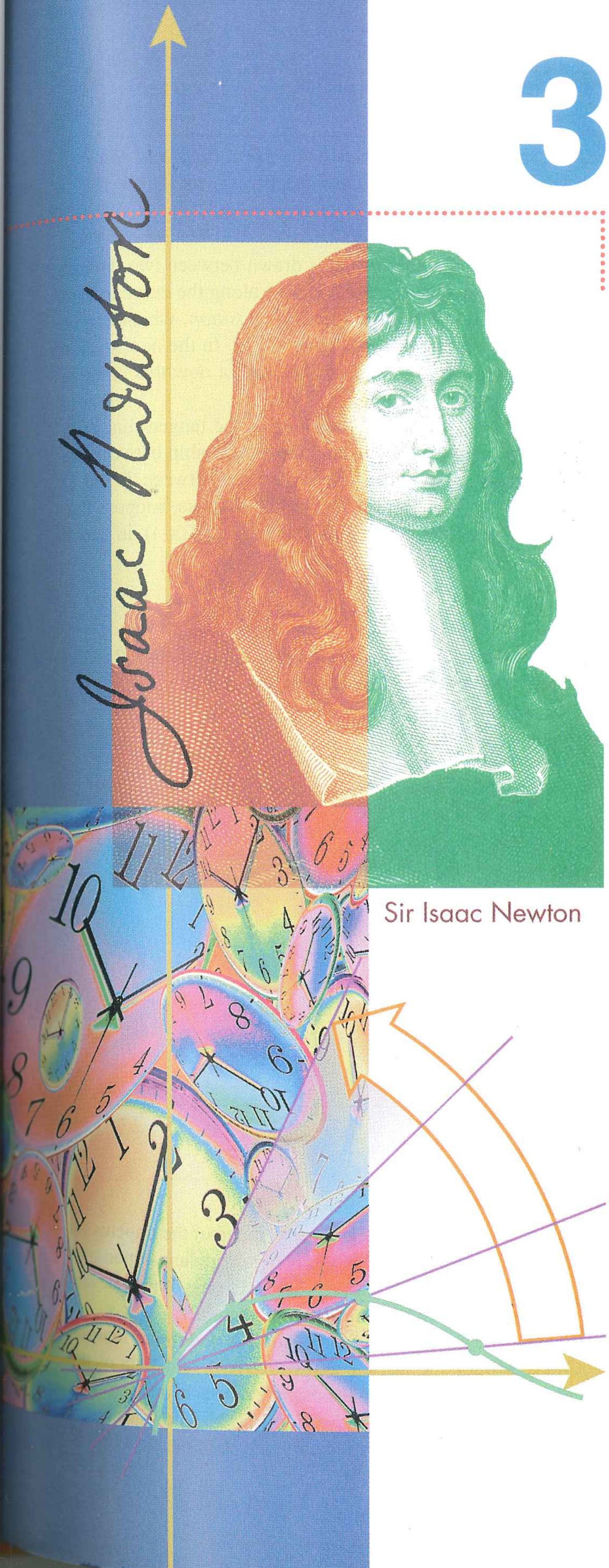


Sir Isaac Newton

3

The Derivative

Many physical phenomena involve changing quantities—the speed of a rocket, the inflation of currency, the number of bacteria in a culture, the shock intensity of an earthquake, the voltage of an electrical signal, and so forth. In this chapter we will develop the concept of a *derivative*, which is the mathematical tool that is used to study rates at which physical quantities change. In Section 3.1 we will show that there is a close relationship between rates of change and tangent lines to graphs, and we will show how the familiar idea of velocity can be viewed as a rate of change. In Sections 3.2 to 3.5 we will define the concept of a derivative precisely and develop the mathematical tools for calculating them.

One of the important themes in applied science is developing methods for approximating quantities that are difficult to calculate exactly. In Section 3.6 we will show how derivatives can be applied to certain kinds of approximation problems.

3.1 TANGENT LINES AND RATES OF CHANGE

In this section we will establish a basic relationship between tangent lines and rates of change. Our work here is intended to be informal and introductory, and all of the ideas that we develop will be revisited in more detail in later sections.

SLOPE OF A TANGENT LINE

In Section 2.1 we observed informally that if a secant line is drawn between two distinct points P and Q on a curve $y = f(x)$, and if Q is allowed to move along the curve toward P, then we can expect the secant line to rotate toward a *limiting position*, which can be regarded as the tangent line to the curve at the point P (Figure 3.1.1). In the next section we will give a precise mathematical definition of a tangent line, but for now this intuitive idea will suffice.

In many problems we will be more concerned with the *slope* of the tangent line than with the tangent line itself, so it will be helpful to understand the relationship between the slope $m_{\tan}$ of the tangent line at P and the slope $m_{\sec}$ of the secant line between P and Q as the point Q moves along the curve $y = f(x)$ toward P. For this purpose, suppose that the secant line passes through the distinct points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$, in which case its slope is

$$m_{\sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{1}$$

(Figure 3.1.2). As this figure suggests, the point Q moves along the curve toward P if and only if x_1 approaches x_0. Thus, from (1) the slope of the tangent line at P is

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

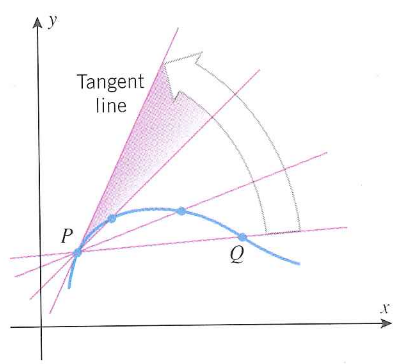


Figure 3.1.1

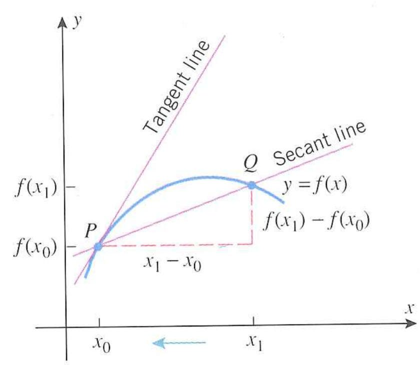


Figure 3.1.2

AVERAGE VERSUS INSTANTANEOUS VELOCITY

Although tangent lines are of interest as a matter of pure geometry, much of the impetus for studying them arose in the seventeenth century when scientists recognized their importance in studying the motion of objects that move with nonconstant velocity. Some of the relevant ideas were discussed in Section 1.5, but it will be helpful to review them here.

Recall that a particle moving along a line, say an s-axis, is said to have ***rectilinear motion***. In the most general kind of rectilinear motion the particle may move back and forth on the line; however, here, as in Section 1.5, we will assume that the particle moves in one direction only—the positive direction of the s-axis. As discussed in Section 1.5, this allows us to use the terms *speed* and *velocity* interchangeably, since there is only one possible direction of motion. General rectilinear motion will be discussed later.

We showed in Section 1.5 that if a particle has uniform rectilinear motion, that is, it moves with constant velocity v along a line, then its position versus time curve is a line of slope v; conversely, if the position versus time curve for a particle in rectilinear motion is a line of slope v, then the particle has constant velocity v. Here, we will consider the more general case of a particle moving in the positive s-direction with *variable* velocity, in which case the position versus time curve need not be linear. For this purpose we will need to examine the meaning of the term *velocity* more critically.

If a car travels 75 miles over a straight road in a 3-hour period, then its average velocity during the trip is $75/3 = 25$ mi/h. However, this does not mean that the car travels at 25 mi/h for the entire trip; it may speed up and slow down at various times. Thus, the average velocity provides information about the velocity of the car over the entire trip but no information about its velocity at specific times during the trip.

Although average velocity is useful for many purposes, there are many situations in which it is of no help. For example, if a car strikes a tree during a trip, the damage sustained is not determined by the average velocity up to the time of impact, but rather by the *instantaneous velocity* at the precise moment of impact. However, the concept of instantaneous velocity is subtle, and a clear understanding of its meaning evaded scientists until the advent of calculus in the seventeenth century.

A nice explanation of the difficulty in defining and calculating instantaneous velocity was given by Morris Kline* who wrote:

> *In contrasting average velocity with instantaneous velocity we implicitly utilize a distinction between interval and instant. . . . An average velocity is one that concerns what happens over an interval of time—3 hours, 5 seconds, one-half second, and so forth. The interval may be small or large, but it does represent the passage of a definite amount of time. We use the word instant, however, to state the fact that something happens so fast that no time elapses. The event is momentary. When we say, for example, that it is 3 o'clock, we refer to an instant, a precise moment. If the lapse of time is pictured by length along a line, then an interval (of time) is represented by a line segment, whereas an instant corresponds to a point. The notion of an instant, although it is used in everyday life, is strictly a mathematical idealization.*
>
> *Our ways of thinking about real events cause us to speak in terms of instants and velocity at an instant, but closer examination shows that the concept of velocity at an instant presents difficulties. Average velocity, which is simply the distance traveled during some interval of time divided by that amount of time, is easily calculated. Suppose, however, that we try to carry over this process to instantaneous velocity. The distance an automobile travels in one instant is* 0 *and the time that elapses during one instant is also* 0. *Hence the distance divided by the time is* $0/0$, *which is meaningless. Thus, although instantaneous velocity is a physical reality, there seems to be a difficulty in calculating it, and unless we can calculate it, we cannot work with it mathematically.*

Our goal, then, is to define the concept of instantaneous velocity in a way that it can be calculated and worked with mathematically. For this purpose, consider a car that moves in a single direction along a straight road, and assume that an s-axis has been introduced with its positive direction in the direction of motion. As shown in Figure 3.1.3, suppose that a clock tracks the elapsed time t, starting at $t = 0$, and that the coordinate of the car as a function of t is $s = f(t)$. The function f is called the ***position function*** of the car, and the graph of $s = f(t)$ is what we have been calling the position versus time curve. The third part of Figure 3.1.3 shows a typical position versus time curve for a car whose coordinate

* MORRIS KLINE (1908–1992). American mathematician, scholar, and educator. Kline made numerous contributions to mathematical thought, wrote extensively on education, especially mathematics education, and taught, lectured, and served as a consultant throughout his very active career. He was the author of many popular books including *Mathematical Thought from Ancient to Modern Times* and *Why Johnny Can't Add: The Failure of the New Mathematics*.

at time $t = 0$ is s_0. Observe that we have drawn the curve so that s increases with t. This is because we have assumed the car to be traveling in the positive direction, and decreasing values of s would imply a motion in the negative direction.

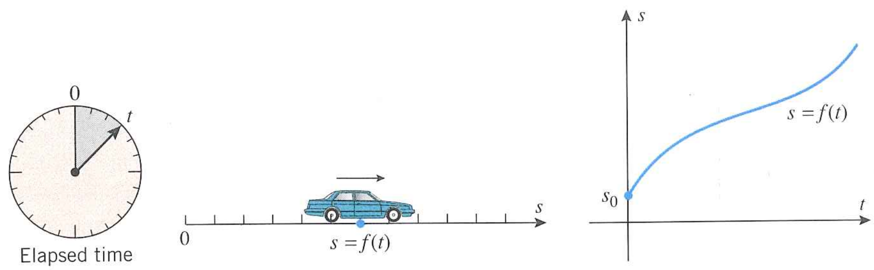


Figure 3.1.3

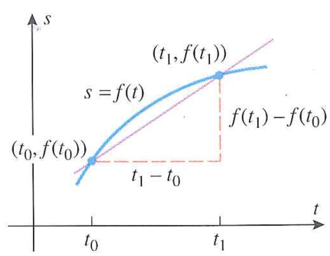


Figure 3.1.4

The position versus time curve provides a simple geometric interpretation of the average velocity of the car over a time interval, say from t_0 to t_1. If the car has a coordinate $s_0 = f(t_0)$ at time t_0 and coordinate $s_1 = f(t_1)$ at time t_1, where $t_1 > t_0$, then the distance traveled during the time interval is $s_1 - s_0$ and the time elapsed is $t_1 - t_0$. Thus, the average velocity during the time interval, denoted by v_{ave}, is

$$v_{\text{ave}} = \frac{s_1 - s_0}{t_1 - t_0} = \frac{f(t_1) - f(t_0)}{t_1 - t_0} \tag{2}$$

which is just the slope of the secant line connecting the points (t_0, s_0) and (t_1, s_1) on the position versus time curve (Figure 3.1.4).

Now suppose that we are interested in the instantaneous velocity of the car at time t_0. Intuition suggests that over a small time interval the velocity of the car cannot vary much, so if t_1 is close to t_0, then the average velocity of the car over the time interval from t_0 to t_1 should closely approximate the instantaneous velocity of the car at time t_0. Moreover, the smaller the time interval between t_0 and t_1, the better the approximation. This suggests that if we let t_1 get closer and closer to t_0, then the average velocity of the car over the time interval from t_0 to t_1 should get closer and closer to the instantaneous velocity at time t_0. Thus, if we denote the instantaneous velocity of the car at time t_0 by v_{inst}, we have

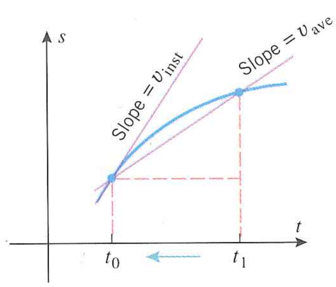


Figure 3.1.5

$$v_{\text{inst}} = \lim_{t_1 \to t_0} v_{\text{ave}} = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0} \tag{3}$$

Since v_{ave} is the slope of the secant line joining the points $(t_0, f(t_0))$ and $(t_1, f(t_1))$ on the position versus time curve $s = f(t)$, and since the point $(t_1, f(t_1))$ moves along this curve toward $(t_0, f(t_0))$ as $t_1 \to t_0$ (Figure 3.1.5), it follows from (3) that v_{inst} can be interpreted as the slope of the tangent line to the position versus time curve at the point $(t_0, f(t_0))$.

These ideas are illustrated numerically in Table 3.1.1. The first part of the table shows the coordinates of a particle moving along an s-axis over the time interval from $t = 4.00$ to

Table 3.1.1

t (s)	4.00	4.50	5.00	5.50	5.80	5.90	5.95	5.98	6.00
s (ft)	1.00	1.25	2.00	3.25	4.24	4.61	4.80	4.92	5.00

TIME INTERVAL	[4.00, 6.00]	[4.50, 6.00]	[5.00, 6.00]	[5.50, 6.00]	[5.80, 6.00]	[5.90, 6.00]	[5.95, 6.00]	[5.98, 6.00]
AVERAGE VELOCITY (ft/s)	1.00	2.50	3.00	3.50	3.80	3.90	4.00	4.00

$t = 6.00$. From these values we can calculate the average velocity of the particle over a succession of shrinking time intervals ending at time $t = 6.00$ s. For example, the calculations for the average velocity over the time interval [4.50, 6.00] are

$$v_{\text{ave}} = \frac{5.00 - 1.25}{6.00 - 4.50} = \frac{3.75}{1.50} = 2.50 \text{ ft/s}$$

The resulting average velocities in the second part of the table suggest that to two decimal places the instantaneous velocity at time $t = 6.00$ s is 4.00 ft/s.

The main ideas in the preceding discussion can be summarized as follows.

3.1.1 GEOMETRIC INTERPRETATION OF AVERAGE VELOCITY. *If a particle moves in the positive direction along an s-axis, and if the position versus time curve is $s = f(t)$, then the average velocity of the particle between times t_0 and t_1 is represented geometrically by the slope of the secant line joining the points $(t_0, f(t_0))$ and $(t_1, f(t_1))$.*

3.1.2 GEOMETRIC INTERPRETATION OF INSTANTANEOUS VELOCITY. *If a particle moves in the positive direction along an s-axis, and if the position versus time curve is $s = f(t)$, then the instantaneous velocity of the particle at time t_0 is represented geometrically by the slope of the tangent line to the curve at the point $(t_0, f(t_0))$.*

AVERAGE AND INSTANTANEOUS RATES OF CHANGE

Velocity can be viewed as a *rate of change*—the rate of change of position with time, or in algebraic terms, the rate of change of s with t. Rates of change occur in many applications. For example:

- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- An engineer might be interested in the rate at which the length of a metal rod changes with temperature.
- An economist might be interested in the rate at which production cost changes with the quantity of a product that is manufactured.
- A medical researcher might be interested in the rate at which the radius of an artery changes with the concentration of alcohol in the bloodstream.

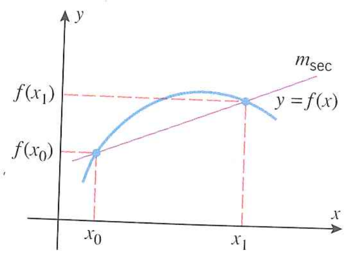
In general, if x and y are any quantities related by an equation $y = f(x)$, we can consider the rate at which y changes with x. As with velocity, we distinguish between an average rate of change represented by the slope of a secant line and an instantaneous rate of change represented by the slope of the tangent line. More precisely, we make the following definitions.

3.1.3 DEFINITION. If $y = f(x)$, then the ***average rate of change of y with respect to x over the interval $[x_0, x_1]$*** is the slope m_{sec} of the secant line joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the graph of f (Figure 3.1.6*a*); that is,

$$m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{4}$$

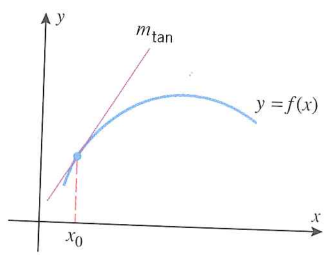
3.1.4 DEFINITION. If $y = f(x)$, then the ***instantaneous rate of change of y with respect to x at the point x_0*** is the slope $m_{\tan}$ of the tangent line to the graph of f at the point x_0 (Figure 3.1.6*b*); that is,

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{5}$$



m_{sec} is the average rate of change of y with respect to x over the interval $[x_0, x_1]$.

(*a*)



$m_{\tan}$ is the instantaneous rate of change of y with respect to x at the point x_0.

(*b*)

Figure 3.1.6

Example 1

Let $y = x^2 + 1$.

(a) Find the average rate of change of y with respect to x over the interval [3, 5].

(b) Find the instantaneous rate of change of y with respect to x at the point $x = -4$.

(c) Find the instantaneous rate of change of y with respect to x at a general point $x = x_0$.

Solution (a). We will apply Formula (4) with $f(x) = x^2 + 1$, $x_0 = 3$, and $x_1 = 5$. This yields

$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{5 - 3} = 8$$

Thus, on the average, y increases 8 units per unit increase in x over the interval [3, 5].

Solution (b). We will apply Formula (5) with $f(x) = x^2 + 1$ and $x_0 = -4$. This yields

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

Because the instantaneous rate of change is negative, y is *decreasing* at the point $x = -4$; it is decreasing at a rate of 8 units per unit increase in x.

Solution (c). We proceed as in part (b).

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{(x_1^2 + 1) - (x_0^2 + 1)}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0} = \lim_{x_1 \to x_0} (x_1 + x_0) = 2x_0$$

Thus, the instantaneous rate of change of y with respect to x at $x = x_0$ is $2x_0$. Observe that the result in part (b) can be obtained from this more general result by letting $x_0 = -4$. ◀

RATES OF CHANGE IN APPLICATIONS

In applied problems, average and instantaneous rates of change must be accompanied by appropriate units. In general, the units for a rate of change of y with respect to x are obtained by "dividing" the units of y by the units of x and then simplifying according to the standard rules of algebra. Here are some examples:

- If y is in degrees Fahrenheit (°F) and x is in inches (in), then a rate of change of y with respect to x has units of degrees Fahrenheit per inch (°F/in).
- If y is in feet per second (ft/s) and x is in seconds (s), then a rate of change of y with respect to x has units of feet per second per second (ft/s/s), which would usually be written as ft/s^2.
- If y is in newton-meters (N·m) and x is in meters (m), then a rate of change of y with respect to x has units of newtons (N), since N·m/m = N.
- If y is in foot-pounds (ft·lb) and x is in hours (h), then a rate of change of y with respect to x has units of foot-pounds per hour (ft·lb/h).

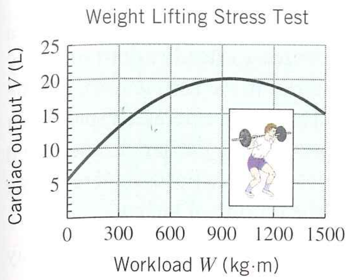


Figure 3.1.7

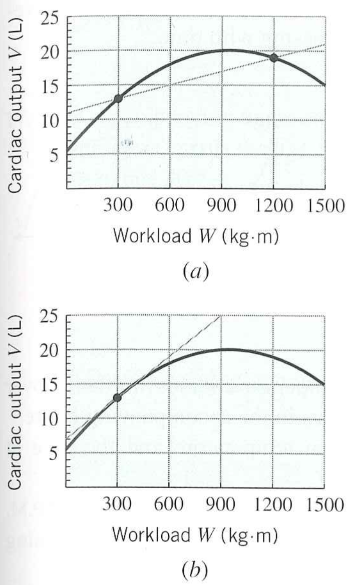


Figure 3.1.8

Example 2

The limiting factor in athletic endurance is cardiac output, that is, the volume of blood that the heart can pump per unit of time during an athletic competition. Figure 3.1.7 shows a stress-test graph of cardiac output V in liters (L) of blood versus workload W in kilogram-meters (kg·m) for 1 minute of weight lifting. This graph illustrates the known medical fact that cardiac output increases with the workload, but after reaching a peak value begins to decrease.

(a) Use the secant line shown in Figure 3.1.8*a* to estimate the average rate of change of cardiac output with respect to workload as the workload increases from 300 to 1200 kg·m.

(b) Use the tangent line shown in Figure 3.1.8*b* to estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 300 kg·m.

Solution (a). Using the estimated points (300, 13) and (1200, 19), the slope of the secant line indicated in Figure 3.1.8*a* is

$$m_{\text{sec}} \approx \frac{19-13}{1200-300} \approx 0.0067 \frac{\text{L}}{\text{kg·m}}$$

Thus, the average rate of change of cardiac output with respect to workload over the interval is approximately 0.0067 L/kg·m. This means that on the average a 1-unit increase in workload produced a 0.0067-L increase in cardiac output over the interval.

Solution (b). Using the estimated tangent line in Figure 3.1.8*b* and the estimated points (0, 7) and (900, 25) on this tangent line, we obtain

$$m_{\tan} \approx \frac{25-7}{900-0} \approx 0.02 \frac{\text{L}}{\text{kg·m}}$$

Thus, the instantaneous rate of change of cardiac output with respect to workload is approximately 0.02 L/kg·m. ◀

EXERCISE SET 3.1

In Exercises 1–4, a function $y = f(x)$ and values of x_0 and x_1 are given.
(a) Find the average rate of change of y with respect to x over the interval $[x_0, x_1]$.
(b) Find the instantaneous rate of change of y with respect to x at the given value of x_0.
(c) Find the instantaneous rate of change of y with respect to x at a general point x_0.
(d) Sketch the graph of $y = f(x)$ together with the secant and tangent lines whose slopes are given by the results in parts (a) and (b).

1. $y = \frac{1}{2}x^2;\ x_0 = 3,\ x_1 = 4$

2. $y = x^3;\ x_0 = 1,\ x_1 = 2$

3. $y = 1/x;\ x_0 = 2,\ x_1 = 3$

4. $y = 1/x^2;\ x_0 = 1,\ x_1 = 2$

In Exercises 5–8, a function f and a value of x_0 are given.
(a) Find the slope of the tangent to the graph of f at a general point x_0.
(b) Use the result in part (a) to find the slope of the tangent line at the given value of x_0.

5. $f(x) = x^2 + 1;\ x_0 = 2$

6. $f(x) = x^2 + 3x + 2;\ x_0 = 2$

7. $f(x) = \sqrt{x};\ x_0 = 1$

8. $f(x) = 1/\sqrt{x};\ x_0 = 4$

9. The accompanying figure shows the position versus time curve for an elevator that moves upward a distance of 60 m and then discharges its passengers.

(a) Estimate the instantaneous velocity of the elevator at $t = 10$ s.

(b) Sketch a velocity versus time curve for the motion of the elevator for $0 \le t \le 20$.

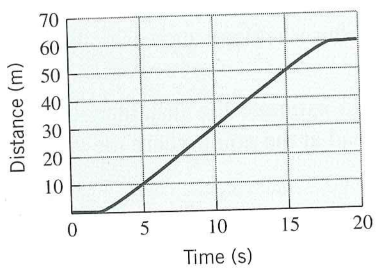


Figure Ex-9

10. The accompanying figure shows the position versus time curve for a certain particle moving along a straight line. Estimate each of the following from the graph:

(a) the average velocity over the interval $0 \le t \le 3$

(b) the values of t at which the instantaneous velocity is zero

(c) the values of t at which the instantaneous velocity is either a maximum or a minimum

(d) the instantaneous velocity when $t = 3$ s.

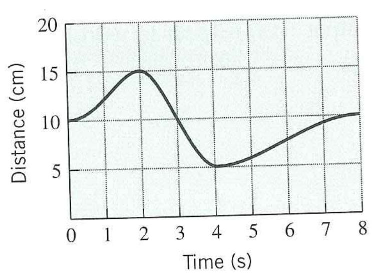


Figure Ex-10

11. The accompanying figure shows the position versus time curve for a certain particle moving on a straight line.

(a) Is the particle moving faster at time t_0 or time t_2? Explain.

(b) At the origin, the tangent is horizontal. What does this tell us about the initial velocity of the particle?

(c) Is the particle speeding up or slowing down in the interval $[t_0, t_1]$? Explain.

(d) Is the particle speeding up or slowing down in the interval $[t_1, t_2]$? Explain.

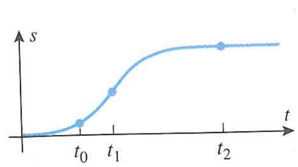


Figure Ex-11

12. An automobile, initially at rest, begins to move along a straight track. The velocity increases steadily until suddenly the driver sees a concrete barrier in the road and applies the brakes sharply at time t_0. The car decelerates rapidly, but it is too late—the car crashes into the barrier at time t_1 and instantaneously comes to rest. Sketch a position versus time curve that might represent the motion of the car.

13. If a particle moves at constant velocity, what can you say about its position versus time curve?

14. The accompanying figure shows the position versus time curves of four different particles moving on a straight line. For each particle, determine whether its instantaneous velocity is increasing or decreasing with time.

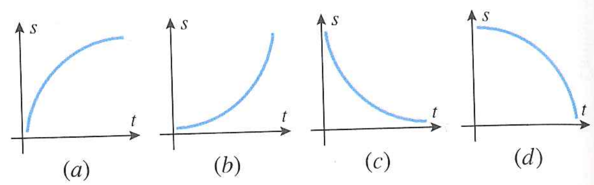


Figure Ex-14

15. Suppose that the outside temperature versus time curve over a 24-hour period is as shown in the accompanying figure.

(a) Estimate the maximum temperature and the time at which it occurs.

(b) The temperature rise is fairly linear from 8 A.M. to 2 P.M. Estimate the rate at which the temperature is increasing during this time period.

(c) Estimate the time at which the temperature is decreasing most rapidly. Estimate the instantaneous rate of change of temperature with respect to time at this instant.

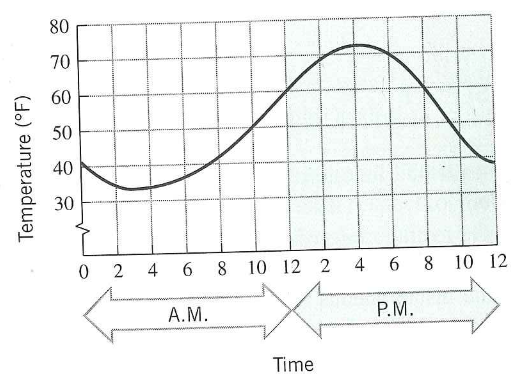


Figure Ex-15

16. The accompanying figure shows the graph of the pressure p in atmospheres (atm) versus the volume V in liters (L) of 1 mole of an ideal gas at a constant temperature of 300 K (kelvins). Use the tangent lines shown in the figure to estimate the rate of change of pressure with respect to volume at the points where $V = 10$ L and $V = 25$ L.

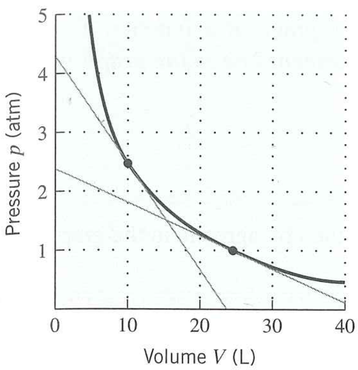


Figure Ex-16

17. The accompanying figure shows the graph of the height h in centimeters versus the age t in years of an individual from birth to age 20.

(a) When is the growth rate greatest?

(b) Estimate the growth rate at age 5.

(c) At approximately what age between 10 and 20 is the growth rate greatest? Estimate the growth rate at this age.

(d) Draw a rough graph of the growth rate versus age.

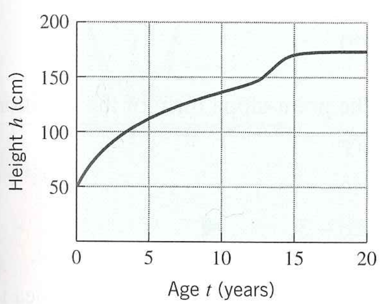


Figure Ex-17

In Exercises 18–21, use Formulas (2) and (3) to find the average and instantaneous velocity.

18. A rock is dropped from a height of 576 ft and falls toward Earth in a straight line. In t seconds the rock drops a distance of $s = 16t^2$ ft.

(a) How many seconds after release does the rock hit the ground?

(b) What is the average velocity of the rock during the time it is falling?

(c) What is the average velocity of the rock for the first 3 s?

(d) What is the instantaneous velocity of the rock when it hits the ground?

19. During the first 40 s of a rocket flight, the rocket is propelled straight up so that in t seconds it reaches a height of $s = 5t^3$ ft.

(a) How high does the rocket travel in 40 s?

(b) What is the average velocity of the rocket during the first 40 s?

(c) What is the average velocity of the rocket during the first 135 ft of its flight?

(d) What is the instantaneous velocity of the rocket at the end of 40 s?

20. A particle moves on a line away from its initial position so that after t hours it is $s = 3t^2 + t$ miles from its initial position.

(a) Find the average velocity of the particle over the interval $[1, 3]$.

(b) Find the instantaneous velocity at $t = 1$.

21. A particle moves in the positive direction along a straight line so that after t minutes its distance is $s = 6t^4$ feet from the origin.

(a) Find the average velocity of the particle over the interval $[2, 4]$.

(b) Find the instantaneous velocity at $t = 2$.

3.2 THE DERIVATIVE

In this section we will introduce the concept of a derivative, *which is the primary mathematical tool that is used to calculate rates of change and slopes of tangent lines.*

TANGENT LINES DEFINED PRECISELY

In the preceding section we showed informally that the slope of the tangent line to the graph of $y = f(x)$ at the point x_0 is given by

$$m_{\tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{1}$$

However, for computational purposes it will be more convenient to express this formula in a different form by introducing a new variable $h = x_1 - x_0$. It follows that $x_1 = x_0 + h$, and consequently $x_1 \to x_0$ as $h \to 0$. Thus, (1) can be expressed as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

(Figure 3.2.1). This suggests the following formal definition of a tangent line.

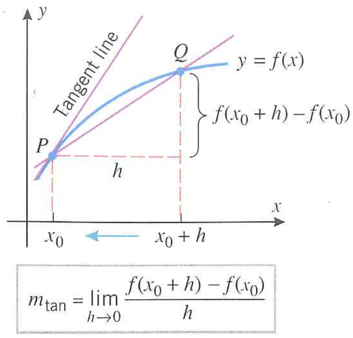


Figure 3.2.1

> **3.2.1 DEFINITION.** If $P(x_0, y_0)$ is a point on the graph of a function f, then the ***tangent line to the graph of f at P***, also called the ***tangent line to the graph of f at x_0***, is defined to be the line through P with slope
>
> $$m_{\tan} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \tag{2}$$
>
> provided this limit exists. If the limit does not exist, then by agreement the graph has no tangent line at P.

It follows from this definition that the point-slope form of the equation of the tangent line at x_0 is

$$y - y_0 = m_{\tan}(x - x_0) \tag{3}$$

Example 1

Find the equation of the tangent line to the graph of $y = x^2 + 1$ at the point (2, 5) (Figure 3.2.2).

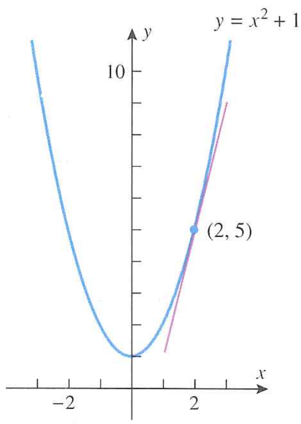


Figure 3.2.2

Solution. First, we will find the slope of the tangent line using (2) with $f(x) = x^2 + 1$ and $x_0 = 2$, and then we will find the equation by using (3). We obtain

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[(2+h)^2+1]-5}{h} \\ &= \lim_{h\to 0} \frac{(5+4h+h^2)-5}{h} = \lim_{h\to 0} \frac{4h+h^2}{h} \\ &= \lim_{h\to 0} (4+h) = 4 \end{aligned}$$

Thus, from (3) with $x_0 = 2$, $y_0 = 5$, and $m_{\tan} = 4$, the point-slope form of the equation of the tangent line is

$$y - 5 = 4(x - 2)$$

which we can write in slope-intercept form as $y = 4x - 3$. ◀

SLOPES OF TANGENT LINES BY ZOOMING

Slopes of tangent lines can be estimated by zooming with graphing utilities. The idea is to zoom in on the point of tangency until the surrounding curve segment appears to be a straight line that nearly coincides with the tangent line (Figure 3.2.3). The utility's trace operation can then be used to estimate the slope. Figure 3.2.4 illustrates this procedure for the tangent line in Example 1. The first part of the figure shows the graph of $y = x^2 + 1$ in the window* $[-6.3, 6.3] \times [0, 6.2]$, and the second part shows the graph after we have zoomed in on the point (2, 5) by a factor of 10. The trace operation produces the points (2.05, 5.2025) and (1.95, 4.8025) on the line, so the slope of the tangent line can be approximated as

$$m \approx \frac{5.2025 - 4.8025}{2.05 - 1.95} = \frac{0.4}{0.1} = 4.0$$

which happens to agree exactly with the result in Example 1. It is important to understand, however, that the exact agreement in this case is accidental; in general, this method will not produce exact results because of roundoff errors in the computations, and also because the magnified curve segment may have a slight curvature, even though it appears to be a straight line.

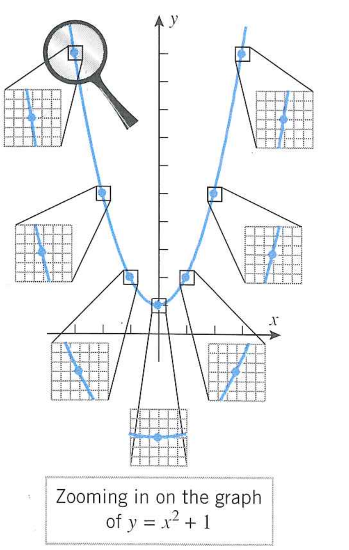


Figure 3.2.3

*The window $[-6.3, 6.3] \times [0, 6.2]$ was chosen because it contains the point of tangency (2, 5) and produces convenient steps on the author's calculator when the trace operation is applied. Books on graphing calculators sometimes call these "friendly windows."

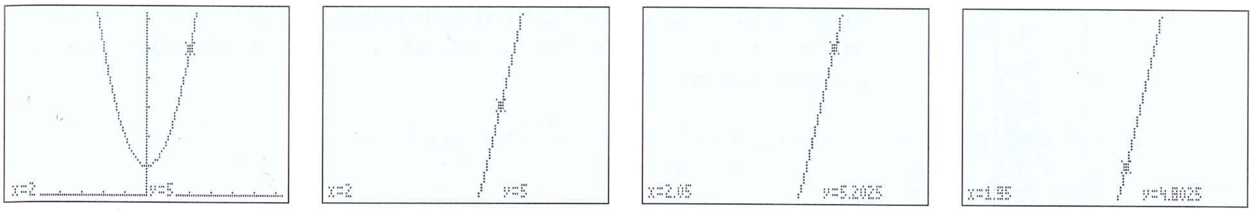


Figure 3.2.4

THE DERIVATIVE

In general, the slope of a tangent line to a curve $y = f(x)$ will depend on the point x at which the slope is being computed; thus, the slope is itself a function of x. To illustrate this, let us use (2) to compute $m_{\tan}$ at a general point x for the curve $y = x^2 + 1$. The computations are similar to those in Example 1, except that now we let x_0 have an arbitrary value $x_0 = x$, whereas in Example 1 we had $x_0 = 2$. We obtain

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) = 2x \end{aligned} \tag{4}$$

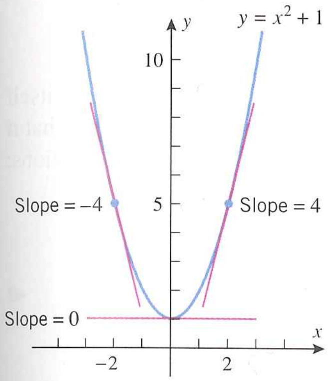


Figure 3.2.5

Now we can use the general formula $m_{\tan} = 2x$ to compute the slope of the tangent line at any point along the curve $y = x^2 + 1$ simply by substituting the appropriate value for x. For example, if $x = 2$, then we obtain $m_{\tan} = 2x = 4$, which agrees with the result in Example 1. Similarly, if $x = 0$, then $m_{\tan} = 0$; and if $x = -2$, then $m_{\tan} = -4$ (Figure 3.2.5).

To generalize this idea, the slope of the tangent line to the graph of $y = f(x)$ at a general point x can be obtained by setting $x_0 = x$ in (2), which yields the formula

$$m_{\tan} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

This "slope-producing function" is so important that it has some notation and terminology associated with it.

3.2.2 DEFINITION. The function f' defined by the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{5}$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x for which the limit exists.

Recalling from the last section that the slope of a tangent line to the graph of $y = f(x)$ can be interpreted as the instantaneous rate of change of y with respect to x, it follows that the derivative of a function f can be interpreted in two ways:

Two interpretations of the Derivative. The derivative f' of a function f can be interpreted either as a function whose value at x is the slope of the tangent line to the graph of $y = f(x)$ at x, or, alternatively, it can be interpreted as a function whose value at x is the instantaneous rate of change of y with respect to x at the point x.

Example 2

(a) Find the derivative with respect to x of $f(x) = x^3 - x$.

(b) Graph f and f' together, and discuss the relationship between the two graphs.

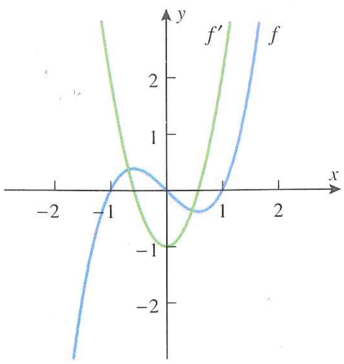


Figure 3.2.6

Solution (a). Later in this chapter we will develop efficient methods for finding derivatives, but for now we will find the derivative directly from Formula (5) in the definition of f'. The computations are as follows:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^3-(x+h)]-(x^3-x)}{h} \\ &= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h} \\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-h}{h} \\ &= \lim_{h\to 0}(3x^2+3xh+h^2-1) = 3x^2-1 \end{aligned}$$

Solution (b). Since $f'(x)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at the point x, the derivative $f'(x)$ is positive where the tangent line $y = f(x)$ has positive slope, it is negative where the tangent line has negative slope, and it is zero where the tangent line is horizontal. We leave it for the reader to verify that this is consistent with the graphs of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$ shown in Figure 3.2.6. ◀

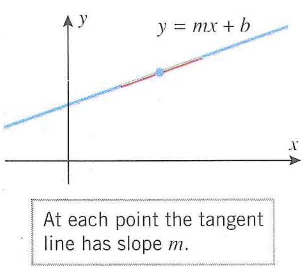


Figure 3.2.7

Example 3

At each point x, the tangent line to a line $y = mx + b$ coincides with the line itself (Figure 3.2.7), and hence all tangent lines have slope m. This suggests geometrically that if $f(x) = mx + b$, then $f'(x) = m$ for all x. This is confirmed by the following computations:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[m(x+h)+b]-(mx+b)}{h} \\ &= \lim_{h\to 0}\frac{mx+mh+b-mx-b}{h} = \lim_{h\to 0}\frac{mh}{h} = \lim_{h\to 0} m = m \end{aligned}$$ ◀

Example 4

(a) Find the derivative with respect to x of $f(x) = \sqrt{x}$.

(b) Find the slope of the tangent line to $y = \sqrt{x}$ at $x = 9$.

(c) Find the limits of $f'(x)$ as $x \to 0^+$ and as $x \to +\infty$, and explain what those limits say about the graph of f.

Solution (a). From Definition 3.2.2,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

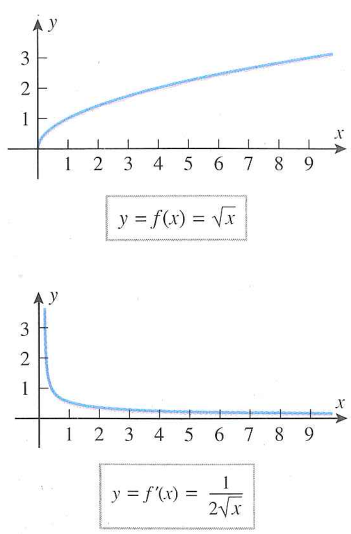


Figure 3.2.8

Solution (b). The slope of the tangent line at $x = 9$ is $f'(9)$, and thus from part (a) this slope is $f'(9) = 1/(2\sqrt{9}) = \frac{1}{6}$.

Solution (c). The graphs of $f(x) = \sqrt{x}$ and $f'(x) = 1/(2\sqrt{x})$ are shown in Figure 3.2.8. Observe that $f'(x) > 0$ if $x > 0$, which means that all tangent lines to the graph of $y = \sqrt{x}$

have positive slope over this interval. Since

$$\lim_{x \to 0^+} \frac{1}{2\sqrt{x}} = +\infty \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{2\sqrt{x}} = 0$$

the tangent lines become more and more vertical as $x \to 0^+$, and they become more and more horizontal as $x \to +\infty$. ◀

FOR THE READER. Use a graphing utility to estimate the slope of the tangent line to $y = \sqrt{x}$ at $x = 9$ by zooming, and compare your result to the exact value obtained in the last example. If you have a CAS, read the documentation to determine how it can be used to find derivatives, and then use it to confirm the derivatives obtained in Examples 2, 3, and 4.

DIFFERENTIABILITY

Recall from Definition 3.2.2 that the derivative of a function f is defined at those points where the limit (5) exists. Points where this limit exists are called ***points of differentiability*** for f, and points where this limit does not exist are called ***points of nondifferentiability*** for f.

If x_0 is a point of differentiability for f, then we say that f is ***differentiable at x_0*** or that ***the derivative of f exists at x_0***; and if x_0 is a point of nondifferentiability for f, then we say that ***the derivative of f does not exist at x_0***. If f is differentiable at every point in an open interval (a, b), then we will say that f is ***differentiable on (a, b)***. This definition also applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the case where f is differentiable on $(-\infty, +\infty)$ we will say that f is ***differentiable everywhere***. If f is differentiable on an open interval but the particular interval is not important for the discussion, then we will say that f is ***differentiable*** (without referencing the interval).

Geometrically, the points of differentiability of f are the points where the curve $y = f(x)$ has a tangent line, and the points of nondifferentiability are the points where the curve does not have a tangent line. Informally stated, the most commonly encountered points of nondifferentiability can be classified as

- Corners
- Points of vertical tangency
- Points of discontinuity

Figure 3.2.9 illustrates each of these situations.

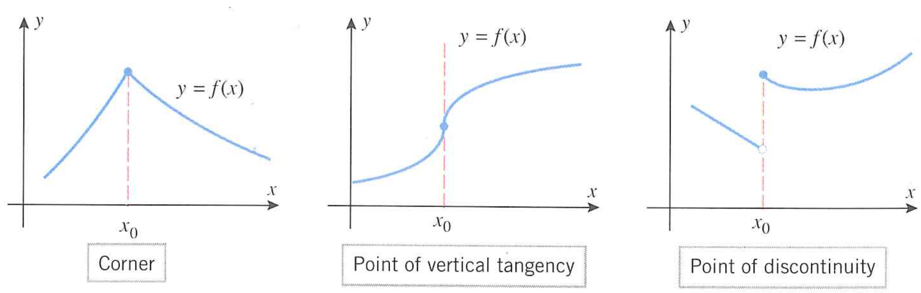


Figure 3.2.9

It makes sense intuitively that corners are points of nondifferentiability, since there is no reasonable way to draw a unique tangent line at such points. For example, Figure 3.2.10*a* shows a typical corner point $P(x_0, f(x_0))$ on the graph of a function f. At this point the secant lines joining P and Q have different limiting positions, depending on whether Q approaches P from the left or right; hence the slopes of the secant lines do not have a two-sided limit.

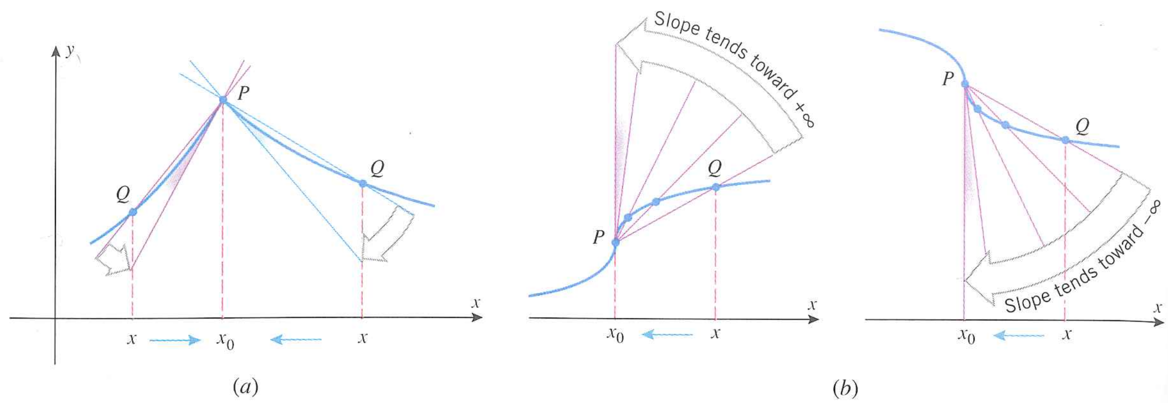


Figure 3.2.10

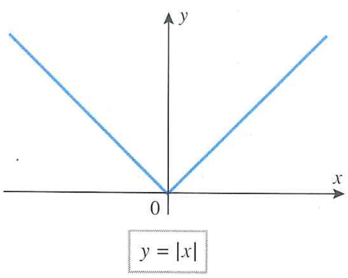


Figure 3.2.11

By a point of *vertical tangency* we mean a place on the curve where the secant lines approach a vertical limiting position. At such points, the only reasonable candidate for the tangent line is the vertical line at the point. But vertical lines have infinite slope, so the derivative (were it to exist) would not have a finite real value there, which explains intuitively why the derivative does not exist at points of vertical tangency (Figure 3.2.10*b*).

Example 5

The graph of $y = |x|$ in Figure 3.2.11 suggests that there is a corner at $x = 0$, and this implies that $f(x) = |x|$ is not differentiable at that point.

(a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$ by showing that the limit in Definition 3.2.2 does not exist at that point.

(b) Find a formula for $f'(x)$.

Solution (a). From Formula (5) with $x = 0$, the value of $f'(0)$, if it were to exist, would be given by

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

But

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases}$$

so that

$$\lim_{h\to 0^-} \frac{|h|}{h} = -1 \quad \text{and} \quad \lim_{h\to 0^+} \frac{|h|}{h} = 1$$

Thus,

$$f'(0) = \lim_{h\to 0} \frac{|h|}{h}$$

does not exist because the one-sided limits are not equal. Consequently, $f(x) = |x|$ is not differentiable at $x = 0$.

Solution (b). A formula for the derivative of $f(x) = |x|$ can be obtained by writing $|x|$ in piecewise form and treating the cases $x > 0$ and $x < 0$ separately. If $x > 0$, then $f(x) = x$ and $f'(x) = 1$; and if $x < 0$, then $f(x) = -x$ and $f'(x) = -1$. Thus,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

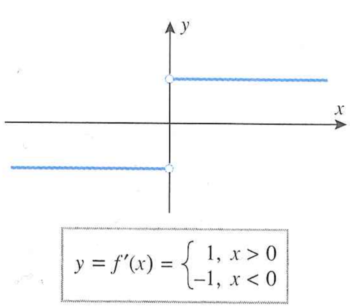


Figure 3.2.12

The graph of f' is shown in Figure 3.2.12. Observe that f' is not a continuous function, so this example shows that the derivative of a continuous function need not be continuous. ◀

RELATIONSHIP BETWEEN DIFFERENTIABILITY AND CONTINUITY

It makes sense intuitively that a function f cannot be differentiable at a point of discontinuity, since there is no reasonable way to draw a unique tangent line at such points. The following theorem shows that a function f must be continuous at each point where it is differentiable (or stated another way, a function f cannot be differentiable at a point of discontinuity).

> **3.2.3 THEOREM.** *If f is differentiable at a point x_0, then f is also continuous at x_0.*

Proof. We are given that f is differentiable at x_0, so it follows from (5) that $f'(x_0)$ exists and is given by

$$f'(x_0) = \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\right] \tag{6}$$

To show that f is continuous at x_0, we must show that $\lim_{x\to x_0} f(x) = f(x_0)$, or equivalently,

$$\lim_{x\to x_0}[f(x)-f(x_0)] = 0$$

Expressing this in terms of the variable $h = x - x_0$, we must prove that

$$\lim_{h\to 0}[f(x_0+h)-f(x_0)] = 0$$

However, this can be proved using (6) as follows:

$$\begin{aligned}\lim_{h\to 0}[f(x_0+h)-f(x_0)] &= \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\cdot h\right]\\ &= \lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\right]\cdot \lim_{h\to 0} h\\ &= f'(x_0)\cdot 0 = 0\end{aligned}$$

■

REMARK. Theorem 3.2.3 shows that differentiability at a point implies continuity at that point. However, the converse is false; that is, *a function may be continuous at a point but not differentiable there*. In fact, this occurs at any point where the function is continuous and has a corner. For example, we saw in Example 5 that the function $f(x) = |x|$ is continuous at $x = 0$, yet not differentiable there.

The relationship between continuity and differentiability was of great historical significance in the development of calculus. In the early nineteenth century mathematicians believed that the graph of a continuous function could not have too many points of nondifferentiability bunched up. They felt that if a continuous function had many points of nondifferentiability, these points, like the tips of a sawblade, would have to be separated from each other and joined by smooth curve segments (Figure 3.2.13). This misconception was shattered by a series of discoveries beginning in 1834. In that year a Bohemian priest, philosopher, and mathematician named Bernhard Bolzano* discovered a procedure for constructing a continuous function that is not differentiable at any point. Later, in 1860, the great

*BERNHARD BOLZANO (1781–1848). Bolzano, the son of an art dealer, was born in Prague, Bohemia (Czech Republic). He was educated at the University of Prague, and eventually won enough mathematical fame to be recommended for a mathematics chair there. However, Bolzano became an ordained Roman Catholic priest, and in 1805 he was appointed to a chair of Philosophy at the University of Prague. Bolzano was a man of great human compassion; he spoke out for educational reform, he voiced the right of individual conscience over government demands, and he lectured on the absurdity of war and militarism. His views so disenchanted Emperor Franz I of Austria that the emperor pressed the Archbishop of Prague to have Bolzano recant his statements. Bolzano refused and was then forced to retire in 1824 on a small pension. Bolzano's main contribution to mathematics was philosophical. His work helped convince mathematicians that sound mathematics must ultimately rest on rigorous proof rather than intuition. In addition to his work in mathematics, Bolzano investigated problems concerning space, force, and wave propagation.

German mathematician, Karl Weierstrass* produced the first formula for such a function. The graphs of such functions are impossible to draw; it is as if the corners are so numerous that any segment of the curve, when suitably enlarged, reveals more corners. The discovery of these pathological functions was important in that it made mathematicians distrustful of their geometric intuition and more reliant on precise mathematical proof. However, they remained only mathematical curiosities until the early 1980s, when applications of them began to emerge. During the past 10 years they have started to play a fundamental role in the study of geometric objects called ***fractals***. Fractals have revealed an order to natural phenomena that were previously dismissed as random and chaotic.

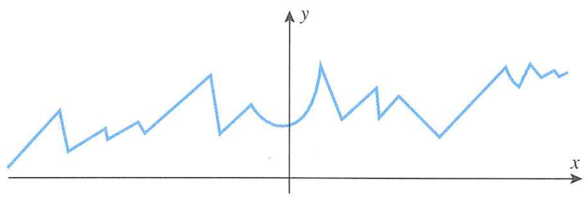


Figure 3.2.13

DERIVATIVE NOTATION

The process of finding a derivative is called ***differentiation***. You can think of differentiation as an operation on functions that associates a function f' with a function f. When the independent variable is x, the differentiation operation is often denoted by

$$\frac{d}{dx}[f(x)]$$

which is read "***the derivative of f (x) with respect to x***." Thus,

$$\frac{d}{dx}[f(x)] = f'(x) \tag{7}$$

For example, with this notation the derivatives obtained in Examples 2, 3, and 4 can be expressed as

$$\frac{d}{dx}[x^3 - x] = 3x^2 - 1, \quad \frac{d}{dx}[mx + b] = m, \quad \frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \tag{8}$$

To denote the value of the derivative at a specific point x_0 with the notation in (7), we would

*KARL WEIERSTRASS (1815–1897). Weierstrass, the son of a customs officer, was born in Ostenfelde, Germany. As a youth Weierstrass showed outstanding skills in languages and mathematics. However, at the urging of his dominant father, Weierstrass entered the law and commerce program at the University of Bonn. To the chagrin of his family, the rugged and congenial young man concentrated instead on fencing and beer drinking. Four years later he returned home without a degree. In 1839 Weierstrass entered the Academy of Münster to study for a career in secondary education, and he met and studied under an excellent mathematician named Christof Gudermann. Gudermann's ideas greatly influenced the work of Weierstrass. After receiving his teaching certificate, Weierstrass spent the next 15 years in secondary education teaching German, geography, and mathematics. In addition, he taught handwriting to small children. During this period much of Weierstrass's mathematical work was ignored because he was a secondary schoolteacher and not a college professor. Then, in 1854, he published a paper of major importance that created a sensation in the mathematics world and catapulted him to international fame overnight. He was immediately given an honorary Doctorate at the University of Königsberg and began a new career in college teaching at the University of Berlin in 1856. In 1859 the strain of his mathematical research caused a temporary nervous breakdown and led to spells of dizziness that plagued him for the rest of his life. Weierstrass was a brilliant teacher and his classes overflowed with multitudes of auditors. In spite of his fame, he never lost his early beer-drinking congeniality and was always in the company of students, both ordinary and brilliant. Weierstrass was acknowledged as the leading mathematical analyst in the world. He and his students opened the door to the modern school of mathematical analysis.

write

$$\frac{d}{dx}[f(x)]\bigg|_{x=x_0} = f'(x_0) \tag{9}$$

For example, from (8)

$$\frac{d}{dx}[x^3 - x]\bigg|_{x=1} = 3(1^2) - 1 = 2, \quad \frac{d}{dx}[mx + b]\bigg|_{x=5} = m, \quad \frac{d}{dx}[\sqrt{x}]\bigg|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Notations (7) and (9) are convenient when no dependent variable is involved. However, if there is a dependent variable, say $y = f(x)$, then (7) and (9) can be written as

$$\frac{d}{dx}[y] = f'(x) \quad \text{and} \quad \frac{d}{dx}[y]\bigg|_{x=x_0} = f'(x_0)$$

It is common to omit the brackets on the left side and write these expressions as

$$\frac{dy}{dx} = f'(x) \qquad \text{and} \qquad \frac{dy}{dx}\bigg|_{x=x_0} = f'(x_0)$$

where dy/dx is read as "the derivative of y with respect to x." For example, if $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad \frac{dy}{dx}\bigg|_{x=x_0} = \frac{1}{2\sqrt{x_0}}, \quad \frac{dy}{dx}\bigg|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

REMARK. Later, the symbols dy and dx will be defined separately. However, for the time being, dy/dx should not be regarded as a ratio; rather, it should be considered as a single symbol denoting the derivative.

When letters other than x and y are used for the independent and dependent variables, then the various notations for the derivative must be adjusted accordingly. For example, if $y = f(u)$, then the derivative with respect to u would be written as

$$\frac{d}{du}[f(u)] = f'(u) \quad \text{and} \quad \frac{dy}{du} = f'(u)$$

In particular, if $y = \sqrt{u}$, then

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}, \quad \frac{dy}{du}\bigg|_{u=u_0} = \frac{1}{2\sqrt{u_0}}, \quad \frac{dy}{du}\bigg|_{u=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

OTHER NOTATIONS

Some writers denote the derivative as $D_x[f(x)] = f'(x)$, but we will not use this notation in this text. In problems where the name of the independent variable is clear from the context, there are some other possible notations for the derivative. For example, if $y = f(x)$, but it is clear from the problem that the independent variable is x, then the derivative with respect to x might be denoted by y' or f'.

Often, you will see Definition 3.2.2 expressed using Δx (delta x) rather than h for the varying quantity, in which case (5) has the form

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{10}$$

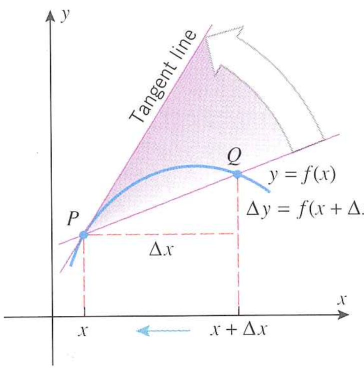
If $y = f(x)$, then it is also common to let

$$\Delta y = f(x + \Delta x) - f(x)$$

in which case

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{11}$$

The geometric interpretations of Δx and Δy are shown in Figure 3.2.14.



$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Figure 3.2.14

DERIVATIVES AT THE ENDPOINTS OF AN INTERVAL

If a function f is defined on a closed interval $[a, b]$ and is not defined outside of that interval, then the derivative $f'(x)$ is not defined at the endpoints a and b because

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

is a two-sided limit and only one-sided limits make sense at the endpoints. To deal with this situation, we define ***derivatives from the left and right***. These are denoted by f'_- and f'_+, respectively, and are defined by

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} \quad \text{and} \quad f'_+(x) = \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h}$$

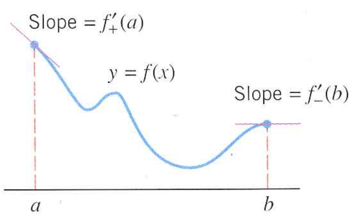


Figure 3.2.15

At points where $f'_+(x)$ exists we say that the function f is ***differentiable from the right***, and at points where $f'_-(x)$ exists we say that f is ***differentiable from the left***. Geometrically, $f'_+(x)$ is the limit of the slopes of the secant lines approaching x from the right, and $f'_-(x)$ is the limit of the slopes of the secant lines approaching x from the left (Figure 3.2.15).

It can be proved that a function f is continuous from the left at those points where it is differentiable from the left and is continuous from the right at those points where it is differentiable from the right.

We will call a function f ***differentiable*** on an interval of the form $[a, b]$, $[a, +\infty)$, $(-\infty, b]$, $[a, b)$, or $(a, b]$ if it is differentiable at all points inside the interval, and it is differentiable at the endpoint(s) from the left or right, as appropriate.

EXERCISE SET 3.2 Graphing Calculator

1. Use the graph of $y = f(x)$ in the accompanying figure to estimate the value of $f'(1)$, $f'(3)$, $f'(5)$, and $f'(6)$.

2. For the function graphed in the accompanying figure, arrange the numbers 0, $f'(-3)$, $f'(0)$, $f'(2)$, and $f'(4)$ in increasing order.

3. (a) If you are given an equation for the tangent line at the point $(a, f(a))$ on a curve $y = f(x)$, how would you go about finding $f'(a)$?

(b) Given that the tangent line to the graph of $y = f(x)$ at the point $(2, 5)$ has the equation $y = 3x + 1$, find $f'(2)$.

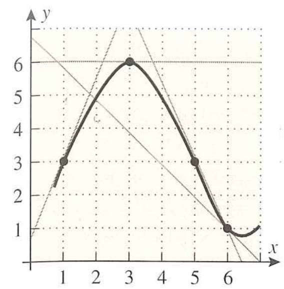


Figure Ex-1

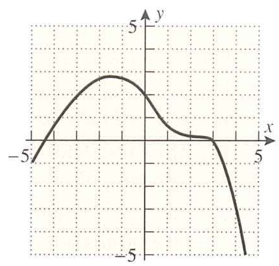


Figure Ex-2

(c) For the equation in part (b), what is the instantaneous rate of change of y with respect to x at $x = 2$?

4. Given that the tangent line to $y = f(x)$ at the point $(-1, 3)$ passes through the point $(0, 4)$, find $f'(-1)$.

5. Sketch the graph of a function f for which $f(0) = 1$, $f'(0) = 0$, $f'(x) > 0$ if $x < 0$, and $f'(x) < 0$ if $x > 0$.

6. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 0$, and $f'(x) > 0$ if $x < 0$ or $x > 0$.

7. Given that $f(3) = -1$ and $f'(3) = 5$, find an equation for the tangent line to the graph of $y = f(x)$ at the point where $x = 3$.

8. Given that $f(-2) = 3$ and $f'(-2) = -4$, find an equation for the tangent line to the graph of $y = f(x)$ at the point where $x = -2$.

In Exercises 9–14, use Definition 3.2.2 to find $f'(x)$, and then find the equation of the tangent line to $y = f(x)$ at the point $x = a$.

9. $f(x) = 3x^2;\ a = 3$

10. $f(x) = x^2 - x;\ a = 2$

11. $f(x) = x^3;\ a = 0$

12. $f(x) = 2x^3 + 1;\ a = -1$

13. $f(x) = \sqrt{x+1};\ a = 8$

14. $f(x) = x^4;\ a = -2$

In Exercises 15–20, use Formula (11) to find dy/dx.

15. $y = \dfrac{1}{x}$

16. $y = \dfrac{1}{x^2}$

17. $y = ax^2 + b$
$(a, b$ constants)

18. $y = \dfrac{1}{x+1}$

19. $y = \dfrac{1}{\sqrt{x}}$

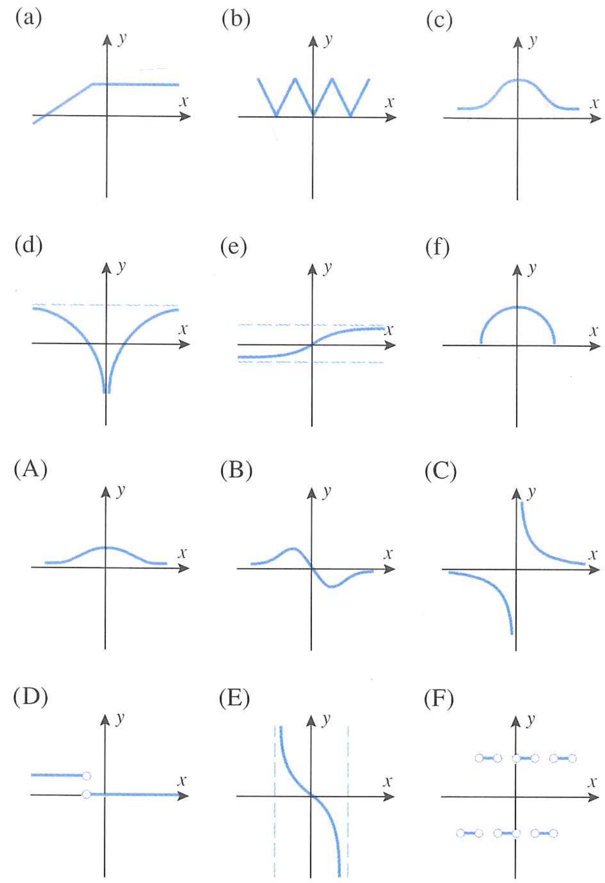
20. $y = x^{1/3}$

In Exercises 21 and 22, use Definition 3.2.2 (with the appropriate change in notation) to obtain the derivative requested.

21. Find $f'(t)$ if $f(t) = 4t^2 + t$.

22. Find dV/dr if $V = \frac{4}{3}\pi r^3$.

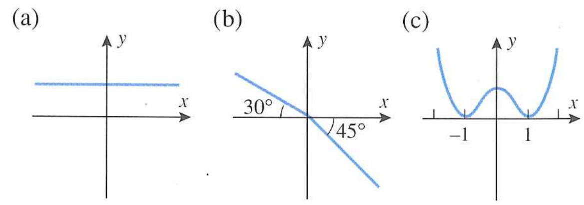
23. Match the graphs of the functions shown in (a)–(f) with the graphs of their derivatives in (A)–(F).



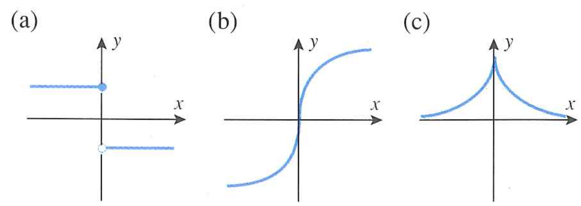
24. Find a function f such that $f'(x) = 1$ for all x, and give an informal argument to justify your answer.

In Exercises 25 and 26, sketch the graph of the derivative of the function whose graph is shown.

25.



26.



In Exercises 27 and 28, the limit represents $f'(a)$ for some function f and some number a. Find $f(x)$ and a in each case.

27. (a) $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$ (b) $\lim_{\Delta x \to 0} \dfrac{\sqrt{1+\Delta x} - 1}{\Delta x}$

28. (a) $\lim_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$ (b) $\lim_{x \to 1} \dfrac{x^7 - 1}{x - 1}$

29. Find $dy/dx|_{x=1}$, given that $y = 4x^2 + 1$.

30. Find $dy/dx|_{x=-2}$, given that $y = (5/x) + 1$.

31. Find an equation for the line that is tangent to the curve $y = x^3 - 2x + 1$ at the point $(0, 1)$, and use a graphing utility to graph the curve and its tangent line on the same screen.

32. Use a graphing utility to graph the following on the same screen: the curve $y = x^2/4$, the tangent line to this curve at $x = 1$, and the secant line joining the points $(0, 0)$ and $(2, 1)$ on this curve.

33. Let $f(x) = 2^x$. Estimate $f'(1)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Definition 3.2.2 by making a table of values for a succession of smaller and smaller values of h.

34. Let $f(x) = \sin x$. Estimate $f'(\pi/4)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Definition 3.2.2 by making a table of values for a succession of smaller and smaller values of h.

35. Suppose that the cost of drilling x feet for an oil well is $C = f(x)$ dollars.
(a) What are the units of $f'(x)$?
(b) In practical terms, what does $f'(x)$ mean in this case?
(c) What can you say about the sign of $f'(x)$?
(d) Estimate the cost of drilling an additional foot, starting at a depth of 300 ft, given that $f'(300) = 1000$.

36. A paint manufacturing company estimates that it can sell $g = f(p)$ gallons of paint at a price of p dollars.
(a) What are the units of dg/dp?
(b) In practical terms, what does dg/dp mean in this case?
(c) What can you say about the sign of dg/dp?
(d) Given that $dg/dp|_{p=10} = -100$, what can you say about the effect of increasing the price from \$10 per gallon to \$11 per gallon?

37. It is a fact that when a flexible rope is wrapped around a rough cylinder, a small force of magnitude F_0 at one end can resist a large force of magnitude F at the other end. The size of F depends on the angle θ through which the rope is wrapped around the cylinder (see the accompanying figure). That figure shows the graph of F (in pounds) versus θ (in radians), where F is the magnitude of the force that can be resisted by a force with magnitude $F_0 = 10$ lb for a certain rope and cylinder.
(a) Estimate the values of F and $dF/d\theta$ when the angle $\theta = 10$ radians.
(b) It can be shown that the force F satisfies the equation $dF/d\theta = \mu F$, where the constant μ is called the ***coefficient of friction***. Use the results in part (a) to estimate the value of μ.

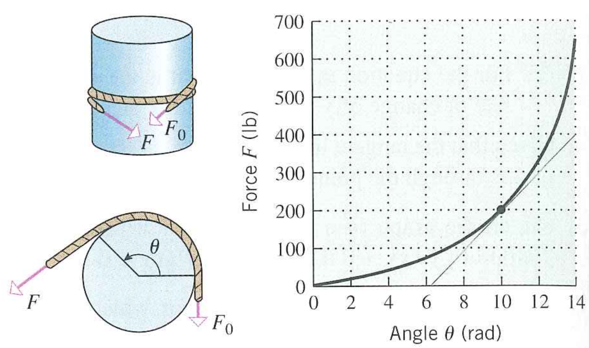


Figure Ex-37

38. According to *The World Almanac and the Book of Facts* (1987), the estimated world population, N, in millions for the years 1850, 1900, 1950, and 1985 was 1175, 1600, 2490, and 4843, respectively. Although the increase in population is not a continuous function of the time t, we can apply the ideas in this section if we are willing to approximate the graph of N versus t by a continuous curve, as shown in the accompanying figure.
(a) Use the estimated tangent line shown in the figure at the point where $t = 1950$ to approximate the value of dN/dt there. Describe your result as a rate of change.
(b) At any instant, the ***growth rate*** is defined as

$$\frac{dN/dt}{N}$$

Use your answer to part (a) to approximate the growth rate in 1950. Express the result as a percentage and include the proper units.

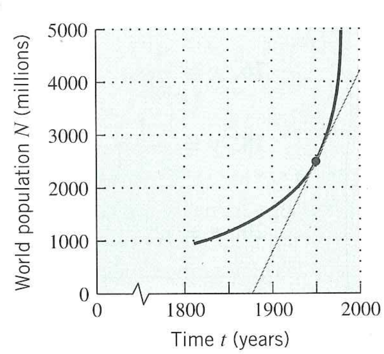


Figure Ex-38

39. According to ***Newton's Law of Cooling***, the rate of change of an object's temperature is proportional to the difference between the temperature of the object and that of the surrounding medium. The accompanying figure shows the graph of the temperature T (in degrees Fahrenheit) versus time t (in minutes) for a cup of coffee, initially with a temperature of 200°F, that is allowed to cool in a room with a constant temperature of 75°F.

(a) Estimate T and dT/dt when $t = 10$ min.

(b) Newton's Law of Cooling can be expressed as

$$\frac{dT}{dt} = k(T - T_0)$$

where k is the constant of proportionality and T_0 is the temperature (assumed constant) of the surrounding medium. Use the results in part (a) to estimate the value of k.

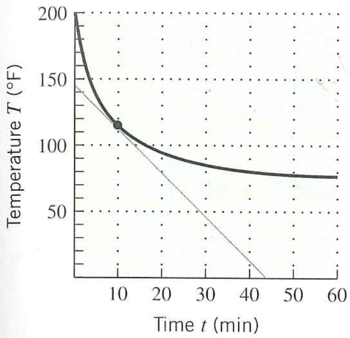


Figure Ex-39

40. Write a paragraph that explains what it means for a function to be differentiable. Include some examples of functions that are not differentiable, and explain the relationship between differentiability and continuity.

41. Show that $f(x) = \sqrt[3]{x}$ is continuous at $x = 0$ but not differentiable at $x = 0$. Sketch the graph of f.

42. Show that $f(x) = \sqrt[3]{(x-2)^2}$ is continuous at $x = 2$ but not differentiable at $x = 2$. Sketch the graph of f.

43. Show that

$$f(x) = \begin{cases} x^2 + 1, & x \le 1 \\ 2x, & x > 1 \end{cases}$$

is continuous and differentiable at $x = 1$. Sketch the graph of f.

44. Show that

$$f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ x + 2, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. Sketch the graph of f.

45. Suppose that a function f is differentiable at $x = 1$ and $\lim_{h\to 0} \frac{f(1+h)}{h} = 5$. Find $f(1)$ and $f'(1)$.

46. Suppose that f is a differentiable function with the property that $f(x+y) = f(x) + f(y) + 5xy$ and $\lim_{h\to 0} \frac{f(h)}{h} = 3$. Find $f(0)$ and $f'(x)$.

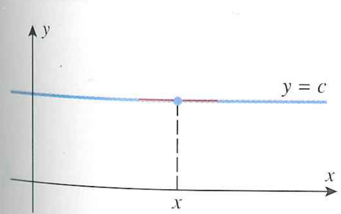
47. Suppose that f has the property $f(x+y) = f(x)f(y)$ for all values of x and y and that $f(0) = f'(0) = 1$. Show that f is differentiable and $f'(x) = f(x)$. [*Hint:* Start by expressing $f'(x)$ as a limit.]

3.3 TECHNIQUES OF DIFFERENTIATION

In the last section we defined the derivative of a function f as a limit, and we used that limit to calculate a few simple derivatives. In this section we will develop some important theorems that will enable us to calculate derivatives more efficiently.

DERIVATIVE OF A CONSTANT

The graph of a constant function $f(x) = c$ is the horizontal line $y = c$, and hence the tangent line to this graph has slope 0 at every point x (Figure 3.3.1). Thus, we should expect the derivative of a constant function to be 0 for all x.



The tangent line to the graph of $f(x) = c$ has slope 0 for all x.

Figure 3.3.1

3.3.1 THEOREM. *The derivative of a constant function is 0; that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0$$

Proof. Let $f(x) = c$. Then from the definition of a derivative,

$$\frac{d}{dx}[c] = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$ ∎

Example 1

If $f(x) = 5$ for all x, then $f'(x) = 0$ for all x; that is,

$$\frac{d}{dx}[5] = 0$$

DERIVATIVE OF x TO A POWER

3.3.2 THEOREM (*The Power Rule*). *If n is a positive integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

Proof. Let $f(x) = x^n$. Thus, from the definition of a derivative and the binomial theorem for expanding the expression $(x+h)^n$, we obtain

$$\frac{d}{dx}[x^n] = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$$= nx^{n-1} + 0 + \cdots + 0 + 0$$

$$= nx^{n-1}$$

REMARK. In words, *to differentiate x to a positive integer power, multiply that power by x raised to the next lower integer power.*

Example 2

$$\frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dx}[x] = 1\cdot x^0 = 1, \quad \frac{d}{dx}[x^{12}] = 12x^{11}$$

DERIVATIVE OF A CONSTANT TIMES A FUNCTION

3.3.3 THEOREM. *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Proof.

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\frac{d}{dx}[f(x)]$$

A constant factor can be moved through a limit sign.

In function notation, Theorem 3.3.3 states

$$(cf)' = cf'$$

REMARK. In words, *a constant factor can be moved through a derivative sign.*

Example 3

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{x}{\pi}\right] = \frac{1}{\pi}\frac{d}{dx}[x] = \frac{1}{\pi}$$

◀

DERIVATIVES OF SUMS AND DIFFERENCES

3.3.4 THEOREM. *If f and g are differentiable at x, then so are f + g and f − g and*

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Proof.

$$\begin{aligned}\frac{d}{dx}[f(x) + g(x)] &= \lim_{h\to 0}\frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\\ &= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\end{aligned}$$

The limit of a sum is the sum of the limits.

The proof for $f - g$ is similar. ■

In function notation, Theorem 3.3.4 states

$$(f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

REMARK. In words, *the derivative of a sum equals the sum of the derivatives*, and *the derivative of a difference equals the difference of the derivatives.*

Example 4

$$\frac{d}{dx}[x^4 + x^2] = \frac{d}{dx}[x^4] + \frac{d}{dx}[x^2] = 4x^3 + 2x$$

$$\frac{d}{dx}[6x^{11} - 9] = \frac{d}{dx}[6x^{11}] - \frac{d}{dx}[9] = 66x^{10} - 0 = 66x^{10}$$

Although Theorem 3.3.4 was stated for sums and differences of two terms, it can be extended to any mixture of finitely many sums and differences of differentiable functions. For example,

$$\frac{d}{dx}[3x^8 - 2x^5 + 6x + 1] = \frac{d}{dx}[3x^8] + \frac{d}{dx}[-2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1]$$
$$= 24x^7 - 10x^4 + 6$$ ◀

DERIVATIVE OF A PRODUCT

3.3.5 THEOREM (***The Product Rule***). *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Proof. The earlier proofs in this section were straightforward applications of the definition of the derivative. However, this proof requires a trick—adding and subtracting the quantity $f(x+h)g(x)$ to the numerator in the derivative definition as follows:

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}\\
&= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\cdot\frac{f(x+h)-f(x)}{h}\right]\\
&= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= [\lim_{h\to 0} f(x+h)]\frac{d}{dx}[g(x)] + [\lim_{h\to 0} g(x)]\frac{d}{dx}[f(x)]\\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]
\end{aligned}$$

[*Note:* In the last step $f(x+h)\to f(x)$ as $h\to 0$ because f is continuous at x by Theorem 3.2.3, and $g(x)\to g(x)$ as $h\to 0$ because $g(x)$ does not involve h and hence remains constant.] ■

The product rule can be written in function notation as

$$(f\cdot g)' = f\cdot g' + g\cdot f'$$

REMARK. In words, *the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.*

WARNING. Note that it is *not* true in general that $(f\cdot g)' = f'\cdot g'$; that is, the derivative of a product is *not* generally the product of the derivatives!

Example 5

Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution. There are two methods that can be used to find dy/dx. We can either use the product rule or we can multiply out the factors in y and then differentiate. We will give both methods.

Method I. *(Using the Product Rule)*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(4x^2-1)(7x^3+x)] \\ &= (4x^2-1)\frac{d}{dx}[7x^3+x] + (7x^3+x)\frac{d}{dx}[4x^2-1] \\ &= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) = 140x^4 - 9x^2 - 1\end{aligned}$$

Method II. *(Multiplying First)*

$$y = (4x^2-1)(7x^3+x) = 28x^5 - 3x^3 - x$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

which agrees with the result obtained using the product rule. ◀

DERIVATIVE OF A QUOTIENT

> **3.3.6 THEOREM (*The Quotient Rule*).** *If f and g are differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x and*
>
> $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

Proof.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

Adding and subtracting $f(x)\cdot g(x)$ in the numerator yields

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)} \\ &= \lim_{h\to 0}\frac{\left[g(x)\cdot\dfrac{f(x+h)-f(x)}{h}\right] - \left[f(x)\cdot\dfrac{g(x+h)-g(x)}{h}\right]}{g(x)\cdot g(x+h)} \\ &= \frac{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} - \lim\limits_{h\to 0} f(x)\cdot\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)} \\ &= \frac{[\lim\limits_{h\to 0} g(x)]\cdot\dfrac{d}{dx}[f(x)] - [\lim\limits_{h\to 0} f(x)]\cdot\dfrac{d}{dx}[g(x)]}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)} \\ &= \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}\end{aligned}$$

[See the note at the end of the proof of Theorem 3.3.5 for an explanation of the last step.] ■

The quotient rule can be written in function notation as

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

REMARK. In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.*

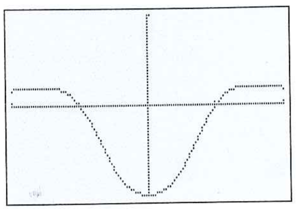
WARNING. Note that it is *not* generally true that $(f/g)' = f'/g'$; that is, the derivative of a quotient is *not* generally the quotient of the derivatives.

Example 6

Let $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$.

(a) Graph $y = f(x)$, and use your graph to make rough estimates of the locations of all horizontal tangent lines.

(b) By differentiating, find the exact locations of the horizontal tangent lines.



$[-2, 2] \times [-1, 1]$
$x\text{Scl} = 1$, $y\text{Scl} = 1$

$y = \dfrac{x^2 - 1}{x^4 + 1}$

Figure 3.3.2

Solution (a). Figure 3.3.2 shows the graph of $y = f(x)$ in the window $[-2, 2] \times [-1, 1]$. This graph suggests that horizontal tangent lines occur at $x = 0$, $x \approx 1.5$, and $x \approx -1.5$.

Solution (b). To find the exact location of the horizontal tangent lines, we must find the points where $dy/dx = 0$. We start by finding dy/dx:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2 - 1}{x^4 + 1}\right] = \frac{(x^4 + 1)\frac{d}{dx}[x^2 - 1] - (x^2 - 1)\frac{d}{dx}[x^4 + 1]}{(x^4 + 1)^2}$$

$$= \frac{(x^4 + 1)(2x) - (x^2 - 1)(4x^3)}{(x^4 + 1)^2}$$

The differentiation is complete. The rest is simplification.

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4 + 1)^2} = -\frac{2x(x^4 - 2x^2 - 1)}{(x^4 + 1)^2}$$

Now we will set $dy/dx = 0$ and solve for x. We obtain

$$-\frac{2x(x^4 - 2x^2 - 1)}{(x^4 + 1)^2} = 0$$

The solutions of this equation are the values of x for which the numerator is 0:

$$2x(x^4 - 2x^2 - 1) = 0$$

The first factor yields the solution $x = 0$. Other solutions can be found by solving the equation

$$x^4 - 2x^2 - 1 = 0$$

This can be treated as a quadratic equation in x^2 and solved by the quadratic formula. This yields

$$x^2 = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

The minus sign yields imaginary values of x, which we ignore since they are not relevant to the problem. The plus sign yields the solutions

$$x = \pm\sqrt{1 + \sqrt{2}}$$

In summary, horizontal tangent lines occur at

$$x = 0, \quad x = \sqrt{1 + \sqrt{2}} \approx 1.55, \quad \text{and} \quad x = -\sqrt{1 + \sqrt{2}} \approx -1.55$$

which is consistent with the rough estimates that we obtained graphically in part (a). ◀

DERIVATIVE OF A RECIPROCAL

The special case of Theorem 3.3.6 in which f is the constant function 1 is of interest in its own right. We leave it for the reader to deduce the following result from Theorem 3.3.6.

3.3.7 THEOREM (*The Reciprocal Rule*). *If g is differentiable at x and $g(x) \neq 0$, then $1/g$ is differentiable at x and*

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = -\frac{\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

The reciprocal rule can be written in function notation as

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

REMARK. In words, *the derivative of the reciprocal of a function is the negative of the derivative of the function divided by the function squared.*

Example 7

$$\frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{\frac{d}{dx}[x]}{x^2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left[\frac{1}{x^3+2x-3}\right] = -\frac{\frac{d}{dx}[x^3+2x-3]}{(x^3+2x-3)^2} = -\frac{3x^2+2}{(x^3+2x-3)^2}$$ ◀

REMARK. The computations in the preceding example could have been done using the quotient rule, but this would have been more work. Where it applies, the reciprocal rule is preferable to the quotient rule.

THE POWER RULE FOR INTEGER EXPONENTS

In Theorem 3.3.2 we established the formula

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for *positive* integer values of n. Eventually, we will show that this formula applies if n is any real number. As our first step in this direction we will show that it applies for *all integer* values of n.

3.3.8 THEOREM. *If n is any integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{1}$$

Proof. The result has already been established in the case where $n > 0$. If $n < 0$, then let $m = -n$ so that

$$f(x) = x^{-m} = \frac{1}{x^m}$$

From Theorem 3.3.7,

$$f'(x) = \frac{d}{dx}\left[\frac{1}{x^m}\right] = -\frac{\frac{d}{dx}[x^m]}{(x^m)^2}$$

Since $n < 0$, it follows that $m > 0$, so x^m can be differentiated using Theorem 3.3.2. Thus,

$$f'(x) = -\frac{mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = -mx^{-m-1} = nx^{n-1}$$

which proves (1). In the case $n = 0$ Formula (1) reduces to

$$\frac{d}{dx}[1] = 0 \cdot x^{-1} = 0$$

which is correct by Theorem 3.3.1. ■

Example 8

$$\frac{d}{dx}[x^{-9}] = -9x^{-9-1} = -9x^{-10}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Note that the last result agrees with that obtained in Example 7. ◀

In Example 4 of Section 3.2 we showed that

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \tag{2}$$

which shows that Formula (1) also works with $n = \frac{1}{2}$, since

$$\frac{d}{dx}[x^{1/2}] = \frac{1}{2x^{1/2}} = \frac{1}{2}x^{-1/2}$$

HIGHER DERIVATIVES

If the derivative f' of a function f is itself differentiable, then the derivative of f' is denoted by f'' and is called the ***second derivative*** of f. As long as we have differentiability, we can continue the process of differentiating derivatives to obtain third, fourth, fifth, and even higher derivatives of f. The successive derivatives of f are denoted by

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

These are called the first derivative, the second derivative, the third derivative, and so forth. Beyond the third derivative, it is too clumsy to continue using primes, so we switch from primes to integers in parentheses to denote the ***order*** of the derivative. In this notation it is easy to denote a derivative of arbitrary order by writing

$f^{(n)}$ The nth derivative of f

The significance of the derivatives of order 2 and higher will be discussed later.

Example 9

If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then

$$\begin{aligned}
f'(x) &= 12x^3 - 6x^2 + 2x - 4 \\
f''(x) &= 36x^2 - 12x + 2 \\
f'''(x) &= 72x - 12 \\
f^{(4)}(x) &= 72 \\
f^{(5)}(x) &= 0 \\
&\vdots \\
f^{(n)}(x) &= 0 \quad (n \geq 5)
\end{aligned}$$

◀

Successive derivatives can also be denoted as follows:

$$f'(x) = \frac{d}{dx}[f(x)]$$

$$f''(x) = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)]$$

$$f'''(x) = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)]$$

$$\vdots \qquad\qquad\qquad \vdots$$

In general, we write

$$f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$$

which is read "the nth derivative of f with respect to x."

When a dependent variable is involved, say $y = f(x)$, then successive derivatives can be denoted by writing

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ldots, \quad \frac{d^ny}{dx^n}, \ldots$$

or more briefly,

$$y', \quad y'', \quad y''', \quad y^{(4)}, \ldots, y^{(n)}, \ldots$$

EXERCISE SET 3.3 Graphing Calculator C CAS

In Exercises 1–12, find dy/dx.

1. $y = 4x^7$
2. $y = -3x^{12}$
3. $y = 3x^8 + 2x + 1$
4. $y = \frac{1}{2}(x^4 + 7)$
5. $y = \pi^3$
6. $y = \sqrt{2}x + (1/\sqrt{2})$
7. $y = -\frac{1}{3}(x^7 + 2x - 9)$
8. $y = \dfrac{x^2 + 1}{5}$
9. $y = ax^3 + bx^2 + cx + d \quad (a, b, c, d \text{ constant})$
10. $y = \dfrac{1}{a}\left(x^2 + \dfrac{1}{b}x + c\right) \quad (a, b, c \text{ constant})$
11. $y = -3x^{-8} + 2\sqrt{x}$
12. $y = 7x^{-6} - 5\sqrt{x}$

In Exercises 13–20, find $f'(x)$.

13. $f(x) = x^{-3} + \dfrac{1}{x^7}$
14. $f(x) = \sqrt{x} + \dfrac{1}{x}$
15. $f(x) = (3x^2 + 6)\left(2x - \frac{1}{4}\right)$
16. $f(x) = (2 - x - 3x^3)(7 + x^5)$
17. $f(x) = (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})$
18. $f(x) = \left(\dfrac{1}{x} + \dfrac{1}{x^2}\right)(3x^3 + 27)$
19. $f(x) = \left(3x^2 + 1\right)^2$
20. $f(x) = \left(x^5 + 2x\right)^2$

In Exercises 21 and 22, find $y'(1)$.

21. $y = \dfrac{1}{5x - 3}$
22. $y = \dfrac{3}{\sqrt{x} + 2}$

In Exercises 23 and 24, find dx/dt.

23. $x = \dfrac{3t}{2t + 1}$
24. $x = \dfrac{t^2 + 1}{3t}$

In Exercises 25–28, find $dy/dx|_{x=1}$.

25. $y = \dfrac{2x - 1}{x + 3}$
26. $y = \dfrac{4x + 1}{x^2 - 5}$
27. $y = \left(\dfrac{3x + 2}{x}\right)(x^{-5} + 1)$
28. $y = (2x^7 - x^2)\left(\dfrac{x - 1}{x + 1}\right)$

In Exercises 29–32, find the indicated derivative.

29. $\dfrac{d}{dt}[16t^2]$
30. $\dfrac{dC}{dr}$, where $C = 2\pi r$
31. $V'(r)$, where $V = \pi r^3$
32. $\dfrac{d}{d\alpha}[2\alpha^{-1} + \alpha]$

33. A spherical balloon is being inflated.
(a) Find a general formula for the instantaneous rate of change of the volume V with respect to the radius r.
(b) Find the rate of change of V with respect to r at the instant when the radius is $r = 5$.

C **34.** Use a CAS to check the answers to the problems you solved in Exercises 1–32.

35. Find $g'(4)$ given that $f(4) = 3$ and $f'(4) = -5$.
(a) $g(x) = \sqrt{x} f(x)$ (b) $g(x) = \dfrac{f(x)}{x}$

36. Find $g'(3)$ given that $f(3) = -2$ and $f'(3) = 4$.
(a) $g(x) = 3x^2 - 5f(x)$ (b) $g(x) = \dfrac{2x+1}{f(x)}$

37. Find $F'(2)$ given that $f(2) = -1$, $f'(2) = 4$, $g(2) = 1$, and $g'(2) = -5$.
(a) $F(x) = 5f(x) + 2g(x)$ (b) $F(x) = f(x) - 3g(x)$
(c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

38. Find an equation for the line that is tangent to the curve $y = (1-x)/(1+x)$ at the point where $x = 2$.

39. Find an equation of the tangent line to the graph of $y = f(x)$ at the point where $x = -3$ if $f(-3) = 2$ and $f'(-3) = 5$.

40. Find $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right]$ (λ_0 is constant).

In Exercises 41 and 42, find d^2y/dx^2.

41. (a) $y = 7x^3 - 5x^2 + x$ (b) $y = 12x^2 - 2x + 3$
(c) $y = \dfrac{x+1}{x}$ (d) $y = (5x^2 - 3)(7x^3 + x)$

42. (a) $y = 4x^7 - 5x^3 + 2x$ (b) $y = 3x + 2$
(c) $y = \dfrac{3x-2}{5x}$ (d) $y = (x^3 - 5)(2x + 3)$

In Exercises 43 and 44, find y'''.

43. (a) $y = x^{-5} + x^5$ (b) $y = 1/x$
(c) $y = ax^3 + bx + c$ (a, b, c constant)

44. (a) $y = 5x^2 - 4x + 7$ (b) $y = 3x^{-2} + 4x^{-1} + x$
(c) $y = ax^4 + bx^2 + c$ (a, b, c constant)

45. Find
(a) $f'''(2)$, where $f(x) = 3x^2 - 2$
(b) $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$, where $y = 6x^5 - 4x^2$
(c) $\left.\dfrac{d^4}{dx^4}[x^{-3}]\right|_{x=1}$

46. Find
(a) $y'''(0)$, where $y = 4x^4 + 2x^3 + 3$
(b) $\left.\dfrac{d^4y}{dx^4}\right|_{x=1}$, where $y = \dfrac{6}{x^4}$.

47. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$.

48. Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

49. Find a general formula for $F''(x)$ if $F(x) = xf(x)$ and f and f' are differentiable at x.

C **50.** Use a CAS to check the answers to the problems you solved in Exercises 41–46.

In Exercises 51 and 52, use a graphing utility to make rough estimates of the locations of all horizontal tangent lines, and then find their exact locations by differentiating.

51. $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$ **52.** $y = \dfrac{x}{x^2+9}$

In Exercises 53 and 54, use Definition 3.2.2 to approximate $f'(1)$ by choosing a small value of h to approximate the limit, and then find the exact value of $f'(1)$ by differentiating.

53. $f(x) = x^3 - 3x + 1$ **54.** $f(x) = x\sqrt{x}$

In Exercises 55 and 56, estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating.

55. $f(x) = \dfrac{x}{x^2+1}$ **56.** $f(x) = \dfrac{x^2-1}{x^2+1}$

57. Find a function $y = ax^2 + bx + c$ whose graph has an x-intercept of 1, a y-intercept of -2, and a tangent line with a slope of -1 at the y-intercept.

58. Find k if the curve $y = x^2 + k$ is tangent to the line $y = 2x$.

59. Find the x-coordinate of the point on the graph of $y = x^2$ where the tangent line is parallel to the secant line that cuts the curve at $x = -1$ and $x = 2$.

60. Find the x-coordinate of the point on the graph of $y = \sqrt{x}$ where the tangent line is parallel to the secant line that cuts the curve at $x = 1$ and $x = 4$.

61. Find the coordinate of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.

62. Show that any two tangent lines to the parabola $y = ax^2$, $a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

63. Suppose that L is the tangent line at $x = x_0$ to the graph of the cubic equation $y = ax^3 + bx$. Find the x-coordinate of the point where L intersects the graph a second time.

64. Show that the segment of the tangent line to the graph of $y = 1/x$ that is cut off by the coordinate axes is bisected by the point of tangency.

65. Show that the triangle that is formed by any tangent line to the graph of $y = 1/x$, $x > 0$, and the coordinate axes has an area of 2 square units.

66. Find conditions on a, b, c, and d so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has
(a) exactly two horizontal tangents
(b) exactly one horizontal tangent
(c) no horizontal tangents.

67. Newton's Law of Gravitation states that the magnitude F of the force exerted by a point with mass M on a point with mass m is

$$F = \frac{GmM}{r^2}$$

where G is a constant and r is the distance between the bodies. Assuming that the points are moving, find a formula for the instantaneous rate of change of F with respect to r.

68. In the temperature range between 0°C and 700°C the resistance R [in ohms (Ω)] of a certain platinum resistance thermometer is given by

$$R = 10 + 0.04124T - 1.779 \times 10^{-5}T^2$$

where T is the temperature in degrees Celsius. Where in the interval from 0°C to 700°C is the resistance of the thermometer most sensitive and least sensitive to temperature changes? [*Hint:* Consider the size of dR/dT in the interval $0 \le T \le 700$.]

In Exercises 69 and 70, use a graphing utility to make rough estimates of the intervals on which $f'(x) > 0$, and then find those intervals exactly by differentiating.

69. $f(x) = x - \dfrac{1}{x}$ **70.** $f(x) = \dfrac{5x}{x^2+4}$

71. Apply the product rule (3.3.5) twice to show that if f, g, and h are differentiable functions, then $f \cdot g \cdot h$ is differentiable, and

$$(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

72. Based on the result in Exercise 71, make a conjecture about a formula for differentiating a product of n functions.

73. Use the formula in Exercise 71 to find

(a) $\dfrac{d}{dx}\left[(2x+1)\left(1+\dfrac{1}{x}\right)(x^{-3}+7)\right]$

(b) $\dfrac{d}{dx}\left[(x^7+2x-3)^3\right]$.

74. Use the formula you obtained in Exercise 72 to find

(a) $\dfrac{d}{dx}\left[x^{-5}(x^2+2x)(4-3x)(2x^9+1)\right]$

(b) $\dfrac{d}{dx}\left[(x^2+1)^{50}\right]$.

In Exercises 75–79, you will have to determine whether a function f is differentiable at a point x_0 where the formula for f changes. Use the following result:
Theorem. *Let f be continuous at x_0 and suppose that*

$$\lim_{x \to x_0^+} f'(x) \quad \textit{and} \quad \lim_{x \to x_0^-} f'(x)$$

exist. Then f is differentiable at x_0 if and only if these limits are equal. Moreover, in the case of equality

$$f'(x_0) = \lim_{x \to x_0^+} f'(x) = \lim_{x \to x_0^-} f'(x)$$

75. Let

$$f(x) = \begin{cases} x^2, & x \le 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there.

76. Let

$$f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < \frac{1}{2} \\ \frac{3}{4}x^2, & x \ge \frac{1}{2} \end{cases}$$

Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

77. Let

$$f(x) = \begin{cases} 3x^2, & x \le 1 \\ ax + b, & x > 1 \end{cases}$$

Find the values of a and b so that f will be differentiable at $x = 1$.

78. (a) Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$$

Show that

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$$

but that $f'(0)$ does not exist.

(b) Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$$

Show that $f'(0)$ exists but $f''(0)$ does not.

79. Find all points where f fails to be differentiable. Justify your answer.
(a) $f(x) = |3x - 2|$ (b) $f(x) = |x^2 - 4|$

80. In each part compute f', f'', f''' and then state the formula for $f^{(n)}$.
(a) $f(x) = 1/x$ (b) $f(x) = 1/x^2$
[*Hint:* The expression $(-1)^n$ has a value of 1 if n is even and -1 if n is odd. Use this expression in your answer.]

81. (a) Prove:

$$\frac{d^2}{dx^2}[cf(x)] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x) + g(x)] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) Do the results in part (a) generalize to nth derivatives? Justify your answer.

82. Prove:

$$(f\cdot g)'' = f''\cdot g + 2f'\cdot g' + f\cdot g''$$

83. (a) Find $f^{(n)}(x)$ if $f(x) = x^n$.
(b) Find $f^{(n)}(x)$ if $f(x) = x^k$ and $n > k$, where k is a positive integer.
(c) Find $f^{(n)}(x)$ if

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

84. Let $f(x) = x^8 - 2x + 3$; find

$$\lim_{h\to 0} \frac{f'(2+h) - f'(2)}{h}$$

85. (a) Prove: If $f''(x)$ exists for each x in (a, b), then both f and f' are continuous on (a, b).
(b) What can be said about the continuity of f and its derivatives if $f^{(n)}(x)$ exists for each x in (a, b)?

3.4 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The main objective of this section is to obtain formulas for the derivatives of trigonometric functions.

DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

For the purpose of finding derivatives of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$, we will assume that x is measured in radians. We will also need the following limits, which were stated in Theorem 2.5.3 (with x rather than h as the variable):

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \quad\text{and}\quad \lim_{h\to 0}\frac{1-\cos h}{h} = 0$$

We begin with the problem of differentiating $\sin x$. From the definition of a derivative we have

$$\begin{aligned}\frac{d}{dx}[\sin x] &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\\ &= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]\\ &= \lim_{h\to 0}\left[\cos x\left(\frac{\sin h}{h}\right) - \sin x\left(\frac{1-\cos h}{h}\right)\right]\end{aligned}$$

Since $\sin x$ and $\cos x$ do not involve h, they remain constant as $h\to 0$; thus,

$$\lim_{h\to 0}(\sin x) = \sin x \quad\text{and}\quad \lim_{h\to 0}(\cos x) = \cos x$$

Consequently,

$$\begin{aligned}\frac{d}{dx}[\sin x] &= \cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right) - \sin x\cdot\lim_{h\to 0}\left(\frac{1-\cos h}{h}\right)\\ &= \cos x\cdot(1) - \sin x\cdot(0) = \cos x\end{aligned}$$

Thus, we have shown that

$$\frac{d}{dx}[\sin x] = \cos x \tag{1}$$

The derivative of $\cos x$ can be obtained similarly, resulting in the formula

$$\frac{d}{dx}[\cos x] = -\sin x \tag{2}$$

The derivatives of the remaining trigonometric functions are

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x\tan x \tag{3–4}$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x \tag{5–6}$$

These can all be obtained from (1) and (2) using the relationships

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

For example,

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x \cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[\cos x]}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

REMARK. The derivative formulas for the trigonometric functions should be memorized. An easy way of doing this is discussed in Exercise 42. Moreover, we emphasize again that in all of the derivative formulas for the trigonometric functions, x is measured in radians.

Example 1

Find $f'(x)$ if $f(x) = x^2 \tan x$.

Solution. Using the product rule and Formula (3), we obtain

$$f'(x) = x^2 \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[x^2] = x^2 \sec^2 x + 2x \tan x$$ ◀

Example 2

Find dy/dx if $y = \dfrac{\sin x}{1 + \cos x}$.

Solution. Using the quotient rule together with Formulas (1) and (2) we obtain

$$\frac{dy}{dx} = \frac{(1+\cos x) \cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[1 + \cos x]}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$ ◀

Example 3

Find $y''(\pi/4)$ if $y(x) = \sec x$.

Solution.

$$y'(x) = \sec x \tan x$$

$$y''(x) = \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x]$$

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

Thus,

$$y''(\pi/4) = \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4)$$

$$= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2}$$ ◀

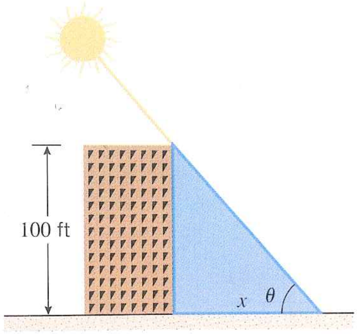


Figure 3.4.1

Example 4

Suppose that the rising Sun passes directly over a building that is 100 feet high, and let θ be the Sun's angle of elevation (Figure 3.4.1). Find the rate at which the length x of the building's shadow is changing with respect to θ when $\theta = 45^\circ$. Express the answer in units of feet/degree.

Solution. The variables x and θ are related by $\tan\theta = 100/x$, or equivalently,

$$x = 100\cot\theta \tag{7}$$

If θ is measured in radians, then Formula (5) is applicable, which yields

$$\frac{dx}{d\theta} = -100\csc^2\theta$$

which is the rate of change of shadow length with respect to the elevation angle θ in units of feet/radian. When $\theta = 45^\circ$ (or equivalently, $\theta = \pi/4$ radians), we obtain

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/4} = -100\csc^2(\pi/4) = -200 \text{ feet/radian}$$

Converting radians (rad) to degrees (deg) yields

$$-200\frac{\text{ft}}{\text{rad}} \cdot \frac{\pi}{180}\frac{\text{rad}}{\text{deg}} = -\frac{10}{9}\pi \approx -3.49 \text{ ft/deg}$$

Thus, when $\theta = 45^\circ$, the shadow length is decreasing (because of the minus sign) at an approximate rate of 3.49 ft/deg increase in the angle of elevation. ◀

EXERCISE SET 3.4 Graphing Calculator C CAS

In Exercises 1–18, find $f'(x)$.

1. $f(x) = 2\cos x - 3\sin x$
2. $f(x) = \sin x\cos x$
3. $f(x) = \dfrac{\sin x}{x}$
4. $f(x) = x^2\cos x$
5. $f(x) = x^3\sin x - 5\cos x$
6. $f(x) = \dfrac{\cos x}{x\sin x}$
7. $f(x) = \sec x - \sqrt{2}\tan x$
8. $f(x) = (x^2+1)\sec x$
9. $f(x) = \sec x\tan x$
10. $f(x) = \dfrac{\sec x}{1+\tan x}$
11. $f(x) = \csc x\cot x$
12. $f(x) = x - 4\csc x + 2\cot x$
13. $f(x) = \dfrac{\cot x}{1+\csc x}$
14. $f(x) = \dfrac{\csc x}{\tan x}$
15. $f(x) = \sin^2 x + \cos^2 x$
16. $f(x) = \dfrac{1}{\cot x}$
17. $f(x) = \dfrac{\sin x\sec x}{1+x\tan x}$
18. $f(x) = \dfrac{(x^2+1)\cot x}{3-\cos x\csc x}$

In Exercises 19–24, find d^2y/dx^2.

19. $y = x\cos x$
20. $y = \csc x$
21. $y = x\sin x - 3\cos x$
22. $y = x^2\cos x + 4\sin x$
23. $y = \sin x\cos x$
24. $y = \tan x$
25. [C] Use a CAS to check the answers to the problems you solved in Exercises 1–24.
26. Find the equation of the line tangent to the graph of $\sin x$ at the point where
    (a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$.
27. Find the equation of the line tangent to the graph of $\tan x$ at the point where
    (a) $x = 0$ (b) $x = \pi/4$ (c) $x = -\pi/4$.
28. (a) Show that $y = \cos x$ and $y = \sin x$ are solutions of the equation $y'' + y = 0$.
    (b) Show that $y = A\sin x + B\cos x$ is a solution for all constants A and B.
29. Find all points in the interval $[-2\pi, 2\pi]$ at which the graph of f has a horizontal tangent line.
    (a) $f(x) = \sin x$ (b) $f(x) = x + \cos x$
    (c) $f(x) = \tan x$ (d) $f(x) = \sec x$
30. [Graphing] (a) Use a graphing utility to make rough estimates of the points in the interval $[0, 2\pi]$ at which the graph of $y = \sin x\cos x$ has a horizontal tangent line.

(b) Find the exact locations of the points where the graph has a horizontal tangent line.

31. A 10-ft ladder leans against a wall at an angle θ with the horizontal, as shown in the accompanying figure. The top of the ladder is x feet above the ground. If the bottom of the ladder is pushed toward the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.

32. An airplane is flying on a horizontal path at a height of 3800 ft, as shown in the accompanying figure. At what rate is the distance s between the airplane and the fixed point P changing with respect to θ when $\theta = 30°$? Express the answer in units of feet/degree.

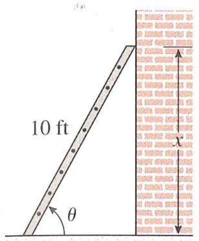


Figure Ex-31

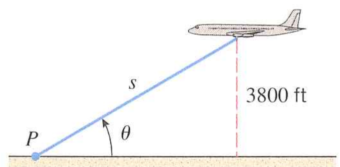


Figure Ex-32

33. A searchlight is located 50 m from a straight wall, as shown in the accompanying figure. Find the rate at which the distance D is changing with θ when $\theta = 45°$. Express the answer in units of meters/degree.

34. An Earth-observing satellite can see only a portion of the Earth's surface. The satellite has horizon sensors that can detect the angle θ shown in the accompanying figure. Let r be the radius of the Earth (assumed spherical) and h the distance of the satellite from the Earth's surface.
(a) Show that $h = r(\csc\theta - 1)$.
(b) Using $r = 6378$ km, and assuming that the satellite is getting closer to the Earth, find the rate at which h is changing with respect to θ when $\theta = 30°$. Express the answer in units of kilometers/degree. [Adapted from *Space Mathematics*, NASA, 1985.]

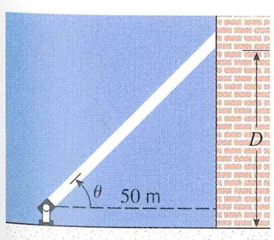


Figure Ex-33

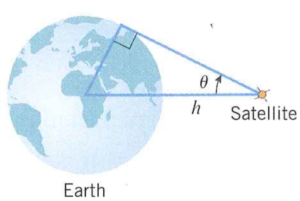


Figure Ex-34

In Exercises 35 and 36, make a conjecture about the derivative by calculating the first few derivatives and observing the resulting pattern.

35. (a) $\dfrac{d^{87}}{dx^{87}}[\sin x]$ (b) $\dfrac{d^{100}}{dx^{100}}[\cos x]$

36. $\dfrac{d^{17}}{dx^{17}}[x \sin x]$

37. In each part, determine where f is differentiable.
(a) $f(x) = \sin x$ (b) $f(x) = \cos x$
(c) $f(x) = \tan x$ (d) $f(x) = \cot x$
(e) $f(x) = \sec x$ (f) $f(x) = \csc x$
(g) $f(x) = \dfrac{1}{1 + \cos x}$ (h) $f(x) = \dfrac{1}{\sin x \cos x}$
(i) $f(x) = \dfrac{\cos x}{2 - \sin x}$

38. (a) Derive Formula (2) using the definition of a derivative.
(b) Use Formulas (1) and (2) to obtain (5).
(c) Use Formula (2) to obtain (4).
(d) Use Formula (1) to obtain (6).

39. Let $f(x) = \cos x$. Find all positive integers n for which $f^{(n)}(x) = \sin x$.

40. (a) Show that $\lim_{h \to 0} \dfrac{\tan h}{h} = 1$.
(b) Use the result in part (a) to help derive the formula for the derivative of $\tan x$ directly from the definition of a derivative.

41. Without using any trigonometric identities, find

$$\lim_{x \to 0} \frac{\tan(x + y) - \tan y}{x}$$

[*Hint:* Relate the given limit to the definition of the derivative of an appropriate function of y.]

42. Let us agree to call the functions $\cos x$, $\cot x$, and $\csc x$ the ***cofunctions*** of $\sin x$, $\tan x$, and $\sec x$, respectively. Convince yourself that the derivative of any cofunction can be obtained from the derivative of the corresponding function by introducing a minus sign and replacing each function in the derivative by its cofunction. Memorize the derivatives of $\sin x$, $\tan x$, and $\sec x$ and then use the above observation to deduce the derivatives of the cofunctions.

43. The derivative formulas for $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ were obtained under the assumption that x is measured in radians. This exercise shows that different (more complicated) formulas result if x is measured in degrees. Prove that if h and x are degree measures, then
(a) $\lim_{h \to 0} \dfrac{\cos h - 1}{h} = 0$ (b) $\lim_{h \to 0} \dfrac{\sin h}{h} = \dfrac{\pi}{180}$
(c) $\dfrac{d}{dx}[\sin x] = \dfrac{\pi}{180} \cos x$.

3.5 THE CHAIN RULE

In this section we will derive a formula that expresses the derivative of a composition $f \circ g$ in terms of the derivatives of f and g. This formula will enable us to differentiate complicated functions using known derivatives of simpler functions.

DERIVATIVES OF COMPOSITIONS

> **3.5.1 PROBLEM.** *If we know the derivatives of f and g, how can we use this information to find the derivative of the composition $f \circ g$?*

The key to solving this problem is to introduce dependent variables

$$y = (f \circ g)(x) = f(g(x)) \quad \text{and} \quad u = g(x)$$

so that $y = f(u)$. We are interested in using the known derivatives

$$\frac{dy}{du} = f'(u) \quad \text{and} \quad \frac{du}{dx} = g'(x)$$

to find the unknown derivative

$$\frac{dy}{dx} = \frac{d}{dx}[f(g(x))]$$

Stated another way, we are interested in using the known rates of change dy/du and du/dx to find the unknown rate of change dy/dx. But intuition suggests that rates of change multiply. For example, if y changes at 4 times the rate of change of u and u changes at 2 times the rate of change of x, then y changes at $4 \times 2 = 8$ times the rate of change of x. Thus, Figure 3.5.1 suggests that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

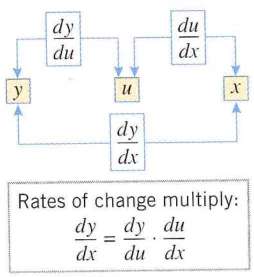


Figure 3.5.1

These ideas are formalized in the following theorem.

> **3.5.2 THEOREM (The Chain Rule).** *If g is differentiable at the point x and f is differentiable at the point $g(x)$, then the composition $f \circ g$ is differentiable at the point x. Moreover, if*
>
> $$y = f(g(x)) \quad \text{and} \quad u = g(x)$$
>
> *then $y = f(u)$ and*
>
> $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

The proof of this result is given in Appendix G.

Example 1

Find dy/dx if $y = 4\cos(x^3)$.

Solution. Let $u = x^3$ so that

$$y = 4\cos u$$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}[4\cos u] \cdot \frac{d}{dx}[x^3]$$
$$= (-4\sin u) \cdot (3x^2) = (-4\sin(x^3)) \cdot (3x^2) = -12x^2\sin(x^3)$$

REMARK. Formula (1) is easy to remember because the left side is exactly what results if we "cancel" the du's on the right side. This "canceling" device provides a good way to remember the chain rule when variables other than x, y, and u are used.

Example 2

Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution. In this case the chain rule takes the form

$$\begin{aligned}\frac{dw}{dt} &= \frac{dw}{dx}\cdot\frac{dx}{dt} = \frac{d}{dx}[\tan x]\cdot\frac{d}{dt}[4t^3+t]\\ &= (\sec^2 x)(12t^2+1) = (12t^2+1)\sec^2(4t^3+t)\end{aligned}$$ ◀

GENERALIZED DERIVATIVE FORMULAS

Although Formula (1) is useful, it is sometimes unwieldy because it involves so many dependent variables. A simpler version of the chain rule can be obtained by noting that $y = f(u)$ in (1), so

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)] \quad\text{and}\quad \frac{dy}{du} = f'(u)$$

Substituting these expressions in (1) yields the following alternative form of the chain rule:

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \tag{2}$$

This very powerful formula vastly extends our differentiation capabilities. For example, to differentiate the function

$$f(x) = \left(x^2 - x + 1\right)^{23} \tag{3}$$

we can let $u = x^2 - x + 1$, so (3) becomes $f(u) = u^{23}$, then apply (2) to obtain

$$\begin{aligned}\frac{d}{dx}\left[(x^2-x+1)^{23}\right] &= \frac{d}{dx}[u^{23}] = \underbrace{23u^{22}}_{f'(u)}\frac{du}{dx}\\ &= 23\left(x^2-x+1\right)^{22}\frac{d}{dx}[x^2-x+1]\\ &= 23\left(x^2-x+1\right)^{22}\cdot(2x-1)\end{aligned}$$

More generally, if u were any other differentiable function of x, the pattern of computations would be virtually the same. For example, if $u = \cos x$, then

$$\begin{aligned}\frac{d}{dx}[\cos^{23} x] &= \frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx} = 23\cos^{22} x\frac{d}{dx}[\cos x]\\ &= 23\cos^{22} x\cdot(-\sin x) = -23\sin x\cos^{22} x\end{aligned}$$

In both of the preceding computations, the chain rule took the form

$$\frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx} \tag{4}$$

This formula is a generalization of the more basic formula

$$\frac{d}{dx}[x^{23}] = 23x^{22} \tag{5}$$

In fact, in the special case where $u = x$, Formula (4) reduces to (5) since

$$\frac{d}{dx}[u^{23}] = 23u^{22}\frac{du}{dx} = 23x^{22}\frac{d[x]}{dx} = 23x^{22}$$

Table 3.5.1 contains a list of ***generalized derivative formulas*** that are consequences of (2).

Table 3.5.1

GENERALIZED DERIVATIVE FORMULAS	
$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$ (n an integer)	$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u \cot u\frac{du}{dx}$

Example 3

Find

(a) $\frac{d}{dx}[\sin(2x)]$ (b) $\frac{d}{dx}[\tan(x^2+1)]$

(c) $\frac{d}{dx}[\sqrt{x^3+\csc x}]$ (d) $\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right]$

Solution (a). Taking $u = 2x$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2 = 2\cos 2x$$

Solution (b). Taking $u = x^2+1$ in the generalized derivative formula for $\tan u$ yields

$$\begin{aligned}\frac{d}{dx}[\tan(x^2+1)] &= \frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}\\ &= \sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1] = \sec^2(x^2+1)\cdot 2x\\ &= 2x\sec^2(x^2+1)\end{aligned}$$

Solution (c). Taking $u = x^3+\csc x$ in the generalized derivative formula for $\sqrt{u}$ yields

$$\begin{aligned}\frac{d}{dx}[\sqrt{x^3+\csc x}] &= \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{x^3+\csc x}}\cdot\frac{d}{dx}[x^3+\csc x]\\ &= \frac{1}{2\sqrt{x^3+\csc x}}\cdot(3x^2-\csc x\cot x) = \frac{3x^2-\csc x\cot x}{2\sqrt{x^3+\csc x}}\end{aligned}$$

Solution (d). Taking $u = 1+x^5\cot x$ in the generalized derivative formula for u^{-8} yields

$$\begin{aligned}\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right] &= \frac{d}{dx}[u^{-8}] = -8u^{-9}\frac{du}{dx}\\ &= -8\left(1+x^5\cot x\right)^{-9}\cdot\frac{d}{dx}[1+x^5\cot x]\\ &= -8\left(1+x^5\cot x\right)^{-9}\cdot(x^5(-\csc^2 x)+5x^4\cot x)\\ &= (8x^5\csc^2 x-40x^4\cot x)\left(1+x^5\cot x\right)^{-9}\end{aligned}$$ ◀

Sometimes you will have to make adjustments in notation or apply the chain rule more than once to calculate a derivative.

Example 4

Find

(a) $\frac{d}{dx}[\sin(\sqrt{1+\cos x})]$ (b) $\frac{d\mu}{dt}$ if $\mu = \sec\sqrt{\omega t}$ (ω constant)

Solution (a). Taking $u = \sqrt{1+\cos x}$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(\sqrt{1+\cos x})] = \frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$$

$$= \cos(\sqrt{1+\cos x}) \cdot \frac{d}{dx}[\sqrt{1+\cos x}]$$

We use the generalized derivative formula for $\sqrt{u}$ with $u = 1+\cos x$.

$$= \cos(\sqrt{1+\cos x}) \cdot \frac{-\sin x}{2\sqrt{1+\cos x}}$$

$$= -\frac{\sin x \cos(\sqrt{1+\cos x})}{2\sqrt{1+\cos x}}$$

Solution (b).

$$\frac{d\mu}{dt} = \frac{d}{dt}[\sec\sqrt{\omega t}] = \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{d}{dt}[\sqrt{\omega t}]$$

We used the generalized derivative formula for $\sec u$ with $u = \sqrt{\omega t}$.

$$= \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{\omega}{2\sqrt{\omega t}}$$

We used the generalized derivative formula for $\sqrt{u}$ with $u = \omega t$. ◀

AN ALTERNATIVE APPROACH TO USING THE CHAIN RULE

As you become more comfortable with the chain rule, you may want to dispense with actually writing out the expression for u in your computations. To accomplish this, it is helpful to express Formula (2) in words. If we call u the "inside function" and f the "outside function" in the composition $f(u)$, then (2) states:

> *The derivative of $f(u)$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.*

For example,

$$\frac{d}{dx}[\cos(x^2+9)] = \underbrace{-\sin(x^2+9)}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{2x}_{\text{Derivative of the inside}}$$

$$\frac{d}{dx}[\tan^2 x] = \frac{d}{dx}\left[(\tan x)^2\right] = \underbrace{(2\tan x)}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{(\sec^2 x)}_{\text{Derivative of the inside}} = 2\tan x\sec^2 x$$

In general, if $f(g(x))$ is a composition of functions in which the inside function g and the outside function f are differentiable, then

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{Derivative of the outside evaluated at the inside}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside}} \qquad (6)$$

DIFFERENTIATING USING COMPUTER ALGEBRA SYSTEMS

Although the chain rule makes it possible to differentiate extremely complicated functions, the computations can be time-consuming to execute by hand. For complicated derivatives engineers and scientists often use computer algebra systems such as *Mathematica*, *Maple*, and *Derive*. For example, although we have all of the mathematical tools to perform the differentiation

$$\frac{d}{dx}\left[\frac{(x^2+1)^{10}\sin^3(\sqrt{x})}{\sqrt{1+\csc x}}\right] \tag{7}$$

by hand, the computations are sufficiently tedious that it would be more efficient to use a computer algebra system.

FOR THE READER. If you have a CAS, use it to obtain the derivatives in Examples 2, 3, and 4, and also to perform the differentiation in (7).

EXERCISE SET 3.5 Graphing Calculator C CAS

In Exercises 1–24, find $f'(x)$.

1. $f(x) = (x^3+2x)^{37}$
2. $f(x) = (3x^2+2x-1)^6$
3. $f(x) = \left(x^3 - \frac{7}{x}\right)^{-2}$
4. $f(x) = \frac{1}{(x^5-x+1)^9}$
5. $f(x) = \frac{4}{(3x^2-2x+1)^3}$
6. $f(x) = \sqrt{x^3-2x+5}$
7. $f(x) = \sqrt{4+\sqrt{3x}}$
8. $f(x) = \sin^3 x$
9. $f(x) = \sin(x^3)$
10. $f(x) = \cos^2(3\sqrt{x})$
11. $f(x) = \tan(4x^2)$
12. $f(x) = 3\cot^4 x$
13. $f(x) = 4\cos^5 x$
14. $f(x) = \csc(x^3)$
15. $f(x) = \sin\left(\frac{1}{x^2}\right)$
16. $f(x) = \tan^4(x^3)$
17. $f(x) = 2\sec^2(x^7)$
18. $f(x) = \cos^3\left(\frac{x}{x+1}\right)$
19. $f(x) = \sqrt{\cos(5x)}$
20. $f(x) = \sqrt{3x-\sin^2(4x)}$
21. $f(x) = [x+\csc(x^3+3)]^{-3}$
22. $f(x) = [x^4-\sec(4x^2-2)]^{-4}$
23. $f(x) = x^2\sqrt{5-x^2}$
24. $f(x) = \frac{x}{\sqrt{1-x^2}}$

In Exercises 25–39, find dy/dx.

25. $y = x^3\sin^2(5x)$
26. $y = \sqrt{x}\tan^3(\sqrt{x})$
27. $y = x^5\sec(1/x)$
28. $y = \frac{\sin x}{\sec(3x+1)}$
29. $y = \cos(\cos x)$
30. $y = \sin(\tan 3x)$
31. $y = \cos^3(\sin 2x)$
32. $y = \frac{1+\csc(x^2)}{1-\cot(x^2)}$
33. $y = (5x+8)^{13}(x^3+7x)^{12}$
34. $y = (2x-5)^2(x^2+4)^3$
35. $y = \left(\frac{x-5}{2x+1}\right)^3$
36. $y = \left(\frac{1+x^2}{1-x^2}\right)^{17}$
37. $y = \frac{(2x+3)^3}{(4x^2-1)^8}$
38. $y = [1+\sin^3(x^5)]^{12}$
39. $y = [x\sin 2x + \tan^4(x^7)]^5$

In Exercises 40–43, find d^2y/dx^2.

40. $y = \sin(3x^2)$
41. $y = x\cos(5x) - \sin^2 x$
42. $y = x\tan\left(\frac{1}{x}\right)$
43. $y = \frac{1+x}{1-x}$

C 44. Use a CAS to check the answers to the problems you solved in Exercises 1–43.

In Exercises 45–48, find an equation for the tangent line to the graph at the specified point.

45. $y = x\cos 3x,\ x = \pi$
46. $y = \sin(1+x^3),\ x = -3$
47. $y = \sec^3\left(\frac{\pi}{2} - x\right),\ x = -\frac{\pi}{2}$
48. $y = \left(x - \frac{1}{x}\right)^3,\ x = 2$

In Exercises 49–52, find the indicated derivative.

49. $y = \cot^3(\pi - \theta)$; find $\frac{dy}{d\theta}$.
50. $\lambda = \left(\frac{au+b}{cu+d}\right)^6$; find $\frac{d\lambda}{du}$ $(a, b, c, d$ constants).
51. $\frac{d}{d\omega}[a\cos^2 \pi\omega + b\sin^2 \pi\omega]$ $(a, b$ constants).

52. $x = \csc^2\left(\frac{\pi}{3} - y\right)$; find $\frac{dx}{dy}$.

53. (a) Use a graphing utility to obtain the graph of the function $f(x) = x\sqrt{4 - x^2}$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f'.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f'.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together.

54. (a) Use a graphing utility to obtain the graph of the function $f(x) = \sin x^2 \cos x$ over the interval $[-\pi/2, \pi/2]$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f' over the interval.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f' over the interval.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together over the interval.

55. If an object suspended from a spring is displaced vertically from its equilibrium position by a small amount and released, and if the air resistance and the mass of the spring are ignored, then the resulting oscillation of the object is called ***simple harmonic motion***. Under appropriate conditions the displacement y from equilibrium in terms of time t is given by

$$y = A\cos\omega t$$

where A is the initial displacement at time $t = 0$, and ω is a constant that depends on the mass of the object and the stiffness of the spring (see the accompanying figure). The constant $|A|$ is called the ***amplitude*** of the motion and ω the ***angular frequency***.
(a) Show that

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

(b) The ***period*** T is the time required to make one complete oscillation. Show that $T = 2\pi/\omega$.
(c) The ***frequency*** f of the vibration is the number of oscillations per unit time. Find f in terms of the period T.
(d) Find the amplitude, period, and frequency of an object that is executing simple harmonic motion given by $y = 0.6\cos 15t$, where t is in seconds and y is in centimeters.

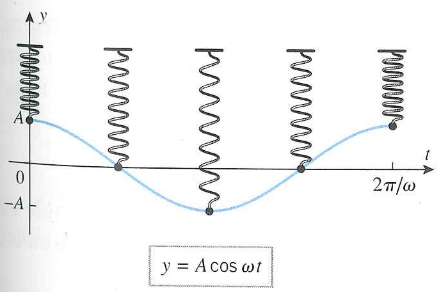


Figure Ex-55

56. Find the value of the constant A so that $y = A\sin 3t$ satisfies the equation

$$\frac{d^2y}{dt^2} + 2y = 4\sin 3t$$

57. The accompanying figure shows the graph of atmospheric pressure p (lb/in^2) versus the altitude h (mi) above sea level.
(a) From the graph and the tangent line at $h = 2$ shown on the graph, estimate the values of p and dp/dh at an altitude of 2 mi.
(b) If the altitude of a space vehicle is increasing at the rate of 0.3 mi/s at the instant when it is 2 mi above sea level, how fast is the pressure changing with time at this instant?

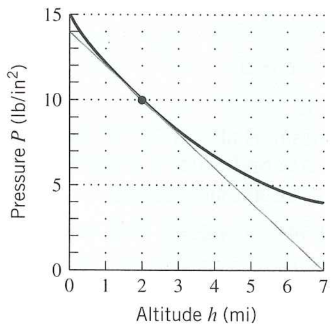


Figure Ex-57

58. The force F (in pounds) acting at an angle θ with the horizontal that is needed to drag a crate weighing W pounds along a horizontal surface at a constant velocity is given by

$$F = \frac{\mu W}{\cos\theta + \mu\sin\theta}$$

where μ is a constant called the ***coefficient of sliding friction*** between the crate and the surface (see the accompanying figure). Suppose that the crate weighs 150 lb and that $\mu = 0.3$.
(a) Find $dF/d\theta$ when $\theta = 30°$. Express the answer in units of pounds/degree.
(b) Find dF/dt when $\theta = 30°$ if θ is decreasing at the rate of $0.5°$/s at this instant.

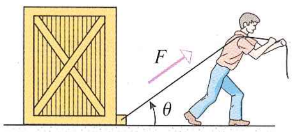


Figure Ex-58

59. Recall that

$$\frac{d}{dx}(|x|) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Use this result and the chain rule to find

$$\frac{d}{dx}(|\sin x|)$$

for nonzero x in the interval $(-\pi, \pi)$.

60. Use the derivative formula for $\sin x$ and the identity

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to obtain the derivative formula for $\cos x$.

61. Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Find $f'(x)$ for $x \neq 0$.
(b) Show that f is continuous at $x = 0$.
(c) Use Definition 3.2.2 to show that $f'(0)$ does not exist.

62. Let

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Find $f'(x)$ for $x \neq 0$.
(b) Show that f is continuous at $x = 0$.
(c) Use Definition 3.2.2 to find $f'(0)$.
(d) Show that f' is not continuous at $x = 0$.

63. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$
2	1	7
8	5	−3

(a) $g'(2)$, where $g(x) = [f(x)]^3$
(b) $h'(2)$, where $h(x) = f(x^3)$

64. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
−1	2	3	2	−3
2	0	4	1	−5

(a) $F'(-1)$, where $F(x) = f(g(x))$
(b) $G'(-1)$, where $G(x) = g(f(x))$

65. Given that $f'(0) = 2$, $g(0) = 0$, and $g'(0) = 3$, find $(f \circ g)'(0)$.

66. Given that $f'(x) = \sqrt{3x+4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.

67. Given that $f'(x) = \dfrac{x}{x^2+1}$ and $g(x) = \sqrt{3x-1}$, find $F'(x)$ if $F(x) = f(g(x))$.

68. Find $f'(x^2)$ if $\dfrac{d}{dx}[f(x^2)] = x^2$.

69. Find $\dfrac{d}{dx}[f(x)]$ if $\dfrac{d}{dx}[f(3x)] = 6x$.

70. Recall that a function f is ***even*** if $f(-x) = f(x)$ and ***odd*** if $f(-x) = -f(x)$, for all x in the domain of f. Assuming that f is differentiable, prove:
(a) f' is odd if f is even
(b) f' is even if f is odd.

71. Draw some pictures to illustrate the results in Exercise 70, and write a paragraph that gives an informal explanation of why the results are true.

72. Let $y = f_1(u)$, $u = f_2(v)$, $v = f_3(w)$, and $w = f_4(x)$. Express dy/dx in terms of dy/du, dw/dx, du/dv, and dv/dw.

73. Find a formula for

$$\frac{d}{dx}[f(g(h(x)))]$$

3.6 LOCAL LINEAR APPROXIMATION; DIFFERENTIALS

Up to now we have been interpreting dy/dx as a single entity representing the derivative of y with respect to x. In this section we will give the quantities dy and dx separate meanings that will allow us to treat dy/dx as a ratio. We will also show how derivatives can be used to approximate functions by simpler linear functions.

INCREMENTS

If the value of a variable changes from one number to another, then the final value minus the initial value is called an ***increment*** in the variable. It is traditional in calculus to denote an increment in a variable x by Δx (read "delta x"). Thus, if the initial value of x is x_0 and the final value is x_1, then

$$\Delta x = x_1 - x_0$$

In this notation the expression Δx is not the product of Δ and x; rather, it is a single entity representing the *change* in the value of x. This notation can be used with any variable; for example, increments in y, t, and θ would be denoted as Δy, Δt, and $\Delta\theta$.

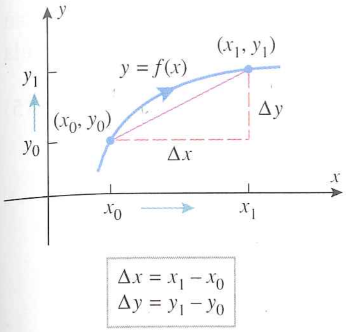


Figure 3.6.1

If $y = f(x)$, and if x changes from an initial value x_0 to a final value x_1, then there is a corresponding change in the value of y from $y_0 = f(x_0)$ to $y_1 = f(x_1)$. Stated another way, the increment $\Delta x = x_1 - x_0$ in x produces a corresponding increment $\Delta y = y_1 - y_0$ in y, where

$$\Delta y = y_1 - y_0 = f(x_1) - f(x_0) \tag{1}$$

(Figure 3.6.1).

Increments can be positive, negative, or zero, depending on the relative positions of the initial and final points—an increment in x is positive if the final point is to the right of the initial point, negative if the final point is to the left of the initial point, and zero if the initial and final points coincide. In Figure 3.6.1, both Δx and Δy are positive.

Observe that the expressions $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$ can be rewritten as

$$x_1 = x_0 + \Delta x \quad \text{and} \quad y_1 = y_0 + \Delta y$$

which simply states that the *final value of a variable is equal to its initial value plus its increment*. With this notation we can express (1) as

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \tag{2}$$

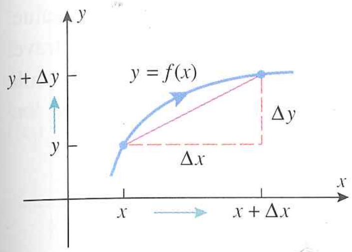


Figure 3.6.2

Sometimes, it is convenient to dispense with subscripts on the initial and final values of a variable, in which case the initial and final values of x would be denoted as x and $x + \Delta x$, and the initial and final values of the variable y would be denoted as y and $y + \Delta y$ (Figure 3.6.2). With this notation the symbols x and y play dual roles—they serve as the names as well as the initial values of the variables. However, this rarely causes any confusion.

With the subscripts omitted, Formula (2) becomes

$$\Delta y = f(x + \Delta x) - f(x) \tag{3}$$

The ratio $\Delta y/\Delta x$ can be interpreted as the slope of the secant line joining the points (x, y) and $(x + \Delta x, y + \Delta y)$, and hence the derivative of y with respect to x can be expressed as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4}$$

(Figure 3.6.3). This is consistent with (11) in Section 3.2.

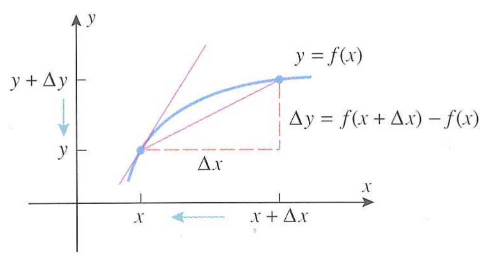


Figure 3.6.3

DIFFERENTIALS

When Newton and Leibniz independently published their discoveries of calculus, they each used different notations for the derivative, and battles raged for more than 50 years over which notation was better. In the end the ***Leibniz notation*** dy/dx won out because it produced correct formulas in a natural way; the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

is a good example.

The symbols "dy" and "dx" that appear in the derivative dy/dx are called ***differentials***, and our next objective is to define these symbols so that dy/dx can actually be treated as a

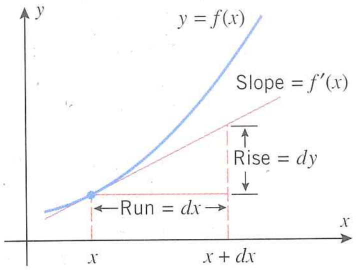


Figure 3.6.4

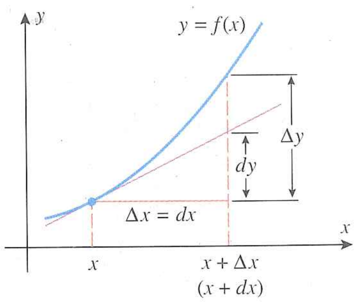


Figure 3.6.5

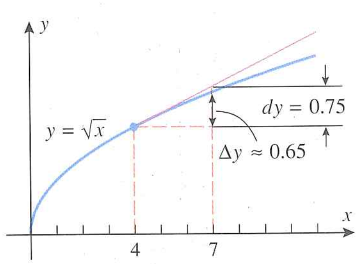


Figure 3.6.6

LOCAL LINEAR APPROXIMATION

ratio. For this purpose, regard x as fixed and *define* dx to be an independent variable that can be assigned an arbitrary value. If f is differentiable at x, then we *define* dy by the formula

$$dy = f'(x)\,dx \tag{5}$$

If $dx \neq 0$, then we can divide both sides of (5) by dx to obtain

$$\frac{dy}{dx} = f'(x)$$

Thus, we have achieved our goal of defining dy and dx so that their ratio is $f'(x)$.

Because

$$\frac{dy}{dx} = f'(x) = m_{\tan}$$

where $m_{\tan}$ is the slope of the tangent to $y = f(x)$ at x, the differentials dy and dx can be viewed as a corresponding rise and run of this tangent line (Figure 3.6.4).

It is important to understand the distinction between the increment Δy and the differential dy. To see the difference, let us assign the independent variables dx and Δx the same value, so $dx = \Delta x$. Then Δy represents the change in y that occurs when we start at x and travel *along the curve* $y = f(x)$ until we have moved Δx $(= dx)$ units in the x-direction, while dy represents the change in y that occurs if we start at x and travel *along the tangent line* until we have moved dx $(= \Delta x)$ units in the x-direction (Figure 3.6.5).

Example 1

If $y = x^2$, then the relation $dy/dx = 2x$ can be written in the *differential form*

$$dy = 2x\,dx$$

When $x = 3$, this becomes

$$dy = 6\,dx$$

This tells us that if we travel along the tangent to the curve $y = x^2$ at $x = 3$, then a change of dx units in x produces a change of $6\,dx$ units in y. For example, if the change in x is $dx = 4$, then the change in y along the tangent is

$$dy = 6(4) = 24 \quad \text{units}$$

◀

Example 2

Let $y = \sqrt{x}$. Find dy and Δy at $x = 4$ with $dx = \Delta x = 3$. Then make a sketch of $y = \sqrt{x}$, showing dy and Δy in the picture.

Solution. From (3) with $f(x) = \sqrt{x}$ we obtain

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{7} - \sqrt{4} \approx 0.65$$

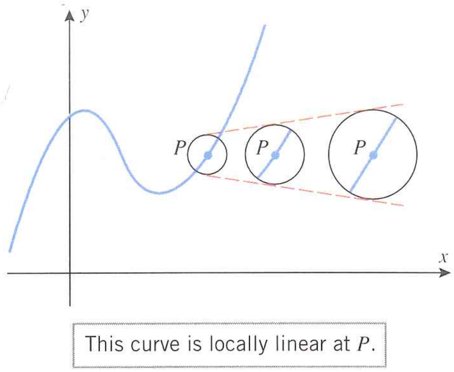
If $y = \sqrt{x}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} = 0.75$$

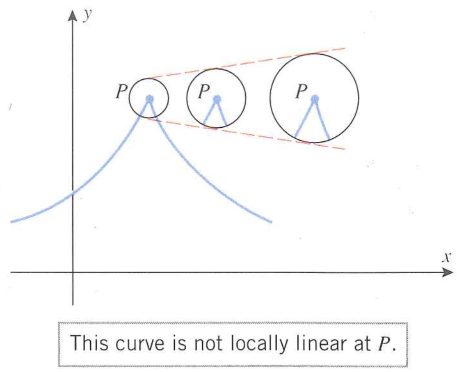
Figure 3.6.6 shows the curve $y = \sqrt{x}$ together with dy and Δy. ◀

Points of differentiability for a function f can be described informally in terms of the behavior of the graph of f under magnification: If P is a point of differentiability for a function f, then stronger and stronger magnifications at P eventually make the curve segment containing P look more and more like a nonvertical line, the line being the tangent line at P. For this reason, a function that is differentiable at a point P is said to be ***locally linear*** at P (Figure 3.6.7).

It follows from the preceding observations that if f is differentiable at x_0, then the tangent line through $(x_0, f(x_0))$ closely approximates the graph of f for values of x near



(a)



(b)

Figure 3.6.7

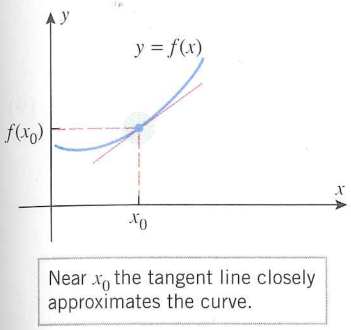


Figure 3.6.8

x_0 (Figure 3.6.8). To capture this intuitive idea analytically, observe that the tangent line through the point $(x_0, f(x_0))$ has slope $f'(x_0)$, so the point-slope form of its equation is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

which we can rewrite as

$$y = f(x_0) + f'(x_0)(x - x_0)$$

To say that this line closely approximates the curve $y = f(x)$ for values of x near x_0, we mean that the approximation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{6}$$

gets better and better as $x \to x_0$. We call (6) the ***local linear approximation of f at x_0***. An alternative version of this formula can be obtained by letting $\Delta x = x - x_0$, in which case (6) can be expressed as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\,\Delta x \tag{7}$$

Example 3

(a) Find the local linear approximation of $f(x) = \sin x$ at $x_0 = 0$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sin 2°$, and compare your approximation to the result produced directly by your calculating device.

Solution (a). Since $f'(x) = \cos x$, it follows from (6) that the local linear approximation of $\sin x$ at a point x_0 is

$$\sin x \approx \sin x_0 + (\cos x_0)(x - x_0)$$

Thus, the local linear approximation at $x_0 = 0$ is

$$\sin x \approx \sin 0 + (\cos 0)(x - 0)$$

which simplifies to

$$\sin x \approx x \tag{8}$$

Solution (b). In (8), the variable x is in radian measure, so we must first convert 2° to radians before we can apply this formula. Since

$$2° = 2(\pi/180) = \pi/90 \text{ radians}$$

it follows from (8) with $x = \pi/90$ that

$$\sin 2° = \sin(\pi/90) \approx \pi/90 \approx 0.0349066$$

This is quite close to the value

$$\sin 2^\circ \approx 0.0348995$$

produced directly on the author's calculator. ◀

Example 4

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to the result produced directly by your calculating device.

Solution (a). Since $f'(x) = 1/(2\sqrt{x})$, it follows from (6) that the local linear approximation of $\sqrt{x}$ at a point x_0 is

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0)$$

Thus, the local linear approximation at $x_0 = 1$ is

$$\sqrt{x} \approx 1 + \tfrac{1}{2}(x - 1) \qquad (9)$$

Solution (b). Applying Formula (9) with $x = 1.1$ yields

$$\sqrt{1.1} \approx 1 + \tfrac{1}{2}(0.1) = 1.05$$

which compares favorably with the approximation

$$\sqrt{1.1} \approx 1.04881$$

produced directly on the author's calculator. ◀

REMARK. In the last two examples we used Formula (6) for the local linear approximation. We could just as well have used Formula (7). For example, with this formula the local linear approximation of $f(x) = \sqrt{x}$ at x_0 is

$$\sqrt{x_0 + \Delta x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}\Delta x$$

Thus, to approximate $\sqrt{1.1}$ with this formula, we take $x_0 = 1$ and $\Delta x = 0.1$, which yields

$$\sqrt{1.1} \approx 1 + \tfrac{1}{2}(0.1) = 1.05$$

This agrees with the result in Example 4.

ERROR IN LOCAL LINEAR APPROXIMATIONS

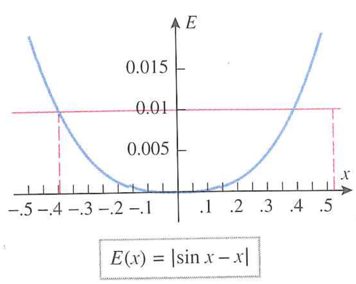


Figure 3.6.9

As a general rule, the accuracy of the local linear approximation to $f(x)$ at a point x_0 will deteriorate as x gets progressively farther from x_0. To illustrate this for approximation (8) in Example 3, let us graph the function

$$E(x) = |\sin x - x|$$

which is the absolute value of the error in the approximation (Figure 3.6.9).

In Figure 3.6.9, the graph shows how the absolute error in the local linear approximation of $\sin x$ at 0 increases as x moves progressively farther from 0 in either the positive or negative direction. The graph also tells us that for values of x between the two vertical lines the absolute error does not exceed 0.01. Thus, for example, we could use the local linear approximation $\sin x \approx x$ for all values of x in the interval $-0.35 < x < 0.35$ (radians) with confidence that the approximation is within ± 0.01 of the exact value.

ERROR PROPAGATION IN APPLICATIONS

In applications, small errors invariably occur in measured quantities. When these quantities are used in computations, those errors are propagated in turn to the computed quantities; this is called ***error propagation***. We will now show how to use a local linear approximation to estimate the error in a computed quantity from estimates of the error in the measured quantity. For this purpose, suppose that

x is the quantity being measured

$y = f(x)$ is the quantity being computed

x_0 is the true value of x

y_0 is the true value of y

Δx is the measurement error in x

Δy is the propagated error in y

Thus, the measured value of x is $x_0 + \Delta x$, and the computed value of y is $y_0 + \Delta y$; and we are interested in using an estimate of Δx to find an estimate of Δy. To do this, we will start with version (7) of the local linear approximation of f at x_0:

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\,\Delta x \tag{10}$$

In this formula, $f(x_0) = y_0$ is the true value of y, and $f(x_0 + \Delta x) = y_0 + \Delta y$ is the computed value of y, so we can rewrite (10) as

$$y_0 + \Delta y \approx y_0 + f'(x_0)\,\Delta x$$

or

$$\Delta y \approx f'(x_0)\,\Delta x$$

Moreover, if we agree to let $dx = \Delta x$, then we can rewrite this as

$$\Delta y \approx f'(x_0)\,dx = dy \tag{11}$$

which tells us that *the propagated error in y can be estimated by the differential of y at x_0 with dx interpreted as the measurement error in x.*

Although Formula (11) looks nice on the surface, it is useless in applied problems because the true value x_0 is unknown! (Keep in mind that the only value of x that is available to the researcher is the measured value $x_0 + dx = x_0 + \Delta x$.) To get around this roadblock researchers use the observed value $x_0 + dx$ rather than the true value x_0 in computing the differential. This is usually satisfactory if dx is small, since x_0 and $x_0 + dx$ are close in value. We will illustrate how this works in the next example, but it will simplify our computations if we drop the subscript in (11) and write the formula as

$$\Delta y \approx dy = f'(x)\,dx \tag{12}$$

Example 5

The radius of a sphere is measured to be 50 cm with a measurement error of ± 0.02 cm. Estimate the error in the computed volume of the sphere.

Solution. The volume of a sphere is

$$V = \tfrac{4}{3}\pi r^3$$

We are given that the error in the radius is $\Delta r = \pm 0.02$, and we want to find the error ΔV in V. If we consider Δr to be small and if we let $dr = \Delta r$, then ΔV can be approximated by dV. Thus, from (12),

$$\Delta V \approx dV = 4\pi r^2\,dr \tag{13}$$

Substituting $r = 50$ and $dr = \pm 0.02$ in (13), we obtain

$$\Delta V \approx 4\pi(2500)(\pm 0.02) \approx \pm 628.32$$

Therefore, the error in the calculated volume is approximately ± 628.32 cubic centimeters (cm^3). ◀

If the true value of a quantity is q and a measurement or calculation produces an error Δq, then $\Delta q/q$ is called the ***relative error*** in the measurement or calculation; when expressed as a percentage, $\Delta q/q$ is called the ***percentage error***. As a practical matter, the true value q is usually unknown, so that the measured or calculated value of q is used instead; and the relative error is approximated by dq/q.

Example 6

The side of a square is measured with a percentage error of $\pm 5\%$. Estimate the percentage error in the calculated area of the square.

Solution. The area A of a square with side x is

$$A = x^2$$

so

$$dA = 2x\,dx$$

We are given that $dx/x = \pm 0.05$, and we want to find dA/A. But it follows from the two preceding formulas that

$$\frac{dA}{A} = \frac{2x\,dx}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x} = 2(\pm 0.05) = \pm 0.1 \qquad (14)$$

Thus, the percentage error in the calculated area of the square is $\pm 10\%$. ◀

FOR THE READER. In (14) we saw that $dA/A = 2(dx/x)$, which tells us that as a rule of thumb the percentage error in the calculated area of a square is twice the percentage error in the measured side. What rule of thumb relates the percentage error in the computed volume of a cube to the percentage error in the measured side? Why?

DIFFERENTIAL FORMULAS

Now that we have defined differentials, every derivative formula has a corresponding differential formula. For example, if $y = \sin x$, then the derivative formula $dy/dx = \cos x$ can also be expressed as

$$dy = \cos x\,dx$$

Moreover, all of the general rules of differentiation have corresponding differential versions:

DERIVATIVE FORMULA	DIFFERENTIAL FORMULA
$\frac{d}{dx}[c] = 0$	$d[c] = 0$
$\frac{d}{dx}[cf] = c\frac{df}{dx}$	$d[cf] = c\,df$
$\frac{d}{dx}[f+g] = \frac{df}{dx} + \frac{dg}{dx}$	$d[f+g] = df + dg$
$\frac{d}{dx}[fg] = f\frac{dg}{dx} + g\frac{df}{dx}$	$d[fg] = f\,dg + g\,df$
$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$	$d\left[\frac{f}{g}\right] = \frac{g\,df - f\,dg}{g^2}$

EXERCISE SET 3.6 Graphing Calculator [C] CAS

1. (a) Let $y = x^2$. Find dy and Δy at $x = 2$ with $dx = \Delta x = 1$.
   (b) Sketch the graph of $y = x^2$, showing dy and Δy in the picture.
2. (a) Let $y = x^3$. Find dy and Δy at $x = 1$ with $dx = \Delta x = 1$.
   (b) Sketch the graph of $y = x^3$, showing dy and Δy in the picture.
3. (a) Let $y = 1/x$. Find dy and Δy at $x = 1$ with $dx = \Delta x = -0.5$.
   (b) Sketch the graph of $y = 1/x$, showing dy and Δy in the picture.
4. (a) Let $y = \sqrt{x}$. Find dy and Δy at $x = 9$ with $dx = \Delta x = -1$.
   (b) Sketch the graph of $y = \sqrt{x}$, showing dy and Δy in the picture.

In Exercises 5–8, find formulas for dy and Δy at a general point x.

5. $y = x^3$
6. $y = 8x - 4$
7. $y = x^2 - 2x + 1$
8. $y = \sin x$

In Exercises 9–12, find the differential dy.

9. (a) $y = 4x^3 - 7x^2$ (b) $y = x \cos x$
10. (a) $y = 1/x$ (b) $y = 5 \tan x$
11. (a) $y = x\sqrt{1-x}$ (b) $y = (1+x)^{-17}$
12. (a) $y = \dfrac{1}{x^3 - 1}$ (b) $y = \dfrac{1 - x^3}{2 - x}$
13. (a) Use Formula (6) to obtain the local linear approximation of x^3 at $x_0 = 1$.
    (b) Use Formula (7) to rewrite the approximation obtained in part (a) in terms of Δx.
    (c) Use the result obtained in part (a) to approximate $(1.02)^3$, and confirm that the formula obtained in part (b) produces the same result.
14. (a) Use Formula (6) to obtain the local linear approximation of $1/x$ at $x_0 = 2$.
    (b) Use Formula (7) to rewrite the approximation obtained in part (a) in terms of Δx.
    (c) Use the result obtained in part (a) to approximate $1/2.05$, and confirm that the formula obtained in part (b) produces the same result.
15. (a) Find the local linear approximation of $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.
    (b) Graph f and its tangent line at x_0 together, and use the graphs to illustrate the relationship between the exact values and the approximations of $\sqrt{0.9}$ and $\sqrt{1.1}$.
16. (a) Find the local linear approximation of $f(x) = 1/\sqrt{x}$ at $x_0 = 4$, and use it to approximate $1/\sqrt{3.9}$ and $1/\sqrt{4.1}$.
    (b) Graph f and its tangent line at x_0 together, and use the graphs to illustrate the relationship between the exact values and the approximations of $1/\sqrt{3.9}$ and $1/\sqrt{4.1}$.

In Exercises 17–20, confirm that the stated formula is the local linear approximation at $x_0 = 0$.

17. $(1+x)^{15} \approx 1 + 15x$
18. $\dfrac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$
19. $\tan x \approx x$
20. $\dfrac{1}{1+x} \approx 1 - x$

In Exercises 21–24, confirm that the stated formula is the local linear approximation of f at $x_0 = 1$, where $\Delta x = x - 1$.

21. $f(x) = x^4;\ (1 + \Delta x)^4 \approx 1 + 4x^3\Delta x$
22. $f(x) = \sqrt{x};\ \sqrt{1 + \Delta x} \approx 1 + \frac{1}{2}\Delta x$
23. $f(x) = \dfrac{1}{2+x};\ \dfrac{1}{3 + \Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$
24. $f(x) = (4+x)^3;\ (5 + \Delta x)^3 \approx 125 + 75\Delta x$
25. (a) Use the local linear approximation of $\sin x$ at $x_0 = 0$ obtained in Example 3 to approximate $\sin 1^\circ$, and compare the approximation to the result produced directly by your calculating device.
    (b) How would you choose x_0 to approximate $\sin 44^\circ$?
    (c) Approximate $\sin 44^\circ$; compare the approximation to the result produced directly by your calculating device.
26. (a) Use the local linear approximation of $\tan x$ at $x_0 = 0$ to approximate $\tan 2^\circ$, and compare the approximation to the result produced directly by your calculating device.
    (b) How would you choose x_0 to approximate $\tan 61^\circ$?
    (c) Approximate $\tan 61^\circ$; compare the approximation to the result produced directly by your calculating device.

In Exercises 27–35, use an appropriate local linear approximation to estimate the value of the given quantity.

27. $(3.02)^4$
28. $(1.97)^3$
29. $\sqrt{65}$
30. $\sqrt{24}$
31. $\sqrt{80.9}$
32. $\sqrt{36.03}$
33. $\sin 0.1$
34. $\tan 0.2$
35. $\cos 31^\circ$
36. The approximation $(1+x)^k \approx 1 + kx$ is commonly used by engineers for quick calculations.
    (a) Derive this result, and use it to make a rough estimate of $(1.001)^{37}$.
    (b) Compare your estimate to that produced directly by your calculating device.
    (c) Show that this formula produces a very bad estimate of $(1.1)^{37}$, and explain why.

In Exercises 37–40, confirm that the formula is a local linear approximation at $x_0 = 0$, and use a graphing utility to estimate an interval of x-values on which the error in the approximation is at most ± 0.1.

37. $\sqrt{x+3} \approx \sqrt{3} + \dfrac{1}{2\sqrt{3}}x$

38. $\dfrac{1}{\sqrt{9-x}} \approx \dfrac{1}{3} + \dfrac{1}{54}x$

39. $\tan x \approx x$

40. $\dfrac{1}{(1+2x)^5} \approx 1 - 10x$

In Exercises 41–44, use dy to approximate Δy when x changes as indicated.

41. $y = \sqrt{3x-2}$; from $x = 2$ to $x = 2.03$

42. $y = \sqrt{x^2+8}$; from $x = 1$ to $x = 0.97$

43. $y = \dfrac{x}{x^2+1}$; from $x = 2$ to $x = 1.96$

44. $y = x\sqrt{8x+1}$; from $x = 3$ to $x = 3.05$

45. The side of a square is measured to be 10 ft, with a possible error of ± 0.1 ft.
(a) Use differentials to estimate the error in the calculated area.
(b) Estimate the percentage errors in the side and the area.

46. The side of a cube is measured to be 25 cm, with a possible error of ± 1 cm.
(a) Use differentials to estimate the error in the calculated volume.
(b) Estimate the percentage errors in the side and volume.

47. The hypotenuse of a right triangle is known to be 10 in exactly, and one of the acute angles is measured to be $30°$, with a possible error of $\pm 1°$.
(a) Use differentials to estimate the errors in the sides opposite and adjacent to the measured angle.
(b) Estimate the percentage errors in the sides.

48. One side of a right triangle is known to be 25 cm exactly. The angle opposite to this side is measured to be $60°$, with a possible error of $\pm 0.5°$.
(a) Use differentials to estimate the errors in the adjacent side and the hypotenuse.
(b) Estimate the percentage errors in the adjacent side and hypotenuse.

49. The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of $\pm 5\%$, use differentials to estimate the percentage error in R. (Assume k is exact.)

50. A 12-foot ladder leaning against a wall makes an angle θ with the floor. If the top of the ladder is h feet up the wall, express h in terms of θ and then use dh to estimate the change in h if θ changes from $60°$ to $59°$.

51. The area of a right triangle with a hypotenuse of H is calculated using the formula $A = \frac{1}{4}H^2 \sin 2\theta$, where θ is one of the acute angles. Use differentials to approximate the error in calculating A if $H = 4$ cm (exactly) and $\theta = 30° \pm 15'$.

52. The side of a square is measured with a possible percentage error of $\pm 1\%$. Use differentials to estimate the percentage error in the area.

53. The side of a cube is measured with a possible percentage error of $\pm 2\%$. Use differentials to estimate the percentage error in the volume.

54. The volume of a sphere is to be computed from a measured value of its radius. Estimate the maximum permissible percentage error in the measurement if the percentage error in the volume must be kept within $\pm 3\%$. ($V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r.)

55. The area of a circle is to be computed from a measured value of its diameter. Estimate the maximum permissible percentage error in the measurement if the percentage error in the area must be kept within $\pm 1\%$.

56. A steel cube with 1-in sides is coated with 0.01 in of copper. Use differentials to estimate the volume of copper in the coating. [*Hint:* Let ΔV be the change in the volume of the cube.]

57. A metal rod 15 cm long and 5 cm in diameter is to be covered (except for the ends) with insulation that is 0.001 cm thick. Use differentials to estimate the volume of insulation. [*Hint:* Let ΔV be the change in volume of the rod.]

58. The time required for one complete oscillation of a pendulum is called its ***period***. If the length L of the pendulum is measured in feet and the period P in seconds, then the period is given by $P = 2\pi\sqrt{L/g}$, where g is a constant called *the acceleration due to gravity*. Use differentials to show that the percentage error in P is approximately half the percentage error in L.

59. If the temperature T of a metal rod of length L is changed by an amount ΔT, then the length will change by the amount $\Delta L = \alpha L \, \Delta T$, where α is called the ***coefficient of linear expansion***. For moderate changes in temperature α is taken as constant.
(a) Suppose that a rod 40 cm long at $20°$C is found to be 40.006 cm long when the temperature is raised to $30°$C. Find α.
(b) If an aluminum pole is 180 cm long at $15°$C, how long is the pole if the temperature is raised to $40°$C? [Take $\alpha = 2.3 \times 10^{-5}/°\text{C}$.]

60. If the temperature T of a solid or liquid of volume V is changed by an amount ΔT, then the volume will change by the amount $\Delta V = \beta V \, \Delta T$, where β is called the ***coefficient of volume expansion***. For moderate changes in temperature β is taken as constant. Suppose that a tank truck loads 4000 gallons of ethyl alcohol at a temperature of $35°$C and delivers its load sometime later at a temperature of $15°$C. Using $\beta = 7.5 \times 10^{-4}/°\text{C}$ for ethyl alcohol, find the number of gallons delivered.

SUPPLEMENTARY EXERCISES

1. State the definition of a derivative, and give two interpretations of it.

2. Explain the difference between average and instantaneous rate of change, and discuss how they are calculated.

3. Given that $y = f(x)$, explain the difference between dy and Δy. Draw a picture that illustrates the relationship between these quantities.

4. Use the definition of a derivative to find dy/dx, and check your answer by calculating the derivative using appropriate derivative formulas.
(a) $y = \sqrt{9-4x}$ (b) $y = \dfrac{x}{x+1}$

In Exercises 5–8, find the values of x at which the curve $y = f(x)$ has a horizontal tangent line.

5. $f(x) = (2x+7)^6(x-2)^5$ **6.** $f(x) = \dfrac{(x-3)^4}{x^2+2x}$

7. $f(x) = \sqrt{3x+1}(x-1)^2$ **8.** $f(x) = \left(\dfrac{3x+1}{x^2}\right)^3$

9. The accompanying figure shows the graph of $y = f'(x)$ for an unspecified function f.
(a) For what values of x does the curve $y = f(x)$ have a horizontal tangent line?
(b) Over what intervals does the curve $y = f(x)$ have tangent lines with positive slope?
(c) Over what intervals does the curve $y = f(x)$ have tangent lines with negative slope?
(d) Given that $g(x) = f(x)\sin x$, and $f(0) = -1$, find $g''(0)$.

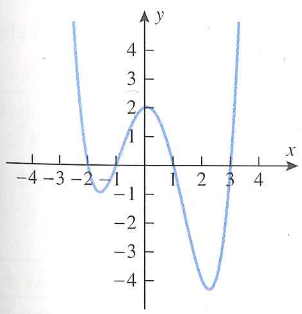


Figure Ex-9

10. In each part, evaluate the expression given that $f(1) = 1$, $g(1) = -2$, $f'(1) = 3$, and $g'(1) = -1$.
(a) $\left.\dfrac{d}{dx}[f(x)g(x)]\right|_{x=1}$ (b) $\left.\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]\right|_{x=1}$
(c) $\left.\dfrac{d}{dx}[\sqrt{f(x)}]\right|_{x=1}$ (d) $\dfrac{d}{dx}[f(1)g'(1)]$

11. Find the equations of all lines through the origin that are tangent to the curve $y = x^3 - 9x^2 - 16x$.

12. Find all values of x for which the tangent line to $y = 2x^3 - x^2$ is perpendicular to the line $x + 4y = 10$.

13. Find all values of x for which the line that is tangent to $y = 3x - \tan x$ is parallel to the line $y - x = 2$.

14. Suppose that $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ k(x-1), & x > 1. \end{cases}$

For what values of k is f
(a) continuous (b) differentiable?

15. Let $f(x) = x^2$. Show that for any distinct values of a and b, the slope of the tangent line to $y = f(x)$ at $x = \frac{1}{2}(a+b)$ is equal to the slope of the secant line through the points (a, a^2) and (b, b^2). Draw a picture to illustrate this result.

16. A car is traveling on a straight road that is 120 mi long. For the first 100 mi the car travels at an average velocity of 50 mi/h. Show that no matter how fast the car travels for the final 20 mi it cannot bring the average velocity up to 60 mi/h for the entire trip.

17. In each part, use the given information to find Δx, Δy, and dy.
(a) $y = 1/(x-1)$; x decreases from 2 to 1.5.
(b) $y = \tan x$; x increases from $-\pi/4$ to 0.
(c) $y = \sqrt{25 - x^2}$; x increases from 0 to 3.

18. Use the formula $V = l^3$ for the volume of a cube of side l to find
(a) the average rate at which the volume of a cube changes with l as l increases from $l = 2$ to $l = 4$
(b) the instantaneous rate at which the volume of a cube changes with l when $l = 5$.

19. The amount of water in a tank t minutes after it has started to drain is given by $W = 100(t-15)^2$ gal.
(a) At what rate is the water running out at the end of 5 min?
(b) What is the average rate at which the water flows out during the first 5 min?

20. Use an appropriate local linear approximation to estimate the value of $\cot 46°$, and compare your answer to the value obtained with a calculating device.

21. The base of the Great Pyramid at Giza is a square that is 230 m on each side.
(a) As illustrated in the accompanying figure, suppose that an archaeologist standing at the center of a side measures the angle of elevation of the apex to be $\phi = 51°$ with an error of $\pm 0.5°$. What can the archaeologist reasonably say about the height of the pyramid?
(b) Use differentials to estimate the allowable error in the elevation angle that will ensure an error in the height is at most ± 5 m.

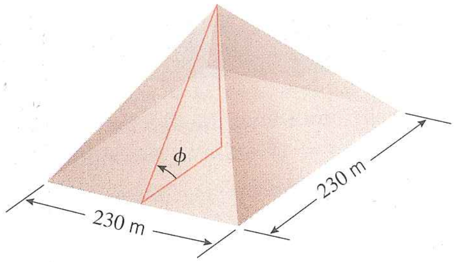


Figure Ex-21

22. The period T of a clock pendulum (i.e., the time required for one back-and-forth movement) is given in terms of its length L by $T = 2\pi\sqrt{L/g}$, where g is the gravitational constant.
   (a) Assuming that the length of a clock pendulum can vary (say, due to temperature changes), find the rate of change of the period T with respect to the length L.
   (b) If L is in meters (m) and T is in seconds (s), what are the units for the rate of change in part (a)?
   (c) If a pendulum clock is running slow, should the length of the pendulum be increased or decreased to correct the problem?
   (d) The constant g generally decreases with altitude. If you move a pendulum clock from sea level to a higher elevation, will it run faster or slower?
   (e) Assuming the length of the pendulum to be constant, find the rate of change of the period T with respect to g.
   (f) Assuming that T is in seconds (s) and g is in meters per second per second (m/s^2), find the units for the rate of change in part (e).

In Exercises 23 and 24, zoom in on the graph of f on an interval containing $x = x_0$ until the graph looks like a straight line. Estimate the slope of this line and then check your answer by finding the exact value of $f'(x_0)$.

23. (a) $f(x) = x^2 - 1,\ x_0 = 1.8$
   (b) $f(x) = \dfrac{x^2}{x-2},\ x_0 = 3.5$

24. (a) $f(x) = x^3 - x^2 + 1,\ x_0 = 2.3$
   (b) $f(x) = \dfrac{x}{x^2+1},\ x_0 = -0.5$

In Exercises 25 and 26, approximate $f'(2)$ by using the limit in Definition 3.2.2 with small values of h. If you have a CAS, see if it can find the exact value of the limit.

25. $f(x) = 2^x$

26. $f(x) = x^{\sin x}$

27. At time $t = 0$ a car moves into the passing lane to pass a slow-moving truck. The average velocity of the car from $t = 1$ to $t = 1 + h$ is

$$v_{\text{ave}} = \frac{3(h+1)^{2.5} + 580h - 3}{10h}$$

Estimate the instantaneous velocity of the car at $t = 1$, where time is in seconds and distance is in feet.

28. A sky diver jumps from an airplane. Suppose that the distance she falls during the period before her parachute opens is $s(t) = 986((0.835)^t - 1) + 176t$, where s is in feet, t is in seconds, and $t \geq 1$. Graph s versus t for $1 \leq t \leq 20$, and use your graph to estimate the instantaneous velocity at $t = 15$.

29. Approximate the values of x at which the tangent line to the graph of $y = x^3 - \sin x$ is horizontal.

30. Use a graphing utility to graph the function

$$f(x) = |x^4 - x - 1| - x$$

and find the values of x where the derivative of this function does not exist.

31. Use a CAS to find the derivative of f from the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and check the result by finding the derivative by hand.
   (a) $f(x) = x^5$
   (b) $f(x) = 1/x$
   (c) $f(x) = 1/\sqrt{x}$
   (d) $f(x) = \dfrac{2x+1}{x-1}$
   (e) $f(x) = \sqrt{3x^2+5}$
   (f) $f(x) = \sin 3x$

In Exercises 32–37: (a) use a CAS to find $f'(x)$, and check the result by hand; (b) use the CAS to find $f''(x)$.

32. $f(x) = x^2 \sin x$

33. $f(x) = \sqrt{x} + \cos^2 x$

34. $f(x) = \dfrac{2x^2 - x + 5}{3x + 2}$

35. $f(x) = \dfrac{\tan x}{1 + x^2}$

36. $f(x) = \dfrac{1}{x} \sin \sqrt{x}$

37. $f(x) = \dfrac{\sqrt{x^4 - 3x + 2}}{x(2 - \cos x)}$