface (use $R = 6378$ km).

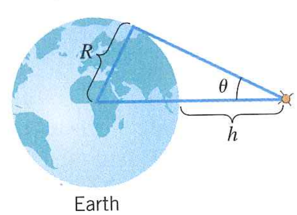


Figure Ex-37

38. The number of hours of daylight on a given day at a given point on the Earth's surface depends on the latitude λ of the point, the angle γ through which the Earth has moved in its orbital plane during the time period from the vernal equinox (March 21), and the angle of inclination ϕ of the Earth's axis of rotation measured from ecliptic north ($\phi \approx 23.55^\circ$). The number of hours of daylight h can be approximated by the formula

$$h = \begin{cases} 24, & D \geq 1 \\ 12 + \frac{2}{15}\sin^{-1} D, & |D| < 1 \\ 0, & D \leq -1 \end{cases}$$

where

$$D = \frac{\sin\phi \sin\gamma \tan\lambda}{\sqrt{1 - \sin^2\phi \sin^2\gamma}}$$

and $\sin^{-1} D$ is in degree measure. Given that Fairbanks, Alaska, is located at a latitude of $\lambda = 65°$ N and also that $\gamma = 90°$ on June 20 and $\gamma = 270°$ on December 20, approximate

(a) the maximum number of daylight hours at Fairbanks to one decimal place

(b) the minimum number of daylight hours at Fairbanks to one decimal place.

[*Note:* This problem was adapted from *TEAM, A Path to Applied Mathematics*, The Mathematical Association of America, Washington, D.C., 1985.]

39. A soccer player kicks a ball with an initial speed of 14 m/s at an angle θ with the horizontal (see the accompanying figure). The ball lands 18 m down the field. If air resistance is neglected, then the ball will have a parabolic trajectory and the horizontal range R will be given by

$$R = \frac{v^2}{g}\sin 2\theta$$

where v is the initial speed of the ball and g is the acceleration due to gravity. Using $g = 9.8\ \text{m/s}^2$, approximate two values of θ, to the nearest degree, at which the ball could have been kicked. Which angle results in the shorter time of flight? Why?

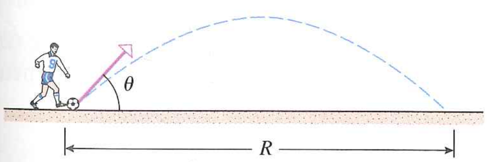


Figure Ex-39

40. The ***law of cosines*** states that

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where a, b, and c are the lengths of the sides of a triangle and θ is the angle formed by sides a and b. Find θ, to the nearest degree, for the triangle with $a = 2$, $b = 3$, and $c = 4$.

41. An airplane is flying at a constant height of 3000 ft above water at a speed of 400 ft/s. The pilot is to release a survival package so that it lands in the water at a sighted point P. If air resistance is neglected, then the package will follow a parabolic trajectory whose equation relative to the coordinate system in the accompanying figure is

$$y = 3000 - \frac{g}{2v^2}x^2$$

where g is the acceleration due to gravity and v is the speed of the airplane. Using $g = 32\ \text{ft/s}^2$, find the "line of sight" angle θ, to the nearest degree, that will result in the package hitting the target point.

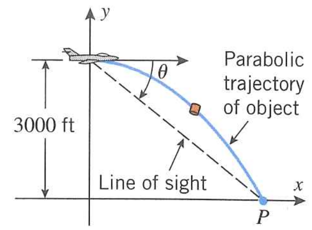


Figure Ex-41

42. A camera is positioned x feet from the base of a missile launching pad (see the accompanying figure). If a missile of length a feet is launched vertically, show that when the base of the missile is b feet above the camera lens, the angle θ subtended at the lens by the missile is

$$\theta = \cot^{-1}\frac{x}{a+b} - \cot^{-1}\frac{x}{b}$$

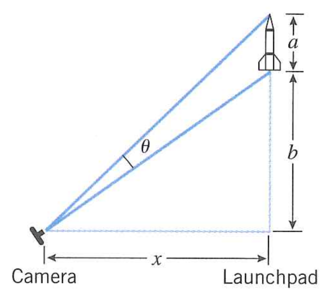


Figure Ex-42

43. Prove:

(a) $\sin^{-1}(-x) = -\sin^{-1} x$

(b) $\tan^{-1}(-x) = -\tan^{-1} x$.

44. Prove:

(a) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(b) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, if $|x| \geq 1$.

45. Prove:

(a) $\sin^{-1} x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

(b) $\cos^{-1} x = \dfrac{\pi}{2} - \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$.

46. Prove:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

provided $-\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi/2$. [*Hint:* Use an identity for $\tan(\alpha + \beta)$.]

47. Use the result in Exercise 46 to show that

(a) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \pi/4$

(b) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \pi/4$.

48. Use identities (4) and (7) to obtain identity (10b).

4.6 RELATED RATES

In this section we will study related rates problems. In such problems one tries to find the rate at which some quantity is changing by relating it to other quantities whose rates of change are known.

RATES OF CHANGE USING THE CHAIN RULE

Figure 4.6.1 shows a liquid draining through a conical filter. As the liquid drains, its volume V, height h, and radius r are functions of the elapsed time t, and at each instant these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h$$

If we differentiate both sides of this equation implicitly with respect to t, then we obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Thus, if the values of r, h, dh/dt, and dr/dt are known, then this equation can be used to find dV/dt. Here are some specific examples that use this basic idea.

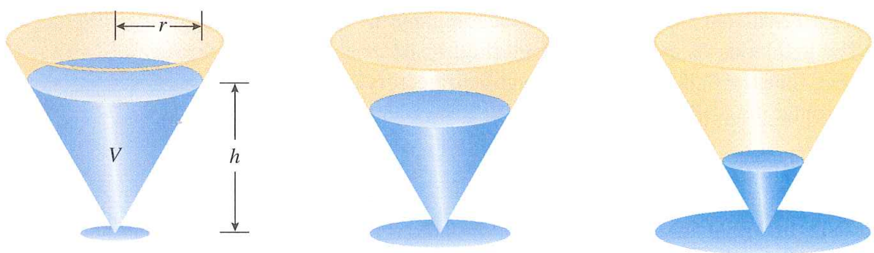


Figure 4.6.1

Example 1

Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

Solution. Let

t = number of seconds elapsed from the time of the spill
r = radius of the spill in feet after t seconds
A = area of the spill in square feet after t seconds

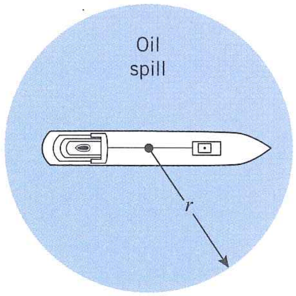


Figure 4.6.2

(Figure 4.6.2). We know the rate at which the radius is increasing, and we want to find the rate at which the area is increasing at the instant when $r = 60$; that is, we want to find

$$\left.\frac{dA}{dt}\right|_{r=60} \quad \text{given that} \quad \frac{dr}{dt} = 2 \text{ ft/s}$$

From the formula for the area of a circle we obtain

$$A = \pi r^2 \tag{1}$$

Because A and r are functions of t, we can differentiate both sides of (1) implicitly with respect to t to obtain

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Thus, when $r = 60$ the area of the spill is increasing at the rate of

$$\left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2/\text{s}$$

or approximately 754 ft^2/s. ◀

With only minor variations, the method used in Example 1 can be used to solve a variety of related rates problems. The method consists of five steps:

A Strategy for Solving Related Rates Problems

Step 1. Draw a figure and label the quantities that vary.

Step 2. Identify the rates of change that are known and the rate of change that is to be found.

Step 3. Find an equation that relates the quantity whose rate of change is to be found to the quantities whose rates of change are known.

Step 4. Differentiate both sides of this equation with respect to time and solve for the derivative that will give the unknown rate of change.

Step 5. Evaluate this derivative at the appropriate point.

Example 2

A baseball diamond is a square whose sides are 90 ft long (Figure 4.6.3). Suppose that a player running from second base to third base has a speed of 30 ft/s at the instant when he is 20 ft from third base. At what rate is the player's distance from home plate changing at that instant?

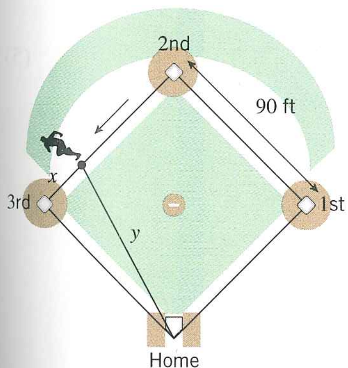


Figure 4.6.3

Solution. Let

t = number of seconds after the player leaves second base
x = distance in feet from third base
y = distance in feet from home plate

(Figure 4.6.3). The rate at which the distance from third base is changing is dx/dt, and the rate at which the distance from home plate is changing is dy/dt. We want to find

$$\left.\frac{dy}{dt}\right|_{x=20} \quad \text{given that} \quad \left.\frac{dx}{dt}\right|_{x=20} = -30 \text{ ft/s}$$

(Note that dx/dt is negative because x is decreasing with respect to t.) From the Theorem of Pythagoras we have

$$x^2 + 90^2 = y^2 \tag{2}$$

Differentiating both sides of this equation with respect to t using the chain rule yields

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt} \quad \text{or} \quad \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \tag{3}$$

When $x = 20$, it follows from (2) that

$$y = \sqrt{20^2 + 90^2} = \sqrt{8500} = 10\sqrt{85}$$

so that (3) yields

$$\left.\frac{dy}{dt}\right|_{x=20} = \frac{20}{10\sqrt{85}}(-30) = -\frac{60}{\sqrt{85}} \approx -6.51 \text{ ft/s}$$

The negative sign in the answer tells us that y is decreasing, which makes sense physically from Figure 4.6.3. ◀

In Figure 4.6.4 we have shown a camera mounted at a point 3000 ft from the base of a rocket launching pad. Let us assume that the rocket rises vertically and the camera is to take a series of photographs of the rocket. Because the rocket will be rising, the elevation angle of the camera will have to vary at just the right rate to keep the rocket in sight. Moreover, because the camera-to-rocket distance will be changing constantly, the camera focusing mechanism will also have to vary at just the right rate to keep the picture sharp. The focusing problem is considered in the exercises, and the elevation problem is addressed in the following example:

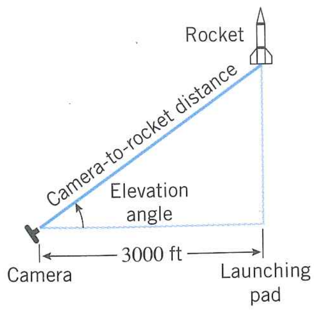


Figure 4.6.4

Example 3

If the rocket shown in Figure 4.6.4 is rising vertically at 880 ft/s when it is 4000 ft up, how fast must the camera elevation angle change at that instant to keep the rocket in sight?

Solution. Let

t = number of seconds elapsed from the time of launch
ϕ = camera elevation angle in radians after t seconds
x = height of the rocket in feet after t seconds

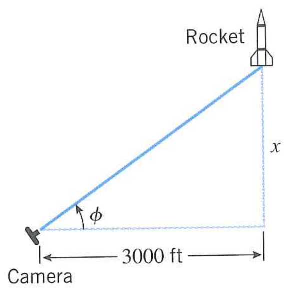


Figure 4.6.5

(Figure 4.6.5). At each instant the rate at which the camera elevation angle must change is $d\phi/dt$, and the rate at which the rocket is rising is dx/dt. We want to find

$$\left.\frac{d\phi}{dt}\right|_{x=4000} \quad \text{given that} \quad \left.\frac{dx}{dt}\right|_{x=4000} = 880 \text{ ft/s}$$

From Figure 4.6.5 we see that

$$\tan\phi = \frac{x}{3000} \tag{4}$$

Because ϕ and x are functions of t, we can differentiate both sides of (4) with respect to t to obtain

$$(\sec^2\phi)\frac{d\phi}{dt} = \frac{1}{3000}\frac{dx}{dt} \quad \text{or} \quad \frac{d\phi}{dt} = \frac{1}{3000\sec^2\phi}\frac{dx}{dt} \tag{5}$$

When $x = 4000$, it follows that

$$\sec\phi = \frac{5000}{3000} = \frac{5}{3}$$

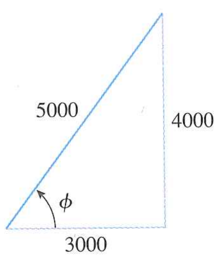


Figure 4.6.6

(Figure 4.6.6), so that from (5)

$$\left.\frac{d\phi}{dt}\right|_{x=4000} = \frac{1}{3000\left(\frac{5}{3}\right)^2}\cdot 880 = \frac{66}{625} \approx 0.11 \text{ radian/s} \approx 6.05 \text{ degrees/s}$$

Alternative Solution. Instead of differentiating both sides of (4), we could have first solved the equation for ϕ and then differentiated:

$$\phi = \tan^{-1}\left(\frac{x}{3000}\right)$$

so

$$\frac{d\phi}{dt} = \frac{1}{1+\left(\frac{x}{3000}\right)^2}\cdot\frac{1}{3000}\frac{dx}{dt}$$

Thus,

$$\left.\frac{d\phi}{dt}\right|_{x=4000} = \frac{1}{1+\left(\frac{4000}{3000}\right)^2}\cdot\frac{880}{3000} = \frac{66}{625} \approx 0.11 \text{ radian/s} \approx 6.05 \text{ degrees/s}$$

which agrees with our previous result. ◀

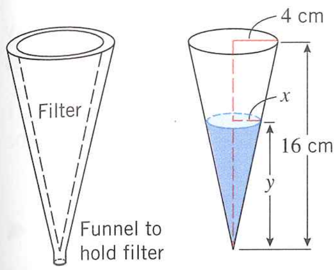


Figure 4.6.7

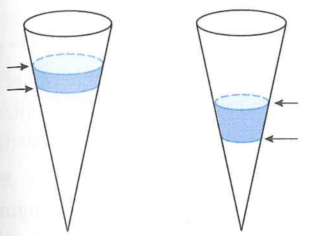

The same volume has drained, but the change in height is greater near the bottom than near the top.

Figure 4.6.8

Example 4

Suppose that liquid is to be cleared of sediment by pouring it through a conical filter that is 16 cm high and has a radius of 4 cm at the top (Figure 4.6.7). Suppose also that the liquid flows out of the cone at a constant rate of 2 cm^3/min.

(a) Do you think that the depth of the liquid will decrease at a constant rate? Give a verbal argument that justifies your conclusion.

(b) Find a formula that expresses the rate of change to the depth of the liquid in terms of the depth, and use that formula to determine whether your conclusion in part (a) is correct.

(c) At what rate is the depth of the liquid changing at the instant when the level is 8 cm deep?

Solution (a). For the volume of liquid to decrease by a *fixed amount*, it requires a greater decrease in depth when the cone is near empty than when it is near full (Figure 4.6.8). This suggests that for the volume to decrease at a constant rate, the depth must decrease at an increasing rate.

Solution (b). Let

t = time elapsed from the initial observation (min)
V = volume of liquid in the cone at time t (cm^3)
y = depth of the liquid in the cone at time t (cm)
x = radius of the liquid surface at time t (cm)

(Figure 4.6.7). At each instant the rate at which the volume of liquid is changing is dV/dt, and the rate at which the depth is changing is dy/dt. We want to express dy/dt in terms of y given that dV/dt has a constant value of $dV/dt = -2$. (We must use a minus sign here because V *decreases* as t increases.)

From the formula for the volume of a cone, the volume V, the radius x, and the depth y are related by

$$V = \tfrac{1}{3}\pi x^2 y \tag{6}$$

If we differentiate both sides of (6) with respect to t, the right side will involve the quantity dx/dt. Since we have no direct information about dx/dt, it is desirable to eliminate x from (6) before differentiating. This can be done using similar triangles. From Figure 4.6.7 we see that

$$\frac{x}{y} = \frac{4}{16} \quad \text{or} \quad x = \frac{1}{4}y$$

Substituting this expression in (6) gives

$$V = \frac{\pi}{48}y^3 \tag{7}$$

Differentiating both sides of (7) with respect to t we obtain

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2\frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{8}$$

which expresses dy/dt in terms of y. The minus sign tells us that y is decreasing with time, and

$$\left|\frac{dy}{dt}\right| = \frac{32}{\pi y^2}$$

tells us how fast y is decreasing. From this formula we see that $|dy/dt|$ increases as y decreases, which confirms our conjecture in part (a) that the depth of the liquid decreases at an increasing rate as the liquid drains through the filter.

Solution (c). The rate at which the depth is changing when the depth is 8 cm can be obtained from (8) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min}$$ ◀

EXERCISE SET 4.6

1. Let A be the area of a square whose sides have length x, and assume that x varies with the time t.
   (a) Draw a picture of the square with the labels A and x placed appropriately.
   (b) Write an equation that relates A and x.
   (c) Use the equation in part (b) to find an equation that relates dA/dt and dx/dt.
   (d) At a certain instant the sides are 3 ft long and increasing at a rate of 2 ft/min. How fast is the area increasing at that instant?

2. Let A be the area of a circle of radius r, and assume that r increases with the time t.
   (a) Draw a picture of the circle with the labels A and r placed appropriately.
   (b) Write an equation that relates A and r.
   (c) Use the equation in part (b) to find an equation that relates dA/dt and dr/dt.
   (d) At a certain instant the radius is 5 cm and increasing at the rate of 2 cm/s. How fast is the area increasing at that instant?

3. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time.
   (a) How are dV/dt, dh/dt, and dr/dt related?
   (b) At a certain instant, the height is 6 in and increasing at 1 in/s, while the radius is 10 in and decreasing at 1 in/s. How fast is the volume changing at that instant? Is the volume increasing or decreasing at that instant?

4. Let l be the length of a diagonal of a rectangle whose sides have lengths x and y, and assume that x and y vary with time.
   (a) How are dl/dt, dx/dt, and dy/dt related?
   (b) If x increases at a constant rate of $\frac{1}{2}$ ft/s and y decreases at a constant rate of $\frac{1}{4}$ ft/s, how fast is the size of the diagonal changing when $x = 3$ ft and $y = 4$ ft? Is the diagonal increasing or decreasing at that instant?

5. Let θ (in radians) be an acute angle in a right triangle, and let x and y, respectively, be the lengths of the sides adjacent and opposite θ. Suppose also that x and y vary with time.
   (a) How are $d\theta/dt$, dx/dt, and dy/dt related?
   (b) At a certain instant, $x = 2$ units and is increasing at 1 unit/s, while $y = 2$ units and is decreasing at $\frac{1}{4}$ unit/s. How fast is θ changing at that instant? Is θ increasing or decreasing at that instant?

6. Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

7. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

8. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

9. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

10. A spherical balloon is inflated so that its volume is increasing at the rate of 3 ft^3/min. How fast is the diameter of the balloon increasing when the radius is 1 ft?

11. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

12. A 17-ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

13. A 13-ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

14. A 10-ft plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being

pushed toward the wall at the rate of 6 in/s, how fast is the acute angle that the plank makes with the ground increasing?

15. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

16. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 mi from the launchpad. How fast is the rocket rising when it is 4 mi high and its distance from the radar station is increasing at a rate of 2000 mi/h?

17. For the camera and rocket shown in Figure 4.6.4, at what rate is the camera-to-rocket distance changing when the rocket is 4000 ft up and rising vertically at 880 ft/s?

18. For the camera and rocket shown in Figure 4.6.4, at what rate is the rocket rising when the elevation angle is $\pi/4$ radians and increasing at a rate of 0.2 radian/s?

19. A satellite is in an elliptical orbit around the Earth. Its distance r (in miles) from the center of the Earth is given by

$$r = \frac{4995}{1 + 0.12 \cos \theta}$$

where θ is the angle measured from the point on the orbit nearest the Earth's surface (see the accompanying figure).

(a) Find the altitude of the satellite at ***perigee*** (the point nearest the surface of the Earth) and at ***apogee*** (the point farthest from the surface of the Earth). Use 3960 mi as the radius of the Earth.

(b) At the instant when θ is $120°$, the angle θ is increasing at the rate of $2.7°$/min. Find the altitude of the satellite and the rate at which the altitude is changing at this instant. Express the rate in units of mi/min.

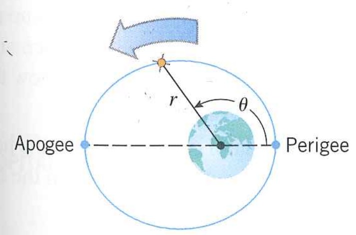


Figure Ex-19

20. An aircraft is flying horizontally at a constant height of 4000 ft above a fixed observation point (see the accompanying figure). At a certain instant the angle of elevation θ is $30°$ and decreasing, and the speed of the aircraft is 300 mi/h.

(a) How fast is θ decreasing at this instant? Express the result in units of degrees/s.

(b) How fast is the distance between the aircraft and the observation point changing at this instant? Express the result in units of ft/s. Use 1 mi = 5280 ft.

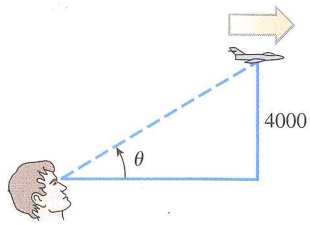


Figure Ex-20

21. A conical water tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows into the tank at a rate of 20 ft^3/min, how fast is the depth of the water increasing when the water is 16 ft deep?

22. Grain pouring from a chute at the rate of 8 ft^3/min forms a conical pile whose altitude is always twice its radius. How fast is the altitude of the pile increasing at the instant when the pile is 6 ft high?

23. Sand pouring from a chute forms a conical pile whose height is always equal to the diameter. If the height increases at a constant rate of 5 ft/min, at what rate is sand pouring from the chute when the pile is 10 ft high?

24. Wheat is poured through a chute at the rate of 10 ft^3/min, and falls in a conical pile whose bottom radius is always half the altitude. How fast will the circumference of the base be increasing when the pile is 8 ft high?

25. An aircraft is climbing at a $30°$ angle to the horizontal. How fast is the aircraft gaining altitude if its speed is 500 mi/h?

26. A boat is pulled into a dock by means of a rope attached to a pulley on the dock (see the accompanying figure). The rope is attached to the bow of the boat at a point 10 ft below the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 125 ft of rope is out?

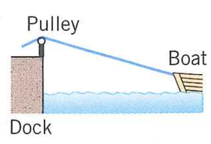


Figure Ex-26

27. For the boat in Exercise 26, how fast must the rope be pulled if we want the boat to approach the dock at a rate of 12 ft/min at the instant when 125 ft of rope is out?

28. A man 6 ft tall is walking at the rate of 3 ft/s toward a streetlight 18 ft high (see the accompanying figure).

(a) At what rate is his shadow length changing?

(b) How fast is the tip of his shadow moving?

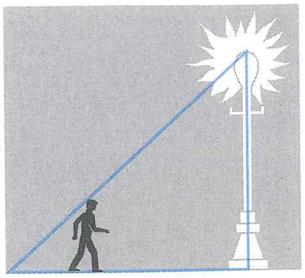


Figure Ex-28

29. A beacon that makes one revolution every 10 s is located on a ship anchored 4 kilometers from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?

30. An aircraft is flying at a constant altitude with a constant speed of 600 mi/h. An antiaircraft missile is fired on a straight line perpendicular to the flight path of the aircraft so that it will hit the aircraft at a point P (see the accompanying figure). At the instant the aircraft is 2 mi from the impact point P the missile is 4 mi from P and flying at 1200 mi/h. At that instant, how rapidly is the distance between missile and aircraft decreasing?

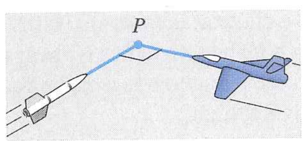


Figure Ex-30

31. Solve Exercise 30 under the assumption that the angle between the flight paths is 120° instead of the assumption that the paths are perpendicular. [*Hint:* Use the law of cosines.]

32. A police helicopter is flying due north at 100 mi/h and at a constant altitude of $\frac{1}{2}$ mi. Below, a car is traveling west on a highway at 75 mi/h. At the moment the helicopter crosses over the highway the car is 2 mi east of the helicopter.
   (a) How fast is the distance between the car and helicopter changing at the moment the helicopter crosses the highway?
   (b) Is the distance between the car and helicopter increasing or decreasing at that moment?

33. A particle is moving along the curve whose equation is

$$\frac{xy^3}{1+y^2} = \frac{8}{5}$$

Assume that the x-coordinate is increasing at the rate of 6 units/s when the particle is at the point (1, 2).
   (a) At what rate is the y-coordinate of the point changing at that instant?
   (b) Is the particle rising or falling at that instant?

34. A point P is moving along the curve whose equation is $y = \sqrt{x^3 + 17}$. When P is at (2, 5), y is increasing at the rate of 2 units/s. How fast is x changing?

35. A point P is moving along the line whose equation is $y = 2x$. How fast is the distance between P and the point (3, 0) changing at the instant when P is at (3, 6) if x is decreasing at the rate of 2 units/s at that instant?

36. A point P is moving along the curve whose equation is $y = \sqrt{x}$. Suppose that x is increasing at the rate of 4 units/s when $x = 3$.
   (a) How fast is the distance between P and the point (2, 0) changing at this instant?
   (b) How fast is the angle of inclination of the line segment from P to (2, 0) changing at this instant?

37. A particle is moving along the curve $y = x \ln x$. Find all values of x at which the rate of change of y with respect to time is three times that of x. [Assume that dx/dt is never zero.]

38. A particle is moving along the curve $16x^2 + 9y^2 = 144$. Find all points (x, y) at which the rates of change of x and y with respect to time are equal. [Assume that dx/dt and dy/dt are never both zero at the same point.]

39. The ***thin lens equation*** in physics is

$$\frac{1}{s} + \frac{1}{S} = \frac{1}{f}$$

where s is the object distance from the lens, S is the image distance from the lens, and f is the focal length of the lens. Suppose that a certain lens has a focal length of 6 cm and that an object is moving toward the lens at the rate of 2 cm/s. How fast is the image distance changing at the instant when the object is 10 cm from the lens? Is the image moving away from the lens or toward the lens?

40. Water is stored in a cone-shaped reservoir (vertex down). Assuming the water evaporates at a rate proportional to the surface area exposed to the air, show that the depth of the water will decrease at a constant rate that does not depend on the dimensions of the reservoir.

41. A meteorite enters the Earth's atmosphere and burns up at a rate that, at each instant, is proportional to its surface area. Assuming that the meteorite is always spherical, show that the radius decreases at a constant rate.

42. On a certain clock the minute hand is 4 in long and the hour hand is 3 in long. How fast is the distance between the tips of the hands changing at 9 o'clock?

43. Coffee is poured at a uniform rate of 20 cm^3/s into a cup whose inside is shaped like a truncated cone (see the accompanying figure). If the upper and lower radii of the cup are 4 cm and 2 cm and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up? [*Hint:* Extend the cup downward to form a cone.]

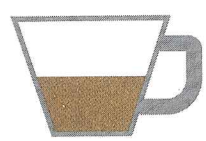


Figure Ex-43

4.7 L'HÔPITAL'S RULE; INDETERMINATE FORMS

In this section we will discuss a general method for using derivatives to find limits. This method will enable us to establish limits with certainty that earlier in the text we were only able to conjecture using numerical or graphical evidence. The method that we will discuss in this section is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

INDETERMINATE FORMS OF TYPE 0/0

In earlier sections we discussed limits that can be determined by inspection or by some appropriate algebraic manipulation. In this section we will be concerned with limits that cannot be obtained by such methods. For example, in Theorem 2.5.3 we were able to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{1}$$

but it required the Squeezing Theorem (2.5.2) and some tricky manipulation of inequalities. Our goal here is to develop a more straightforward method.

What makes the limit in (1) bothersome is the fact that the numerator and denominator both approach 0 as $x \to 0$. Such limits are called ***indeterminate forms of type* 0/0**. In limits of this type there are two tendencies working against each other: as the numerator approaches 0 it tends to drive the ratio toward 0, and as the denominator approaches 0 it tends to drive the ratio toward $+\infty$ or $-\infty$. What happens in (1) is that these conflicting tendencies offset each other in such a way that the limit is 1.

Although the limit in (1) is not self-evident, it can be conjectured from numerical evidence, as in Table 2.1.2. However, it can also be conjectured from the local linear approximation of $\sin x$ at 0. To see this, recall from Formula (5) of Section 3.6 that if a function f is differentiable at a point x_0, then for values of x near x_0, the values of $f(x)$ can be approximated as

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

where the approximation tends to get better and better as $x \to x_0$. In particular, we showed in Example 3 of Section 3.6 that the local linear approximation of $\sin x$ at $x_0 = 0$ is

$$\sin x \approx x$$

This suggests that the value of $(\sin x)/x$ gets closer and closer to 1 as $x \to 0$, and hence we can reasonably conclude that

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{x} = 1$$

L'HÔPITAL'S RULE

The idea of using local linear approximations to evaluate indeterminate forms of type 0/0 can be used to motivate a more general procedure for finding such limits. For this purpose, suppose that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)}$$

is an indeterminate form of type 0/0, that is,

$$\lim_{x\to x_0} f(x) = 0 \quad \text{and} \quad \lim_{x\to x_0} g(x) = 0 \tag{2}$$

For simplicity, let us also assume that f and g are differentiable at $x = x_0$ and that f' and g' are continuous at $x = x_0$. The differentiability of f and g at $x = x_0$ implies that f and g are continuous at $x = x_0$, and hence from (2)

$$f(x_0) = \lim_{x\to x_0} f(x) = 0 \quad \text{and} \quad g(x_0) = \lim_{x\to x_0} g(x) = 0 \tag{3}$$

Moreover, the continuity of f' and g' at $x = x_0$ implies that

$$\lim_{x\to x_0} f'(x) = f'(x_0) \quad \text{and} \quad \lim_{x\to x_0} g'(x) = g'(x_0) \tag{4}$$

Thus, from (3) and (4) and the local linear approximations of f and g at $x = x_0$, we have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f(x_0) + f'(x_0)(x - x_0)}{g(x_0) + g'(x_0)(x - x_0)}$$

$$= \lim_{x\to x_0} \frac{f'(x_0)(x - x_0)}{g'(x_0)(x - x_0)} = \frac{f'(x_0)}{g'(x_0)}$$

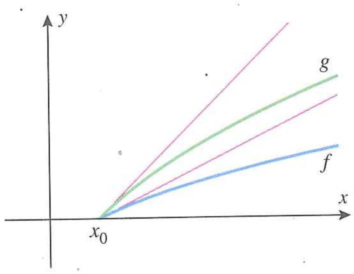
which from (4) can be expressed as

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \tag{5}$$

This result, called ***L'Hôpital's** rule**, converts an indeterminate form of type 0/0 into a new limit involving derivatives that in many situations can be evaluated by inspection or by algebraic methods. For example,

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[1 - \cos x]}{\frac{d}{dx}[x]} = \lim_{x\to 0} \frac{\sin x}{1} = \sin 0 = 0$$

which agrees with the result in Theorem 2.5.3.



The graphs of f and g together with their local linear approximations at the point x_0

Figure 4.7.1

REMARK. Figure 4.7.1 provides a geometric explanation of (5). That figure shows the graphs f and g and the graphs of their local linear approximations at x_0. Note that $f(x)\to 0$ and $g(x)\to 0$ as $x\to x_0$ in the figure because the limit of f/g is an indeterminate form of type 0/0. The figure strongly suggests that for values of x near x_0 there is little difference between the ratio of $f(x)$ and $g(x)$, and the ratio of the corresponding values in the local linear approximations, which is what we showed algebraically.

Although we motivated Formula (5) by assuming that f and g have continuous derivatives at $x = x_0$, the result is true without this assumption. Moreover, the result is also valid for one-sided limits and limits at $+\infty$ and $-\infty$. We omit the formal proof.

4.7.1 THEOREM (*L'Hôpital's Rule for Form* 0/0). *Let* lim *stand for one of the limits* $\lim_{x\to a}$, $\lim_{x\to a^+}$, $\lim_{x\to a^-}$, $\lim_{x\to +\infty}$, *or* $\lim_{x\to -\infty}$, *and suppose that* $\lim f(x) = 0$ *and* $\lim g(x) = 0$. *If* $\lim [f'(x)/g'(x)]$ *has a finite value* L, *or if this limit is* $+\infty$ *or* $-\infty$, *then*

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

REMARK. Note that in L'Hôpital's rule the numerator and denominator are differentiated separately, which is not the same as differentiating $f(x)/g(x)$.

* GUILLAUME FRANCOIS ANTOINE DE L'HÔPITAL (1661–1704). French mathematician. L'Hôpital, born to parents of the French high nobility, held the title of Marquis de Sainte-Mesme Comte d'Autrement. He showed mathematical talent quite early and at age 15 solved a difficult problem about cycloids posed by Pascal. As a young man he served briefly as a cavalry officer, but resigned because of nearsightedness. In his own time he gained fame as the author of the first textbook ever published on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696). L'Hôpital's rule appeared for the first time in that book. Actually, L'Hôpital's rule and most of the material in the calculus text were due to John Bernoulli, who was L'Hôpital's teacher. L'Hôpital dropped his plans for a book on integral calculus when Leibniz informed him that he intended to write such a text. L'Hôpital was apparently generous and personable, and his many contacts with major mathematicians provided the vehicle for disseminating major discoveries in calculus throughout Europe.

In the following examples we will apply L'Hôpital's rule using the following three-step process:

Step 1. Check that $\lim f(x)/g(x)$ is an indeterminate form. If it is not, then L'Hôpital's rule cannot be used.

Step 2. Differentiate f and g separately.

Step 3. Find $\lim f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to $\lim f(x)/g(x)$.

Example 1

In each part confirm that the limit is an indeterminate form of type $0/0$, and evaluate it using L'Hôpital's rule.

(a) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x-2}$ (b) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{x}$ (c) $\displaystyle\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}$ (d) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x^3}$

(e) $\displaystyle\lim_{x\to 0^-}\frac{\tan x}{x^2}$ (f) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (g) $\displaystyle\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$

Solution (a). The numerator and denominator have a limit of 0, so L'Hôpital's rule applies and yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\dfrac{d}{dx}[x^2-4]}{\dfrac{d}{dx}[x-2]}=\lim_{x\to 2}\frac{2x}{1}=4$$

Observe that this particular limit could also have been obtained by factoring

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Solution (b). The numerator and denominator have a limit of 0, so L'Hôpital's rule applies and yields

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[\sin 2x]}{\dfrac{d}{dx}[x]}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

Observe that this result agrees with that obtained by substitution in Example 2(b) of Section 2.5.

Solution (c). The numerator and denominator have a limit of 0, so L'Hôpital's rule applies and yields

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2}\frac{\dfrac{d}{dx}[1-\sin x]}{\dfrac{d}{dx}[\cos x]}=\lim_{x\to \pi/2}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

Solution (d). The numerator and denominator have a limit of 0, so L'Hôpital's rule applies and yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[e^x-1]}{\dfrac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (e). The numerator and denominator have a limit of 0, so L'Hôpital's rule applies and yields

$$\lim_{x\to 0^-} \frac{\tan x}{x^2} = \lim_{x\to 0^-} \frac{\sec^2 x}{2x} = -\infty$$

Solution (f). The numerator and denominator have a limit of 0, so L'Hôpital's rule applies and yields

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Solution (g). The numerator and denominator have a limit of 0, so L'Hôpital's rule applies and yields

$$\lim_{x\to +\infty} \frac{x^{-4/3}}{\sin(1/x)} = \lim_{x\to +\infty} \frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)} = \lim_{x\to +\infty} \frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)} = \frac{0}{1} = 0$$ ◀

WARNING. Applying L'Hôpital's rule to limits that are not indeterminate forms can lead to incorrect results. For example, in the limit

$$\lim_{x\to 0} \frac{x+6}{x+2} = \frac{6}{2} = 3$$

the numerator approaches 6 and the denominator approaches 2, so the limit is not an indeterminate form of type 0/0. However, if we ignore this and blindly apply L'Hôpital's rule, we reach the following *erroneous* conclusion:

$$\lim_{x\to 0} \frac{\frac{d}{dx}[x+6]}{\frac{d}{dx}[x+2]} = \lim_{x\to 0} \frac{1}{1} = 1$$

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or the one-sided limits) of a function are $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x) = \infty \quad \text{means} \quad \lim_{x\to a^+} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty$$

$$\lim_{x\to +\infty} f(x) = \infty \quad \text{means} \quad \lim_{x\to +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x) = -\infty$$

$$\lim_{x\to a} f(x) = \infty \quad \text{means} \quad \lim_{x\to a^+} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type*** ∞/∞. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

4.7.2 THEOREM (***L'Hôpital's Rule for Form*** ∞/∞). *Let* lim *stand for one of the limits* $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to +\infty}$, *or* $\lim\limits_{x\to -\infty}$, *and suppose that* $\lim f(x) = \infty$ *and* $\lim g(x) = \infty$. *If* $\lim [f'(x)/g'(x)]$ *has a finite value* L, *or if this limit is* $+\infty$ *or* $-\infty$, *then*

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

Example 2

In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

(a) $\displaystyle\lim_{x\to+\infty}\frac{x}{e^x}$ (b) $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{\csc x}$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to+\infty}\frac{x}{e^x}=\lim_{x\to+\infty}\frac{1}{e^x}=0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x}=\lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x} \tag{6}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (6) can be rewritten as

$$\lim_{x\to 0^+}\left(-\frac{\sin x}{x}\tan x\right)=-\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\tan x=-(1)(0)=0$$

Thus,

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x}=0$$

◀

ANALYZING THE GROWTH OF EXPONENTIAL FUNCTIONS USING L'HÔPITAL'S RULE

If n is any positive integer, then $x^n\to+\infty$ as $x\to+\infty$. Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example, we know that $e^x\to+\infty$ as $x\to+\infty$ and that the growth of e^x is very rapid (Table 4.2.3); however, the growth of x^n is also rapid when n is a high power, so it is reasonable to ask whether high powers of x grow more or less rapidly than e^x. One way to investigate this is to examine the behavior of the ratio x^n/e^x as $x\to+\infty$. For example, Figure 4.7.2*a* shows the graph of $y=x^5/e^x$. This graph suggests that $x^5/e^x\to 0$ as $x\to+\infty$, and this implies that the growth of the function e^x is sufficiently rapid that its values eventually overtake those of x^5 and force the ratio toward zero. Stated informally, "e^x eventually grows more rapidly than x^5." The same conclusion could have been reached by putting e^x on top and examining the behavior of e^x/x^5 as $x\to+\infty$ (Figure 4.7.2*b*). In this case the values of e^x eventually overtake those of x^5 and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that e^x *eventually grows more rapidly than any positive integer power of* x, that is,

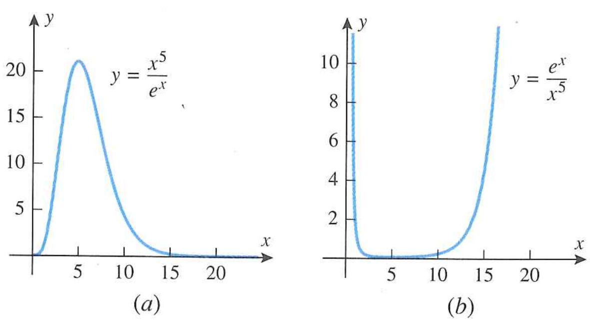


Figure 4.7.2

$$\lim_{x \to +\infty} \frac{x^n}{e^x} = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty \tag{7–8}$$

Both limits are indeterminate forms of type ∞/∞ that can be evaluated using L'Hôpital's rule. For example, to establish (7), we will need to apply L'Hôpital's rule n times. For this purpose, observe that successive differentiations of x^n reduce the exponent by 1 each time, thus producing a constant for the nth derivative. For example, the successive derivatives of x^3 are $3x^2$, $6x$, and 6. In general, the nth derivative of x^n is the constant $n(n-1)(n-2)\cdots 1 = n!$ (verify).* Thus, applying L'Hôpital's rule n times to (7) yields

$$\lim_{x \to +\infty} \frac{x^n}{e^x} = \lim_{x \to +\infty} \frac{n!}{e^x} = 0$$

Limit (8) can be established similarly.

INDETERMINATE FORMS OF TYPE $0 \cdot \infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x) \cdot g(x), \quad f(x)^{g(x)}, \quad f(x) - g(x), \quad f(x) + g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x \to 0^+} x \ln x$$

is an ***indeterminate form of type* $0 \cdot \infty$** because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x \to +\infty} [\sqrt{x}(1 - x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

WARNING. It is tempting to argue that an indeterminate form of type $0 \cdot \infty$ has value 0 since "zero times anything is zero." However, this is fallacious since $0 \cdot \infty$ is not a product of numbers, but rather a statement about limits. For example, the following limits are of the form $0 \cdot \infty$:

$$\lim_{x \to 0^+} x \cdot \frac{1}{x} = 1, \quad \lim_{x \to 0^+} x^2 \cdot \frac{1}{x} = 0, \quad \lim_{x \to 0^+} \sqrt{x} \cdot \frac{1}{x} = +\infty$$

Indeterminate forms of type $0 \cdot \infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

Example 3

Evaluate

(a) $\lim_{x \to 0^+} x \ln x$ (b) $\lim_{x \to \pi/4} (1 - \tan x) \sec 2x$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0 \cdot \infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} \quad \text{or} \quad \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of

*Recall that for $n \geq 1$ the expression $n!$ is read ***n-factorial*** and denotes the product of the first n integers.

type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type ∞/∞:

$$\lim_{x\to\pi/4} (1-\tan x)\sec 2x = \lim_{x\to\pi/4} \frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4} \frac{1-\tan x}{\cos 2x}$$

$$= \lim_{x\to\pi/4} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1$$ ◀

INDETERMINATE FORMS OF TYPE $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty)-(+\infty), \quad (-\infty)-(-\infty),$$
$$(+\infty)+(-\infty), \quad (-\infty)+(+\infty)$$

is called an ***indeterminate form of type*** $\infty - \infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty)+(+\infty), \quad (+\infty)-(-\infty),$$
$$(-\infty)+(-\infty), \quad (-\infty)-(+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty-\infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

Example 4

Evaluate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty-\infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right)$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right) = \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$$ ◀

INDETERMINATE FORMS OF TYPE 0^0, ∞^0, 1^∞

Limits of the form

$$\lim f(x)^{g(x)}$$

give rise to ***indeterminate forms of the types*** $\mathbf{0^0}$, $\boldsymbol{\infty^0}$, ***and*** $\mathbf{1^\infty}$. (The meaning of these symbols should be clear.) For example, the limit

$$\lim_{x\to 0^+}(1+x)^{1/x}$$

whose value we know to be e [see Formula (5) of Section 4.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then calculating the limit of $\ln y$ by expressing it as

$$\lim \ln y = \lim [\ln(f(x)^{g(x)})] = \lim [g(x) \ln f(x)]$$

Once the limit of $\ln y$ is known, the limit of $y = f(x)^{g(x)}$ itself can generally be obtained by a method that we will illustrate in the next example.

Example 5

Show that $\lim_{x \to 0} (1 + x)^{1/x} = e$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1 + x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1 + x)^{1/x} = \frac{1}{x} \ln(1 + x) = \frac{\ln(1 + x)}{x}$$

Thus,

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + x)}{x}$$

which is an indeterminate form of type 0/0, so by L'Hôpital's rule

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \frac{1/(1 + x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus,

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

◀

EXERCISE SET 4.7 Graphing Calculator C CAS

In Exercises 1 and 2, evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule.

1. (a) $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + 2x - 8}$ (b) $\lim_{x \to +\infty} \frac{2x - 5}{3x + 7}$

2. (a) $\lim_{x \to 0} \frac{\sin x}{\tan x}$ (b) $\lim_{x \to 1} \frac{x^2 - 1}{x^3 - 1}$

In Exercises 3–36, find the limit.

3. $\lim_{x \to 1} \frac{\ln x}{x - 1}$

4. $\lim_{x \to 0} \frac{\sin 2x}{\sin 5x}$

5. $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$

6. $\lim_{x \to 3} \frac{x - 3}{3x^2 - 13x + 12}$

7. $\lim_{\theta \to 0} \frac{\tan \theta}{\theta}$

8. $\lim_{t \to 0} \frac{te^t}{1 - e^t}$

9. $\lim_{x \to \pi^+} \frac{\sin x}{x - \pi}$

10. $\lim_{x \to 0^+} \frac{\sin x}{x^2}$

11. $\lim_{x \to +\infty} \frac{\ln x}{x}$

12. $\lim_{x \to +\infty} \frac{e^{3x}}{x^2}$

13. $\lim_{x \to 0^+} \frac{\cot x}{\ln x}$

14. $\lim_{x \to 0^+} \frac{1 - \ln x}{e^{1/x}}$

15. $\lim_{x \to +\infty} \frac{x^{100}}{e^x}$

16. $\lim_{x \to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$

17. $\lim_{x \to 0} \frac{\sin^{-1} 2x}{x}$

18. $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3}$

19. $\lim_{x \to +\infty} xe^{-x}$

20. $\lim_{x \to \pi^-} (x - \pi) \tan \frac{1}{2}x$

21. $\lim_{x \to +\infty} x \sin \frac{\pi}{x}$

22. $\lim_{x \to 0^+} \tan x \ln x$

23. $\lim_{x \to \pi/2^-} \sec 3x \cos 5x$

24. $\lim_{x \to \pi} (x - \pi) \cot x$

25. $\lim_{x \to +\infty} (1 - 3/x)^x$

26. $\lim_{x \to 0} (1 + 2x)^{-3/x}$

27. $\lim_{x \to 0} (e^x + x)^{1/x}$

28. $\lim_{x \to +\infty} (1 + a/x)^{bx}$

29. $\lim_{x \to 1} (2 - x)^{\tan(\pi/2)x}$

30. $\lim_{x \to +\infty} [\cos(2/x)]^{x^2}$

31. $\lim_{x \to 0} (\csc x - 1/x)$

32. $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos 3x}{x^2} \right)$

33. $\lim_{x \to +\infty} (\sqrt{x^2 + x} - x)$

34. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

35. $\lim_{x \to +\infty} [x - \ln(x^2 + 1)]$

36. $\lim_{x \to +\infty} [\ln x - \ln(1 + x)]$

37. Use a CAS to check the answers you obtained in Exercises 31–36.

38. Show that for any positive integer n

(a) $\lim_{x \to +\infty} \frac{\ln x}{x^n} = 0$ (b) $\lim_{x \to +\infty} \frac{x^n}{\ln x} = +\infty$

39. (a) Find the error in the following calculation:

$$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim_{x \to 1} \frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim_{x \to 1} \frac{6x - 2}{6x - 2} = 1$$

(b) Find the correct answer.

40. Find $\lim_{x \to 1} \frac{x^4 - 4x^3 + 6x^2 - 4x + 1}{x^4 - 3x^3 + 3x^2 - x}$.

In Exercises 41–44, make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule.

41. $\lim_{x \to +\infty} \frac{\ln(\ln x)}{\sqrt{x}}$

42. $\lim_{x \to 0^+} x^x$

43. $\lim_{x \to 0^+} (\sin x)^{3/\ln x}$

44. $\lim_{x \to (1/2)\pi^-} \frac{4 \tan x}{1 + \sec x}$

In Exercises 45–48, make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule.

45. $y = \ln x - e^x$

46. $y = x - \ln(1 + 2e^x)$

47. $y = (\ln x)^{1/x}$

48. $y = \left(\frac{x + 1}{x + 2} \right)^x$

49. Limits of the type

$$0/\infty, \ \infty/0, \ 0^\infty, \ \infty \cdot \infty, \ +\infty + (+\infty),$$
$$+\infty - (-\infty), \ -\infty + (-\infty), \ -\infty - (+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x \to 0^+} \frac{x}{\ln x}$ (b) $\lim_{x \to +\infty} \frac{x^3}{e^{-x}}$

(c) $\lim_{x \to (1/2)\pi^-} (\cos x)^{\tan x}$ (d) $\lim_{x \to 0^+} (\ln x) \cot x$

(e) $\lim_{x \to 0^+} \left(\frac{1}{x} - \ln x \right)$ (f) $\lim_{x \to -\infty} (x + x^3)$

50. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types 0^0, ∞^0, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^0, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples, which were transmitted to me by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:

(a) $\lim_{x \to 0^+} \left[x^{(\ln a)/(1 + \ln x)} \right] = 0^0 = a$

(b) $\lim_{x \to +\infty} \left[x^{(\ln a)/(1 + \ln x)} \right] = \infty^0 = a$

(c) $\lim_{x \to 0} \left[(x + 1)^{(\ln a)/x} \right] = 1^\infty = a.$

Prove these results.

In Exercises 51–54, verify that L'Hôpital's rule is of no help in finding the limit, then find the limit, if it exists, by some other method.

51. $\lim_{x \to +\infty} \frac{x + \sin 2x}{x}$

52. $\lim_{x \to +\infty} \frac{2x - \sin x}{3x + \sin x}$

53. $\lim_{x \to +\infty} \frac{x(2 + \sin 2x)}{x + 1}$

54. $\lim_{x \to +\infty} \frac{x(2 + \sin x)}{x^2 + 1}$

55. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t = 0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e., $R \to 0^+$)?

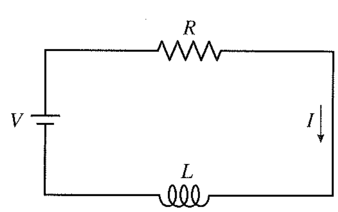


Figure Ex-55

56. (a) Show that $\lim_{x \to \pi/2} (\pi/2 - x) \tan x = 1$.

(b) Show that

$$\lim_{x \to \pi/2} \left(\frac{1}{\pi/2 - x} - \tan x \right) = 0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2 - x}$$

should be good for values of x near $\pi/2$. Use a calculator to find $\tan x$ and $1/(\pi/2 - x)$ for $x = 1.57$; compare the results.

C 57. (a) Use a CAS to show that if k is a positive constant, then

$$\lim_{x \to +\infty} x(k^{1/x} - 1) = \ln k$$

(b) Confirm this result using L'Hôpital's rule. [*Hint:* Express the limit in terms of $t = 1/x$.]

(c) If n is a positive integer, then it follows from part (a) with $x = n$ that the approximation

$$n(\sqrt[n]{k} - 1) \approx \ln k$$

should be good when n is large. Use this result and the square root key on a calculator to approximate the values of $\ln 0.3$ and $\ln 2$ with $n = 1024$, then compare the values obtained with values of the logarithms generated directly from the calculator. [*Hint:* The nth roots for which n is a power of 2 can be obtained as successive square roots.]

58. Let $f(x) = x^2 \sin(1/x)$.

(a) Are the limits $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ indeterminate forms?

(b) Use a graphing utility to generate the graph of f, and use the graph to make conjectures about the limits in part (a).

(c) Use the Squeezing Theorem (2.5.2) to confirm that your conjectures in part (b) are correct.

59. Find all values of k and l such that

$$\lim_{x \to 0} \frac{k + \cos lx}{x^2} = -4$$

60. (a) Explain why L'Hôpital's rule does not apply to the problem

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin x}$$

(b) Find the limit.

61. Find $\lim_{x \to 0^+} \dfrac{x \sin(1/x)}{\sin x}$ if it exists.

SUPPLEMENTARY EXERCISES

1. (a) State conditions under which two functions, f and g, will be inverses, and give several examples of such functions.

(b) In words, what is the relationship between the graphs of $y = f(x)$ and $y = g(x)$ when f and g are inverse functions?

(c) What is the relationship between the domains and ranges of inverse functions f and g?

(d) What condition must be satisfied for a function f to have an inverse? Give some examples of functions that do not have inverses.

(e) If f and g are inverse functions and f is continuous, must g be continuous? Give a reasonable informal argument to support your answer.

(f) If f and g are inverse functions and f is differentiable, must g be differentiable? Give a reasonable informal argument to support your answer.

2. (a) State the restrictions on the domains of $\sin x$, $\cos x$, $\tan x$, and $\sec x$ that are imposed to make those functions one-to-one in the definitions of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} x$.

(b) Sketch the graphs of the restricted trigonometric functions in part (a) and their inverses.

3. (a) Under what conditions will a limit of the form

$$\lim\,[f(x)/g(x)]$$

be an indeterminate form?

(b) If $\lim g(x) = 0$, must $\lim\,[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your answer.

4. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim\,[f(x) - g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

5. In each part, find $f^{-1}(x)$ if the inverse exists.

(a) $f(x) = 8x^3 - 1$ (b) $f(x) = x^2 - 2x + 1$

(c) $f(x) = (e^x)^2 + 1$ (d) $f(x) = (x + 2)/(x - 1)$

6. Let $f(x) = (ax + b)/(cx + d)$. What conditions on a, b, c, d guarantee that f^{-1} exists? Find $f^{-1}(x)$.

7. In each part, find the equation of the tangent line at the specified point.

(a) $x^{2/3} - y^{2/3} - y = 1;\ (1, -1)$

(b) $\sin xy = y;\ (\pi/2, 1)$

8. In each part, find the exact numerical value of the given expression.
(a) $\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)]$
(b) $\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)]$

9. Express the following function as a rational function of x:

$$3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1)$$

10. Suppose that $y = Ce^{kt}$, where C and k are constants, and let $Y = \ln y$. Show that the graph of Y versus t is a line, and state its slope and Y-intercept.

11. In each part, find the limit.

(a) $\lim_{x\to+\infty}(e^x - x^2)$ (b) $\lim_{x\to 1}\sqrt{\dfrac{\ln x}{x^4-1}}$

(c) $\lim_{x\to 0}\dfrac{a^x-1}{x}$, $a > 0$

12. Show that the function $y = e^{ax}\sin bx$ satisfies

$$y'' - 2ay' + (a^2+b^2)y = 0$$

for any real constants a and b.

13. Show that the function $y = \tan^{-1} x$ satisfies

$$y'' = -2\sin y\cos^3 y$$

14. Show that the rate of change of $y = 3^{2x}5^{7x}$ is proportional to y.

15. The hypotenuse of a right triangle is growing at a rate of a cm/s and one leg is decreasing at a rate of b cm/s. How fast is the acute angle between the hypotenuse and the other leg changing at the instant when both legs are 1 cm?

16. In each part, find $(f^{-1})'(x)$ using Formula (21) of Section 4.3, and check your answer by differentiating f^{-1} directly.
(a) $f(x) = 3/(x+1)$ (b) $f(x) = \sqrt{e^x}$

17. (a) Sketch the curves $y = \pm e^{-x/2}$ and $y = e^{-x/2}\sin 2x$ for $-\pi/2 \le x \le 3\pi/2$ in the same coordinate system, and check your work using a graphing utility.
(b) Find all x-intercepts of the curve $y = e^{-x/2}\sin 2x$ in the stated interval, and find the x-coordinates of all points where this curve intersects the curves $y = \pm e^{-x/2}$.

18. In each part, sketch the graph, and check your work with a graphing utility.
(a) $f(x) = 3\sin^{-1}(x/2)$
(b) $f(x) = \cos^{-1} x - \pi/2$
(c) $f(x) = 2\tan^{-1}(-3x)$
(d) $f(x) = \cos^{-1} x + \sin^{-1} x$

19. In each part, use any appropriate method to find dy/dx.
(a) $y = (1+x)^{1/x}$ (b) $y = x^{(e^x)}$
(c) $y = e^{\ln(x^3+1)}$ (d) $y = \dfrac{a}{1+be^{-x}}$
(e) $xy^{2/3} + yx^{2/3} = x^2$ (f) $y = \ln\left(\dfrac{\sqrt{x}\sqrt[3]{x+1}}{\sin x\sec x}\right)$

20. (a) Suppose that the graph of $y = \log x$ is drawn with equal scales of 1 inch per unit in both the x- and y-directions. If a bug wants to walk along the graph until it reaches a height of 5 ft above the x-axis, how many miles to the right of the origin will it have to travel?
(b) Suppose that the graph of $y = 10^x$ is drawn with equal scales of 1 inch per unit in both the x- and y-directions. If a bug wants to walk along the graph until it reaches a height of 100 mi above the x-axis, how many feet to the right of the origin will it have to travel?

21. (a) Show that the graphs of $y = \ln x$ and $y = x^{0.2}$ intersect.
(b) Approximate the solution(s) of the equation $\ln x = x^{0.2}$ to three decimal places.

22. (a) Show that for $x > 0$ and $k \neq 0$ the equations

$$x^k = e^x \quad \text{and} \quad \frac{\ln x}{x} = \frac{1}{k}$$

have the same solutions.
(b) Use the graph of $y = (\ln x)/x$ to determine the values of k for which the equation $x^k = e^x$ has two distinct positive solutions.
(c) Find the positive solution(s) of $x^8 = e^x$.

23. Find the value of b so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

24. In each part, find the value of k for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
(a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

EXPANDING THE CALCULUS HORIZON

For additional material relating to this chapter, visit the Anton Website at http://www.wiley.com/college/anton