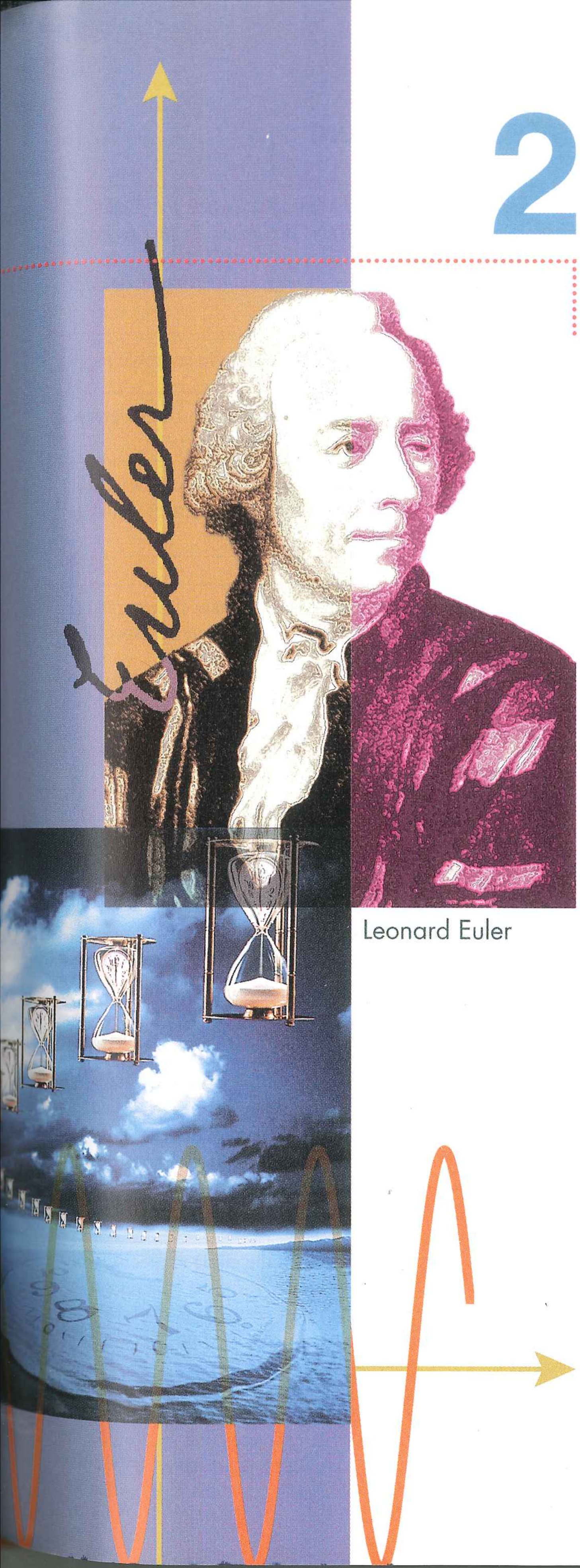


Leonard Euler

2

LIMITS AND CONTINUITY

The development of calculus was stimulated by two geometric problems: finding areas of plane regions and finding tangent lines to curves. As discussed in the Introduction, both of these problems require a “limit process” for their general solution. However, limit processes occur in many other applications as well—so many, in fact that the concept of a “limit” is the fundamental building block on which all other calculus concepts are based.

In this chapter we will develop the concept of a limit in stages: In Section 2.1 we will develop the basic ideas informally, relying on our intuition; in Section 2.2 we will discuss methods for calculating limits; and in Section 2.3 we will give the precise mathematical definition of a limit. In Sections 2.4 and 2.5 we will apply limits to the study “continuous” curves. Such curves are important because they model the idea of a smooth flow without breaks or interruptions—the flow of time, the motion of an object in flight, or the gradual warming of a room on a sunny day, for example.

2.1 LIMITS (AN INTUITIVE INTRODUCTION)

As discussed in the introduction to this chapter, the concept of a limit is the fundamental building block on which all other calculus concepts are based. In this section we will study limits informally, with the goal of developing an "intuitive feel" for the basic ideas. In the next two sections we will focus on the computational methods and precise definitions.

THE TANGENT LINE, AREA, AND VELOCITY PROBLEMS

Many of the basic ideas in calculus can be motivated by the following three problems.

THE TANGENT LINE PROBLEM. Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P (Figure 2.1.1).

THE AREA PROBLEM. Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis (Figure 2.1.2).

THE INSTANTANEOUS VELOCITY PROBLEM. Given the position versus time curve for a particle moving along a coordinate line, find the velocity of the particle at a specified instant of time.

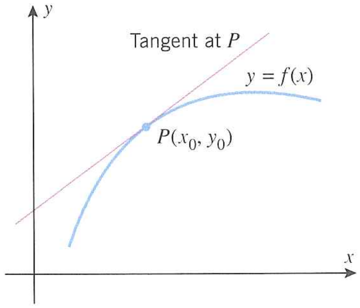


Figure 2.1.1

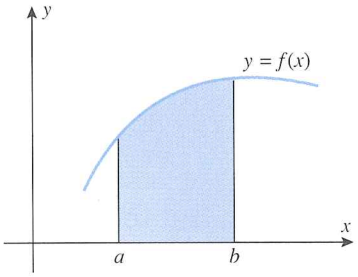


Figure 2.1.2

Traditionally, that portion of calculus arising from the tangent line problem is called ***differential calculus*** and that arising from the area problem is called ***integral calculus***. However, we will see later that the tangent line and area problems are so closely related that the distinction between differential and integral calculus is often hard to discern.

In order to solve the three problems posed above, it is necessary to have a more precise understanding of what the terms *tangent line*, *area*, and *velocity at an instant* actually mean. Let us begin with the notion of a tangent line.

TANGENT LINES AND LIMITS

In plane geometry, a line is called *tangent* to a circle if it meets the circle at precisely one point (Figure 2.1.3a). However, this definition is not appropriate for more general curves. For example, in Figure 2.1.3b, the line meets the curve exactly once but is obviously not what we would regard to be a tangent line; and in Figure 2.1.3c, the line appears to be tangent to the curve, yet it intersects the curve more than once.

To obtain a definition of a tangent line that applies to curves other than circles, we must view tangent lines another way. For this purpose, suppose that we are interested in the tangent line at a point P on a curve in the xy-plane and that Q is any point that lies on the curve and is different from P. The line through P and Q is called a ***secant line*** for the curve at P. Intuition suggests that if we move the point Q along the curve toward P, then the

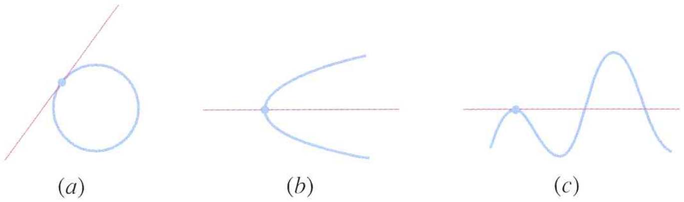


Figure 2.1.3

secant line will rotate toward a *limiting position*. The line in this limiting position is what we will consider to be the ***tangent line*** at P (Figure 2.1.4a). As suggested by Figure 2.1.4b, this new concept of a tangent line coincides with the traditional concept when applied to circles.

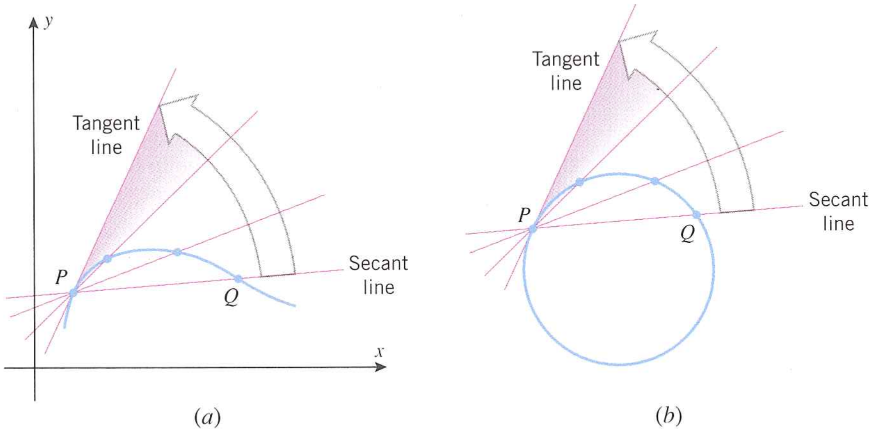


Figure 2.1.4

AREAS AND LIMITS

Just as the general notion of a tangent line leads to the concept of *limit*, so does the general notion of area. For many plane regions with straight-line boundaries, areas can be calculated by subdividing the region into rectangles or triangles and adding the areas of the constituent parts (Figure 2.1.5). However, for regions with curved boundaries, such as that in Figure 2.1.6a, a more general approach is needed. One such approach is to begin by approximating the area of the region by inscribing a number of rectangles of equal width under the curve and adding the areas of these rectangles (Figure 2.1.6b). Intuition suggests that if we repeat that approximation process using more and more rectangles, then the rectangles will tend to fill in the gaps under the curve, and the approximations will get closer and closer to the exact area under the curve (Figure 2.1.6c). This suggests that we can define the area under the curve to be the *limiting value* of these approximations.

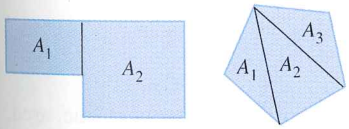


Figure 2.1.5

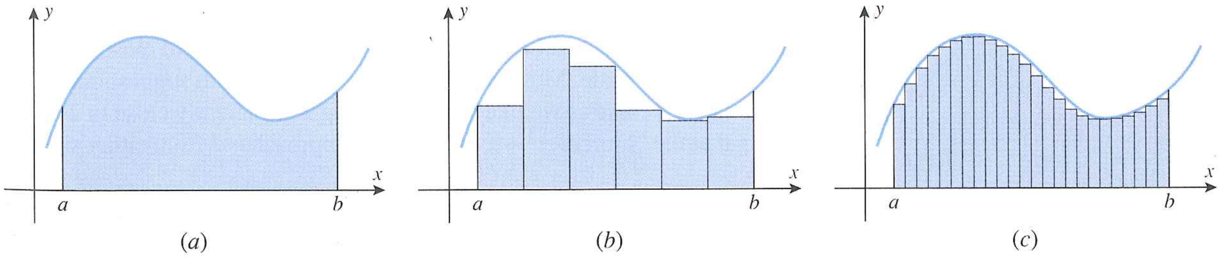


Figure 2.1.6

INSTANTANEOUS VELOCITY AND LIMITS

Recall from Formula (11) of Section 1.5 that if a particle moves along an s-axis, then its average velocity v_{ave} over the time interval from t_0 to t_1 is defined as

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{s_1 - s_0}{t_1 - t_0} \tag{1}$$

where s_0 and s_1 are the coordinates of the particle at times t_0 and t_1, respectively. Geometrically, v_{ave} is the slope of the secant line joining the points (t_0, s_0) and (t_1, s_1) on the position versus time curve for the particle (Figure 2.1.7).

Suppose, however, that we are not interested in the average velocity over a time interval, but rather the velocity v_{inst} at a specific instant of time. It is not a simple matter of applying Formula (1), since the displacement and the elapsed time in an instant are both 0. However, intuition suggests that over a sufficiently small time interval, the velocity of the particle will not vary much; thus, there should not be much difference between the instantaneous velocity at an instant of time, say $t = t_0$, and the average velocity over a time interval from $t = t_0$ to $t = t_1$, provided that the time interval is small. This suggests that we can approximate v_{inst} as

$$v_{\text{inst}} \approx v_{\text{ave}} = \frac{s_1 - s_0}{t_1 - t_0} \tag{2}$$

Moreover, the closer t_1 is to t_0, the better the approximation. However, as t_1 gets closer and closer to t_0, the slope of the secant line in Figure 2.1.8 will approach the slope of the tangent line to the curve at time $t = t_0$; and this suggests that we can *define* the instantaneous velocity of the particle at time $t = t_0$ to be the slope of the tangent line to the position versus time curve at that point. Thus, once we know how to calculate slopes of tangent lines, we will have a method for calculating instantaneous velocities.

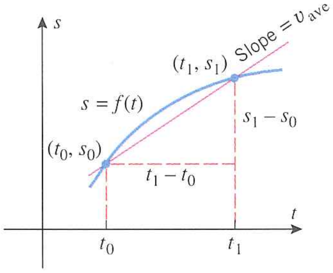


Figure 2.1.7

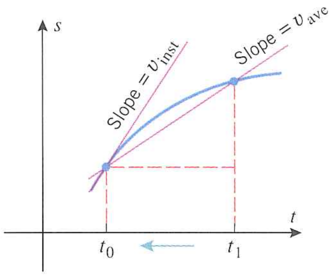


Figure 2.1.8

LIMITS

Now that we have seen how the concept of a limit enters into solving the tangent line, area, and instantaneous velocity problems, let us focus on the limit concept itself.

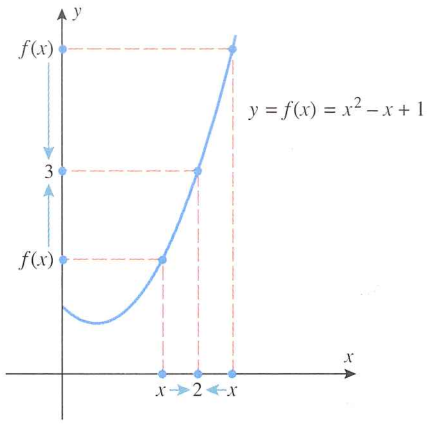
The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

as x gets closer and closer to 2. It is evident from the graph and table in Figure 2.1.9 that the values of $f(x)$ get closer and closer to 3 as x gets closer and closer to 2 from either the left side or the right side. Moreover, the graph and table both suggest that we can make the values of $f(x)$ as close as we like to 3 by making x sufficiently close to 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3 \tag{3}$$

Observe that in this limit analysis we are only concerned with the values of f *near* the point $x = 2$ and not the value of f *at* the point $x = 2$.



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

Figure 2.1.9

This leads us to the following general idea.

2.1.1 LIMITS (AN INFORMAL VIEW). If the values of $f(x)$ can be made as close as we like to L by making x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{4}$$

which is read "the limit of $f(x)$ as x approaches a is L."

Expression (4) is also commonly written as

$$f(x) \to L \quad \text{as} \quad x \to a$$

With this notation we can express (3) as

$$x^2 - x + 1 \to 3 \quad \text{as} \quad x \to 2$$

Example 1

Make a conjecture about the value of the limit

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1} \tag{5}$$

Solution. Observe that this function is undefined at $x = 0$. However, this has no bearing on the limit, since the limit is concerned with the behavior of f for x near, but not equal to, 0. Table 2.1.1 shows successions of x-values approaching 0 from the left side and the right side. In both cases the values of $f(x)$, calculated to six decimal places, appear to get closer

Table 2.1.1

x	−0.01	−0.001	−0.0001	−0.00001	0	0.00001	0.0001	0.001	0.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

Left side → ← Right side

and closer to 2, and hence we conjecture that

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1} = 2 \tag{6}$$

However, it should be kept in mind that this conjecture is based on a limited amount of numerical evidence; we are *guessing* that if we were to extend the table and continue to let x get closer and closer to 0 from either side, then the values of $f(x)$ would continue to get closer and closer to 2. Fortunately, in this example we have other ways of confirming our conjecture. One possibility is to simplify Formula (5) algebraically by rationalizing the denominator. This yields

$$f(x) = \frac{x}{\sqrt{x+1}-1} = \frac{x(\sqrt{x+1}+1)}{(x+1)-1} = \sqrt{x+1}+1 \quad (x \neq 0) \tag{7}$$

It is evident from this alternative formula for f that as x gets closer and closer to 0, the values of $f(x) = \sqrt{x+1}+1$ get closer and closer to 2, confirming (6). Yet another confirmation of (6) can be obtained from the graph of f. It follows from (7) that the graph of f is identical to the graph of $y = \sqrt{x+1}+1$, except for a hole at $x = 0$, where f is undefined (Figure 2.1.10). This figure suggests that as x moves along the x-axis toward 0 from either side, the values of $y = f(x)$ get closer and closer to 2, which again agrees with (6). ◀

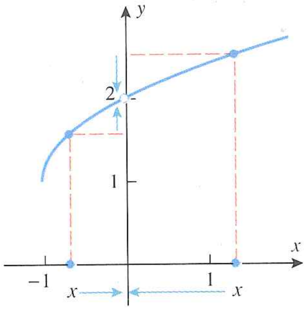


Figure 2.1.10

Example 2

Make a conjecture about the value of the limit

$$\lim_{x\to 0} \frac{\sin x}{x}$$

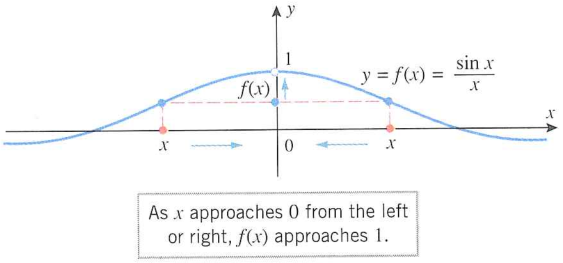
Solution. The function $f(x) = (\sin x)/x$ is undefined at $x = 0$, but, as discussed previously, this has no bearing on the limit. With the help of a calculating utility set to radian measure, we obtain Table 2.1.2, which suggests that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{8}$$

This result is consistent with the graph of $f(x) = (\sin x)/x$ shown in Figure 2.1.11; but unlike the preceding example, where we were able to confirm the limit algebraically by simplifying the formula for the function, that is not possible here. However, later in this chapter we will give a geometric argument to prove that our conjecture is correct. ◀

Table 2.1.2

x (RADIANS)	$y = \frac{\sin x}{x}$
±1.0	0.84147
±0.9	0.87036
±0.8	0.89670
±0.7	0.92031
±0.6	0.94107
±0.5	0.95885
±0.4	0.97355
±0.3	0.98507
±0.2	0.99335
±0.1	0.99833
±0.01	0.99998



As x approaches 0 from the left or right, $f(x)$ approaches 1.

Figure 2.1.11

FOR THE READER. Use a calculating utility to confirm limit (8). Does the limit change if x is in degrees?

NUMERICAL PITFALLS

Although numerical evidence is helpful for guessing at limits, it can lead to incorrect conclusions. For example, Table 2.1.3 shows values of $f(x) = \sin(\pi/x)$ at selected values of x on both sides of 0. The numerical data in that table suggest that

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$$

However, this conclusion is incorrect, as evidenced by the graph of f shown in Figure 2.1.12. This graph shows that as $x \to 0$, the values of f oscillate between -1 and 1 with increasing rapidity, and hence do not approach a limit. The numerical data in Table 2.1.3 deceived us into believing the limit to be zero because we happened to choose values of x that were all x-intercepts.

Table 2.1.3

x (RADIANS)	$\frac{\pi}{x}$	$f(x) = \sin\left(\frac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
$\vdots$	$\vdots$	$\vdots$

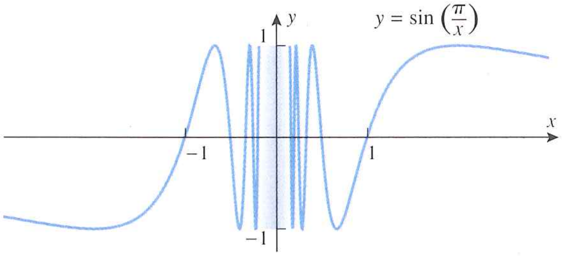


Figure 2.1.12

Numerical evidence can also lead to incorrect conclusions about limits because of round-off error or because the table of values used to find the limit is not extensive enough to reveal the behavior of the function completely. Thus, when a limit is conjectured from numerical data it is important to look for corroborating graphical or algebraic evidence to support the conjecture.

ONE-SIDED LIMITS

The limit in (4) is commonly called a ***two-sided limit*** because it requires the values of $f(x)$ to get closer and closer to L as x approaches a from *either* side. However, some functions exhibit different behaviors on the two sides of a point a, in which case it is necessary to distinguish whether x is near a on the left side or the right side for purposes of investigating the limiting behavior. For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

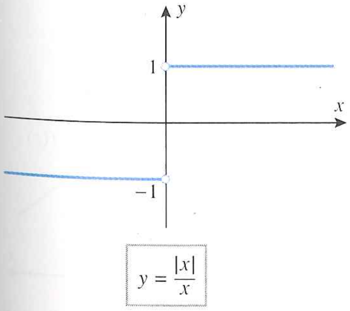


Figure 2.1.13

(Figure 2.1.13). As x approaches 0 from the right side, the values of $f(x)$ approach 1 (in fact, they are exactly 1 for all such x), and as x approaches 0 from the left side, the values of $f(x)$ approach -1. We describe these two statements by saying that "the limit of $f(x) = |x|/x$ is 1 as x approaches 0 from the right" and that "the limit of $f(x) = |x|/x$ is -1 as x approaches 0 from the left"; we denote these limits by writing

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{|x|}{x} = -1 \tag{9–10}$$

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

This leads us to the following general idea:

2.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW). If the values of $f(x)$ can be made as close as we like to L by making x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \tag{11}$$

which is read "the limit of $f(x)$ as x approaches a from the right is L." Similarly, if the values of $f(x)$ can be made as close as we like to L by making x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \tag{12}$$

Expressions (11) and (12), which are called ***one-sided limits***, are also commonly written a

$$f(x) \to L \text{ as } x \to a^+ \quad \text{and} \quad f(x) \to L \text{ as } x \to a^-$$

respectively. With this notation (9) and (10) can be expressed as

$$\frac{|x|}{x} \to 1 \text{ as } x \to 0^+ \quad \text{and} \quad \frac{|x|}{x} \to -1 \text{ as } x \to 0^-$$

THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS

In general, there is no guarantee that a function will have a limit at a specified point, and ther is some terminology to describe such situations. If the values of $f(x)$ do not get closer an closer to some *single* number L as $x \to a$, then we say that the limit of $f(x)$ as x approache a ***does not exist*** (and similarly for one-sided limits). For example, the two-sided limit o $f(x) = |x|/x$ does not exist as $x \to 0$ because the values of $f(x)$ do not approach a singl number—the values approach -1 from the left and 1 from the right.

In general, the following condition must be satisfied for the two-sided limit of a functio to exist.

2.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS. The two-sided limit of a function f exists at a point a if and only if the one-sided limits exist at that point and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

REMARK. Sometimes, one or both of the one-sided limits may fail to exist (which, i turn, implies that the two-sided limit does not exist). For example, we saw earlier that th one-sided limits of $f(x) = \sin(\pi/x)$ do not exist as x approaches 0 because the functio keeps oscillating between -1 and 1, failing to settle in on a single value; and this implie that the two-sided limit does not exist as x approaches 0.

Example 3

For the functions in Figure 2.1.14, find the one-sided and two-sided limits at $x = a$ if the exist.

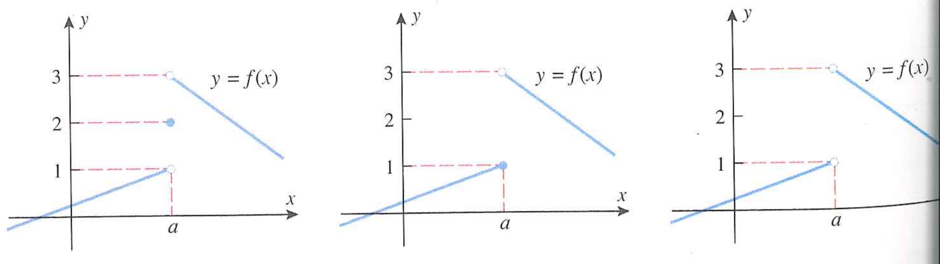


Figure 2.1.14

Solution. The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are

$$\lim_{x\to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x\to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as $x \to a$ because the one-sided limits are not equal. ◀

Example 4

For the functions in Figure 2.1.15, find the one-sided and two-sided limits at $x = a$ if they exist.

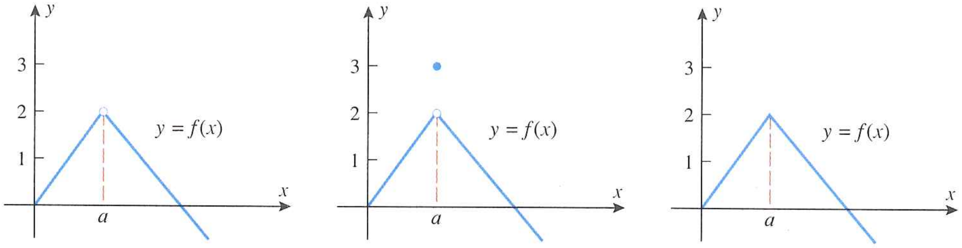


Figure 2.1.15

Solution. As in the preceding example, the value of f at $x = a$ has no bearing on the limits as $x \to a$, so that in all three cases we have

$$\lim_{x\to a^+} f(x) = 2 \quad \text{and} \quad \lim_{x\to a^-} f(x) = 2$$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x\to a} f(x) = 2$$

◀

A FIRST LOOK AT CONTINUITY

Plane curves can be divided into two categories—those that have breaks or holes and those that do not. Breaks or holes in a curve are called ***discontinuities***; a curve with no discontinuities is called ***continuous*** (Figure 2.1.16).

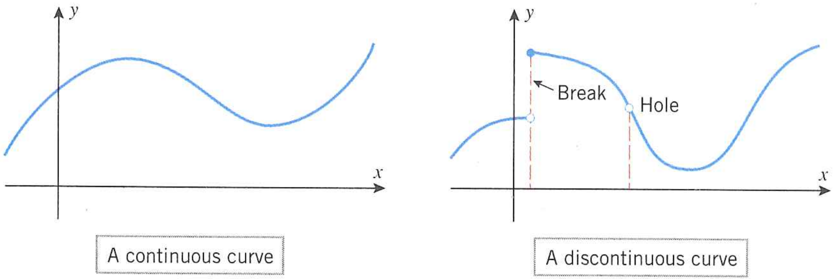


Figure 2.1.16

Examples 3 and 4 provide some useful insight into what it means for the graph of a function to be continuous. Of the six functions in those examples, only the last one does not have a break or hole in its graph at $x = a$. For the functions in Example 3, the break in the graph at $x = a$ results from the fact that the one-sided limits at that point have different values. A break of this type is called a ***jump discontinuity*** in the graph. For the first two functions in Example 4, the hole in the graph is caused by a mismatch between the value of the function at $x = a$ and the two-sided limit as x approaches a. In the first graph, the function is simply undefined at $x = a$, leaving a hole; and in the second graph, $f(a)$ is defined, but its value is different from the limit, resulting in a point that is displaced from the main part of the graph. A break due to a hole or a displaced point is called a ***removable discontinuity*** in the graph. The third graph is continuous at $x = a$, since the value of f at $x = a$ is the same as the two-sided limit of f as x approaches a, thereby ensuring that there is no jump or hole.

All of this suggests that three conditions must be satisfied to ensure that the graph of a function does not have a discontinuity at a given point:

- The function must be defined at the point.
- The two-sided limit must exist at the point.
- The value of the function and the value of the two-sided limit must be the same.

There will be more on this later.

INFINITE LIMITS AND VERTICAL ASYMPTOTES

Sometimes one-sided or two-sided limits will fail to exist because the values of the function increase or decrease indefinitely. For example, consider the behavior of the function $f(x) = 1/x$ as x gets closer and closer to 0. It is evident from the table and graph in Figure 2.1.17 that as x gets closer and closer to 0 from the right, the values of $f(x) = 1/x$ are positive and increase indefinitely; and as x gets closer and closer to 0 from the left, the values of $f(x)$ are negative and decrease indefinitely. We denote these limiting behaviors by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

More generally:

2.1.4 INFINITE LIMITS (AN INFORMAL VIEW). If the values of $f(x)$ increase indefinitely as x approaches a from the right or left, then we write

$$\lim_{x \to a^+} f(x) = +\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = +\infty$$

as appropriate, and we say that $f(x)$ ***increases without bound*** as $x \to a^+$ or $x \to a^-$. Similarly, if the values of $f(x)$ decrease indefinitely as x approaches a from the right or left, then we write

$$\lim_{x \to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = -\infty$$

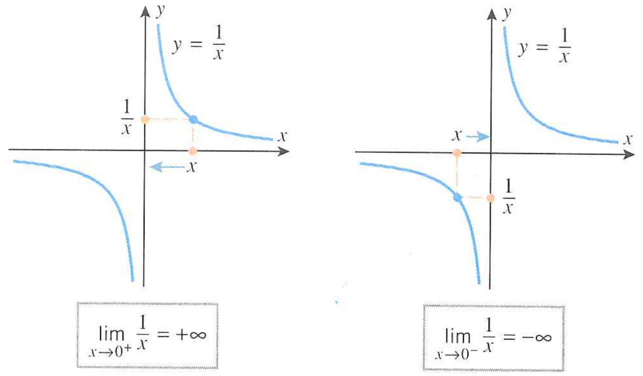
as appropriate, and say that $f(x)$ ***decreases without bound*** as $x \to a^+$ or $x \to a^-$. Moreover, if both one-sided limits are $+\infty$, then we write

$$\lim_{x \to a} f(x) = +\infty$$

and if both one-sided limits are $-\infty$, then we write

$$\lim_{x \to a} f(x) = -\infty$$

REMARK. It should be emphasized that the symbols $+\infty$ and $-\infty$, as used here, describe the particular way in which the limits fail to exist; they are not numerical limits and consequently cannot be manipulated using rules of algebra. For example, it is *not* correct to write $(+\infty) - (+\infty) = 0$.



x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

Left side → ← Right side

Figure 2.1.17

Example 5

For the functions in Figure 2.1.18, describe the limits at $x = a$ in appropriate limit notation.

Solution (a). In Figure 2.1.18a, the function increases indefinitely as x approaches a from the right and decreases indefinitely as x approaches a from the left. Thus,

$$\lim_{x\to a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x\to a^-} \frac{1}{x-a} = -\infty$$

Solution (b). In Figure 2.1.18b, the function increases indefinitely as x approaches a from both the left and right. Thus,

$$\lim_{x\to a} \frac{1}{(x-a)^2} = \lim_{x\to a^+} \frac{1}{(x-a)^2} = \lim_{x\to a^-} \frac{1}{(x-a)^2} = +\infty$$

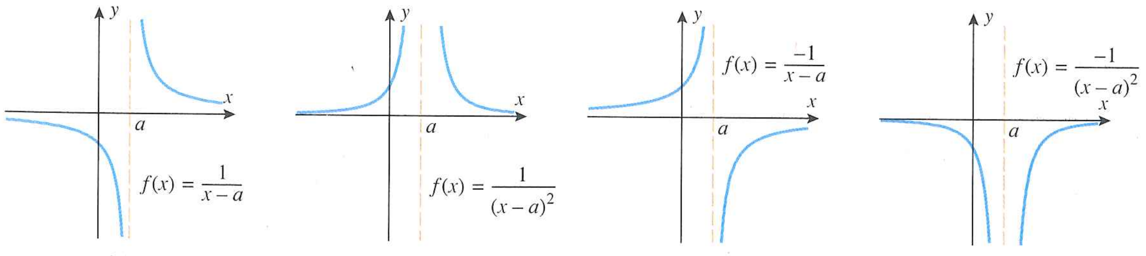
Solution (c). In Figure 2.1.18c, the function decreases indefinitely as x approaches a from the right and increases indefinitely as x approaches a from the left. Thus,

$$\lim_{x\to a^+} \frac{-1}{x-a} = -\infty \quad \text{and} \quad \lim_{x\to a^-} \frac{-1}{x-a} = +\infty$$

Solution (d). In Figure 2.1.18d, the function decreases indefinitely as x approaches a from both the left and right. Thus,

$$\lim_{x\to a} \frac{-1}{(x-a)^2} = \lim_{x\to a^+} \frac{-1}{(x-a)^2} = \lim_{x\to a^-} \frac{-1}{(x-a)^2} = -\infty$$

◀



(*a*) (*b*) (*c*) (*d*)

Figure 2.1.18

Geometrically, if $f(x) \to +\infty$ as x approaches a from the left or right, then the graph of $y = f(x)$ eventually gets closer and closer to the line $x = a$ as the graph is traversed in the positive y-direction; and if $f(x) \to -\infty$ as x approaches a from the left or right, then the graph of $y = f(x)$ eventually gets closer and closer to the line $x = a$ as the graph is traversed in the negative y-direction. We call this line a *vertical asymptote* (from the Greek *asymptotos*, meaning "nonintersecting").

2.1.5 DEFINITION. A line $x = a$ is called a ***vertical asymptote*** of the graph of a function f if $f(x)$ approaches $+\infty$ or $-\infty$ as x approaches a from the left or right.

Example 6

The four functions graphed in Figure 2.1.18 all have a vertical asymptote at $x = a$, which is indicated by the dashed vertical lines in the figure. ◀

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

Thus far, we have used limits to describe the behavior of $f(x)$ as x approaches a point $x = a$. However, sometimes we will not be concerned with the behavior of $f(x)$ near a specific point, but rather with how the values of $f(x)$ behave as x increases without bound or decreases without bound. This is sometimes called the ***end behavior*** of the function because it describes how the function behaves for values of x that are far from the origin. For example, it is evident from the table and graph in Figure 2.1.19 that as x increases without bound, the values of $f(x) = 1/x$ are positive, but get closer and closer to 0; and similarly, as x decreases without bound, the values of $f(x) = 1/x$ are negative, but also get closer and closer to 0. We denote these limiting behaviors by writing

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

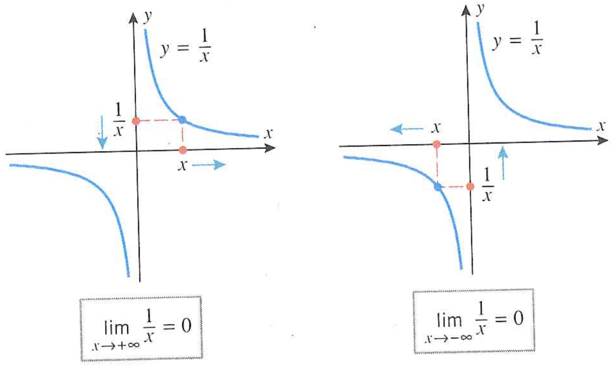
More generally:

2.1.6 LIMITS AT INFINITY (AN INFORMAL VIEW). If the values of $f(x)$ eventually get closer and closer to a number L as x increases without bound, then we write

$$\lim_{x \to +\infty} f(x) = L \tag{13}$$

Similarly, if the values of $f(x)$ eventually get closer and closer to a number L as x decreases without bound, then we write

$$\lim_{x \to -\infty} f(x) = L \tag{14}$$



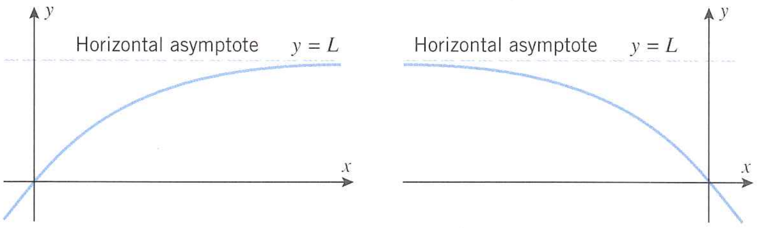
x	−10,000	−1000	−100	−10	−1		1	10	100	1000	10,000
$f(x)$	−0.0001	−0.001	−0.01	−0.1	−1		1	0.1	0.01	0.001	0.0001

← x decreasing without bound

x increasing without bound →

Figure 2.1.19

Geometrically, if $f(x) \to L$ as $x \to +\infty$, then the graph of $y = f(x)$ eventually gets closer and closer to the line $y = L$ as the graph is traversed in the positive direction (Figure 2.1.20*a*); and if $f(x) \to L$ as $x \to -\infty$, then the graph of $y = f(x)$ eventually gets closer and closer to the line $y = L$ as the graph is traversed in the negative x-direction (Figure 2.1.20*b*). In either case we call the line $y = L$ a *horizontal asymptote* of the graph of f. For example, the four functions in Figure 2.1.18 all have $y = 0$ as a horizontal asymptote.



(*a*) (*b*)

Figure 2.1.20

2.1.7 DEFINITION. A line $y = L$ is called a ***horizontal asymptote*** of the graph of a function f if $f(x) \to L$ as $x \to +\infty$ or as $x \to -\infty$.

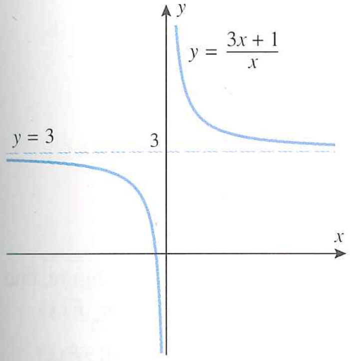


Figure 2.1.21

Sometimes the existence of a horizontal asymptote of a function f will be readily apparent from the formula for f. For example, it is evident that the function

$$f(x) = \frac{3x+1}{x} = 3 + \frac{1}{x}$$

has a horizontal asymptote at $y = 3$ (Figure 2.1.21), since the value of $1/x$ approaches 0 as $x \to +\infty$ or $x \to -\infty$. For more complicated functions, algebraic manipulations or special techniques that we will study in the next section may have to be applied to confirm the existence of horizontal asymptotes.

HOW LIMITS AT INFINITY CAN FAIL TO EXIST

Limits at infinity can fail to exist for various reasons. One possibility is that the values of $f(x)$ may increase or decrease without bound as $x \to +\infty$ or as $x \to -\infty$. For example, the values of $f(x) = x^3$ increase without bound as $x \to +\infty$ and decrease without bound as $x \to -\infty$; and for $f(x) = -x^3$ the values decrease without bound as $x \to +\infty$ and increase without bound as $x \to -\infty$ (Figure 2.1.22). We denote this by writing

$$\lim_{x \to +\infty} x^3 = +\infty, \qquad \lim_{x \to -\infty} x^3 = -\infty, \qquad \lim_{x \to +\infty} (-x^3) = -\infty, \qquad \lim_{x \to -\infty} (-x^3) = +\infty$$

More generally:

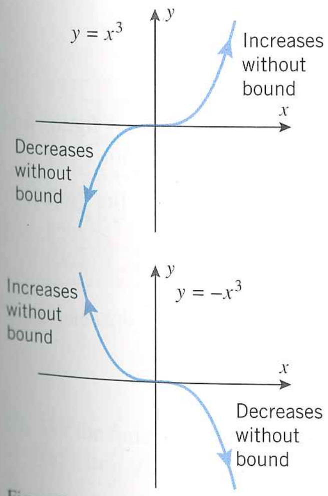


Figure 2.1.22

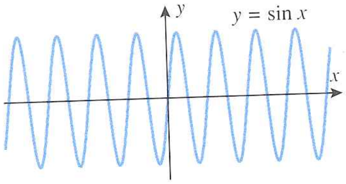
2.1.8 INFINITE LIMITS AT INFINITY (AN INFORMAL VIEW). If the values of $f(x)$ increase without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

$$\lim_{x \to +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = +\infty$$

as appropriate; and if the values of $f(x)$ decrease without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

$$\lim_{x \to +\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = -\infty$$

as appropriate.



There is no limit as $x \to +\infty$ or $x \to -\infty$.

Figure 2.1.23

Limits at infinity can also fail to exist because the graph of the function oscillates indefinitely in such a way that the values of the function do not approach a fixed number and do not increase or decrease without bound; the trigonometric functions $\sin x$ and $\cos x$ have this property, for example (Figure 2.1.23). In such cases we say that the limit ***fails to exist because of oscillation***.

EXERCISE SET 2.1 Graphing Calculator C CAS

1. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$ (c) $\lim_{x\to 3} f(x)$
(d) $f(3)$ (e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

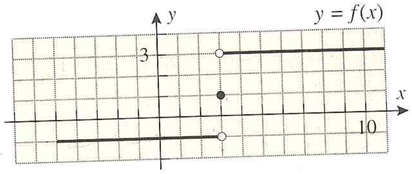


Figure Ex-1

2. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$
(d) $f(2)$ (e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

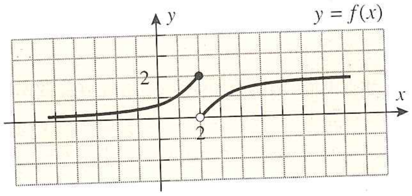


Figure Ex-2

3. For the function g graphed in the accompanying figure, find
(a) $\lim_{x\to 4^-} g(x)$ (b) $\lim_{x\to 4^+} g(x)$ (c) $\lim_{x\to 4} g(x)$
(d) $g(4)$ (e) $\lim_{x\to -\infty} g(x)$ (f) $\lim_{x\to +\infty} g(x)$.

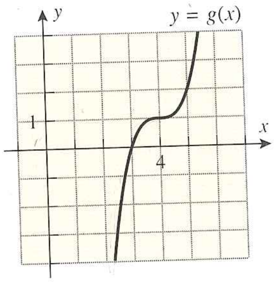


Figure Ex-3

4. For the function g graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} g(x)$ (b) $\lim_{x\to 0^+} g(x)$ (c) $\lim_{x\to 0} g(x)$
(d) $g(0)$ (e) $\lim_{x\to -\infty} g(x)$ (f) $\lim_{x\to +\infty} g(x)$.

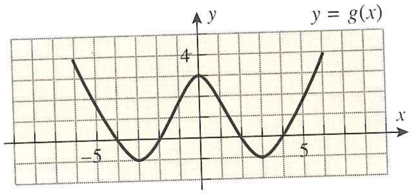


Figure Ex-4

5. For the function F graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} F(x)$ (b) $\lim_{x\to -2^+} F(x)$ (c) $\lim_{x\to -2} F(x)$
(d) $F(-2)$ (e) $\lim_{x\to -\infty} F(x)$ (f) $\lim_{x\to +\infty} F(x)$.

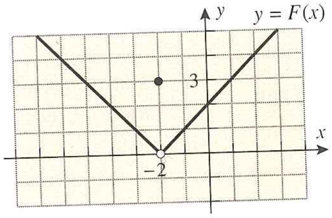


Figure Ex-5

6. For the function F graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} F(x)$ (b) $\lim_{x\to 3^+} F(x)$ (c) $\lim_{x\to 3} F(x)$
(d) $F(3)$ (e) $\lim_{x\to -\infty} F(x)$ (f) $\lim_{x\to +\infty} F(x)$.

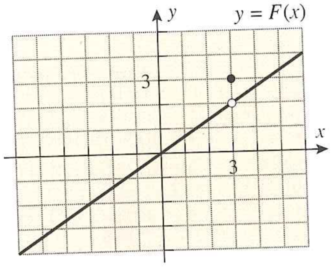


Figure Ex-6

7. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} \phi(x)$ (b) $\lim_{x\to -2^+} \phi(x)$ (c) $\lim_{x\to -2} \phi(x)$
(d) $\phi(-2)$ (e) $\lim_{x\to -\infty} \phi(x)$ (f) $\lim_{x\to +\infty} \phi(x)$.

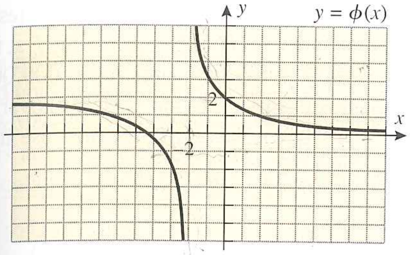


Figure Ex-7

8. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to 4^-} \phi(x)$ (b) $\lim_{x\to 4^+} \phi(x)$ (c) $\lim_{x\to 4} \phi(x)$
(d) $\phi(4)$ (e) $\lim_{x\to -\infty} \phi(x)$ (f) $\lim_{x\to +\infty} \phi(x)$.

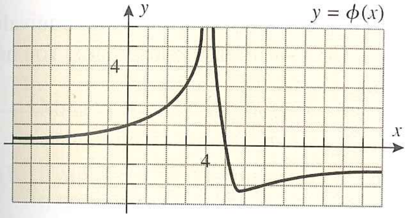


Figure Ex-8

9. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$ (c) $\lim_{x\to 3} f(x)$
(d) $f(3)$ (e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

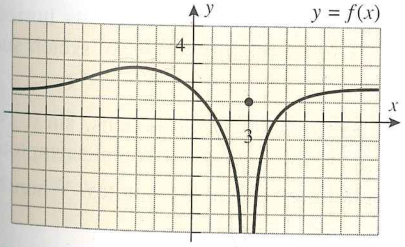


Figure Ex-9

10. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$
(d) $f(0)$ (e) $\lim_{x\to -\infty} f(x)$ (f) $\lim_{x\to +\infty} f(x)$.

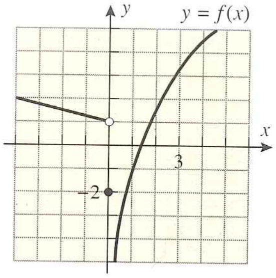


Figure Ex-10

11. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$ (c) $\lim_{x\to 0} G(x)$
(d) $G(0)$ (e) $\lim_{x\to -\infty} G(x)$ (f) $\lim_{x\to +\infty} G(x)$.

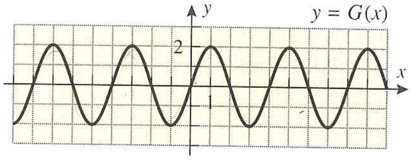


Figure Ex-11

12. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$ (c) $\lim_{x\to 0} G(x)$
(d) $G(0)$ (e) $\lim_{x\to -\infty} G(x)$ (f) $\lim_{x\to +\infty} G(x)$.

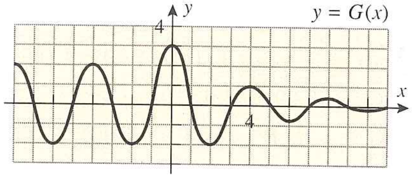


Figure Ex-12

13. Consider the function g graphed in the accompanying figure. For what values of x_0 does $\lim_{x\to x_0} g(x)$ exist?

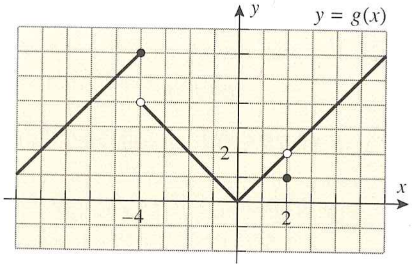


Figure Ex-13

14. Consider the function f graphed in the accompanying figure. For what values of x_0 does $\lim_{x \to x_0} f(x)$ exist?

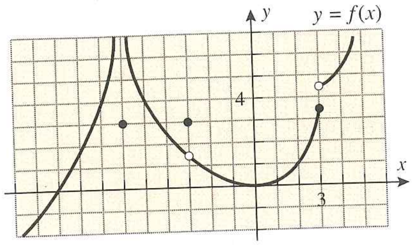


Figure Ex-14

> In Exercises 15 and 16, find all points of discontinuity for the function, and for each such point state which of the three condition(s) for continuity fail to hold.

15. (a) The function f in Exercise 1
(b) The function F in Exercise 5
(c) The function f in Exercise 9

16. (a) The function f in Exercise 2
(b) The function F in Exercise 6
(c) The function f in Exercise 10

> In Exercises 17–20: (i) Make a guess at the limit (if it exists) by evaluating the function at the specified points. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to set your calculating and graphing utilities to the radian mode.]

$\boxed{\text{C}}$ **17.** (a) $\displaystyle\lim_{x \to 1} \frac{x-1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 0, 0.5, 0.9, 0.99, 0.999$

(b) $\displaystyle\lim_{x \to 1^+} \frac{x+1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 1.0001$

(c) $\displaystyle\lim_{x \to 1^-} \frac{x+1}{x^3-1}$; $x = 0, 0.5, 0.9, 0.99, 0.999, 0.9999$

$\boxed{\text{C}}$ **18.** (a) $\displaystyle\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\displaystyle\lim_{x \to 0^+} \frac{\sqrt{x+1}+1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$

(c) $\displaystyle\lim_{x \to 0^-} \frac{\sqrt{x+1}+1}{x}$; $x = -0.25, -0.1, -0.001, -0.0001$

$\boxed{\text{C}}$ **19.** (a) $\displaystyle\lim_{x \to 0} \frac{\sin 3x}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\displaystyle\lim_{x \to -1} \frac{\cos x}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

$\boxed{\text{C}}$ **20.** (a) $\displaystyle\lim_{x \to -1} \frac{\tan(x+1)}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

(b) $\displaystyle\lim_{x \to 0} \frac{\sin(5x)}{\sin(2x)}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

> In Exercises 21 and 22: (i) Approximate the y-coordinates of all horizontal asymptotes of $y = f(x)$ by evaluating f at the points $\pm 10, \pm 100, \pm 1000, \pm 100{,}000$, and $\pm 100{,}000{,}000$. (ii) Confirm your conclusions by graphing $y = f(x)$ over an appropriate interval. (iii) If you have a CAS, then use it to find the horizontal asymptotes.

$\boxed{\text{C}}$ **21.** (a) $f(x) = \dfrac{2x+3}{x+4}$ (b) $f(x) = \left(1 + \dfrac{3}{x}\right)^x$

(c) $f(x) = \dfrac{x^2+1}{x+1}$

$\boxed{\text{C}}$ **22.** (a) $f(x) = \dfrac{x^2-1}{5x^2+1}$ (b) $f(x) = \left(2 + \dfrac{1}{x}\right)^x$

(c) $f(x) = \dfrac{\sin x}{x}$

> In Exercises 23 and 24, express the limit as an equivalent limit in which $x \to 0^+$ or $x \to 0^-$, as appropriate. [You need not evaluate the limit.]

23. (a) $\displaystyle\lim_{x \to +\infty} x \sin\left(\frac{1}{x}\right)$ (b) $\displaystyle\lim_{x \to +\infty} \frac{1-x}{1+x}$

(c) $\displaystyle\lim_{x \to -\infty} \left(1 + \frac{2}{x}\right)^x$

24. (a) $\displaystyle\lim_{x \to +\infty} \frac{\cos(\pi/x)}{\pi/x}$ (b) $\displaystyle\lim_{x \to +\infty} \frac{x}{1+x}$

(c) $\displaystyle\lim_{x \to -\infty} (1+2x)^{1/x}$

25. (a) Sketch the graph of a function that has two horizontal asymptotes.

(b) Can the graph of a function intersect its horizontal asymptotes? If not, explain why. If so, sketch such a graph.

26. (a) Do any of the trigonometric functions, $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, have horizontal asymptotes?

(b) Do any of them have vertical asymptotes? Where?

$\boxed{\text{C}}$ **27.** (a) Let

$$f(x) = x^3 - \frac{3^x}{2000}$$

Make a conjecture about the limit of f as $x \to 0^+$ by evaluating f at the points $x = 1, 0.75, 0.5, 0.25, 0.1, 0.05$.

(b) Evaluate f at the points $x = 0.01, 0.001, 0.0001, 0.00001, 0.000001$, and make another conjecture.

(c) What flaw does this reveal about using numerical evidence to make conjectures about limits?

(d) If you have a CAS, use it to show that the exact value of the limit is $-1/2000$.

> Roundoff error is one source of inaccuracy in calculator and computer computations. Another source of error, called ***catastrophic subtraction***, occurs when two nearly equal numbers are subtracted, and the result is used as part of another calculation. For example, by hand calculation we have
>
> $$(0.123456789012345 - 0.123456789012344) \times 10^{15} = 1$$
>
> However, the author's calculator produces a value of 0 for this computation because it can only store 14 decimal digits, and the numbers being subtracted are identical in the first 14 decimal digits. Catastrophic subtraction can sometimes be avoided by rearranging formulas algebraically, but your best defense is to be aware that it can occur. Watch out for it in the next exercise.

C **28.** (a) Let

$$f(x) = \frac{x - \sin x}{x^3}$$

Make a conjecture about the limit of f as $x \to 0^+$ by evaluating f at the points $x = 0.1, 0.01, 0.001, 0.0001$.

(b) Evaluate f at the points $x = 0.00001, 0.0000001, 0.00000001, 0.000000001, 0.0000000001$, and make another conjecture.

(c) What flaw does this reveal about using numerical evidence to make conjectures about limits?

(d) If you have a CAS, use it to show that the exact value of the limit is $\frac{1}{6}$.

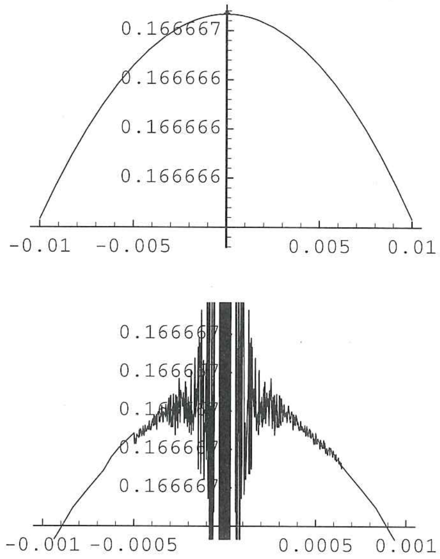
29. (a) The accompanying figure shows graphs of the function from Exercise 28 over two different intervals. What is happening?

(b) Use your graphing utility to generate the graphs, and see whether the same problem occurs.

(c) Would you expect a similar problem to occur in the vicinity of $x = 0$ for the function

$$f(x) = \frac{1 - \cos x}{x}?$$

See if it does.



Erratic graph generated by Mathematica

Figure Ex-29

2.2 LIMITS (COMPUTATIONAL TECHNIQUES)

In the last section we discussed limits informally, focusing on the basic ideas. In this section, we will discuss algebraic methods for finding limits, reserving the discussion of the underlying theory behind these methods for the next section.

SOME BASIC LIMITS

Our strategy for finding limits algebraically has two parts:

- First we will establish the limits of some simple functions.
- Then we will develop a repertoire of theorems that will enable us to use the limits of those simple functions as building blocks for finding limits of more complicated functions.

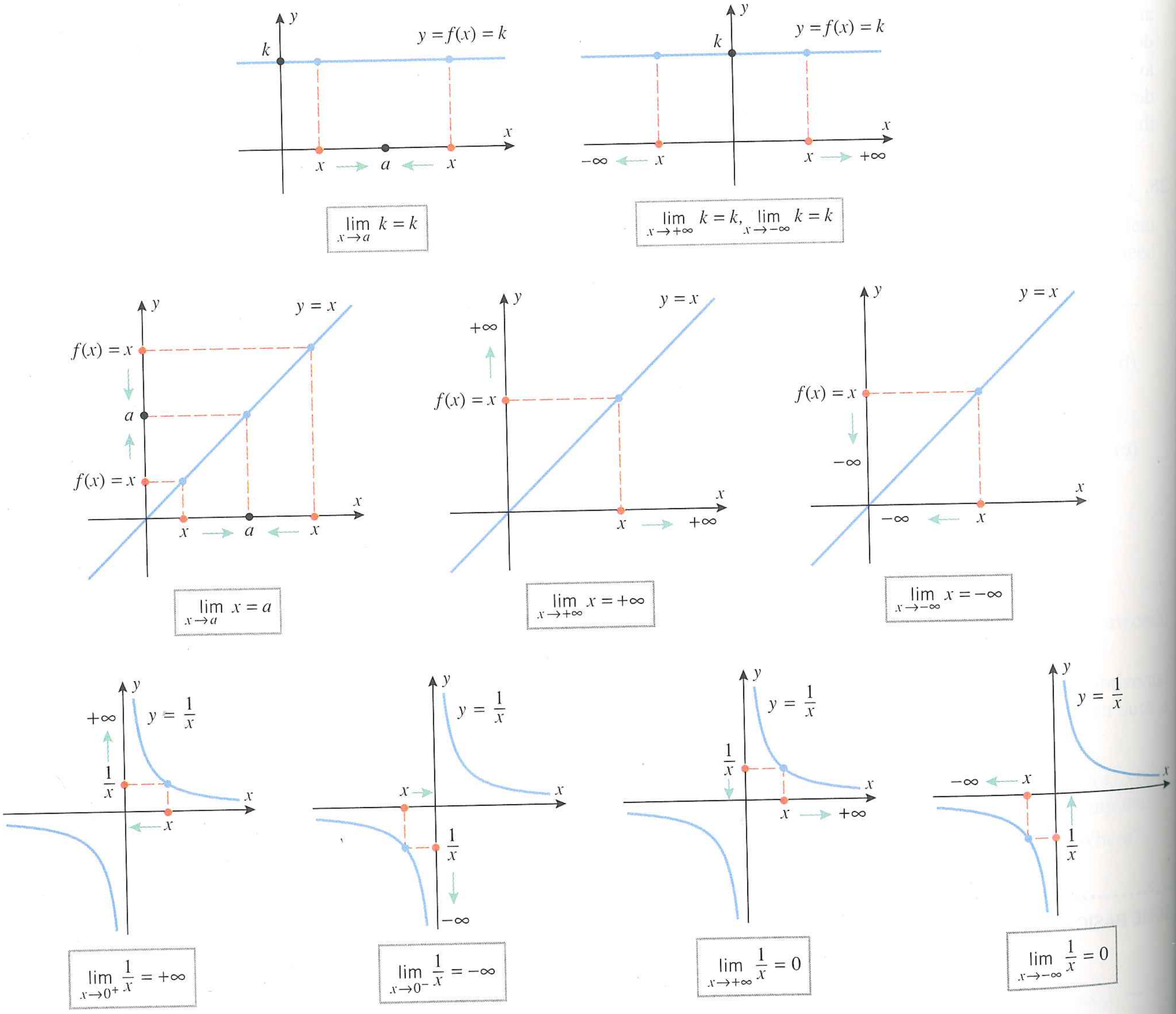
The ten limits in the following theorem, all of which should be evident from Figure 2.2.1, will form our building blocks—three involve the constant function $f(x) = k$, three involve the linear function $f(x) = x$, and four involve the rational function $f(x) = 1/x$.

2.2.1 THEOREM.

$$\lim_{x\to a} k = k \qquad \lim_{x\to +\infty} k = k \qquad \lim_{x\to -\infty} k = k$$

$$\lim_{x\to a} x = a \qquad \lim_{x\to +\infty} x = +\infty \qquad \lim_{x\to -\infty} x = -\infty$$

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to +\infty} \frac{1}{x} = 0 \qquad \lim_{x\to -\infty} \frac{1}{x} = 0$$



$\lim_{x\to a} k = k$

$\lim_{x\to +\infty} k = k, \lim_{x\to -\infty} k = k$

$\lim_{x\to a} x = a$

$\lim_{x\to +\infty} x = +\infty$

$\lim_{x\to -\infty} x = -\infty$

$\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\lim_{x\to +\infty} \frac{1}{x} = 0$

$\lim_{x\to -\infty} \frac{1}{x} = 0$

Figure 2.2.1

In the case of the constant function $f(x) = k$, the values of $f(x)$ do not change as x varies, which explains why the limit of $f(x)$ is k, regardless of whether the limit is computed at a point a or as x approaches $+\infty$ or $-\infty$. For example,

$$\lim_{x\to 2} 3 = 3, \quad \lim_{x\to -2} 3 = 3, \quad \lim_{x\to +\infty} 3 = 3, \quad \lim_{x\to +\infty} 0 = 0, \quad \lim_{x\to -\infty} 3 = 3, \quad \lim_{x\to -\infty} 0 = 0$$

The limits of the function $f(x) = 1/x$ should make sense to you intuitively, based on your experience with fractions: making the denominator closer to zero increases the numerical size of the fraction (i.e., increases its absolute value), and increasing the numerical size of the denominator makes the numerical size of the fraction closer to zero. This is illustrated in Table 2.2.1.

Table 2.2.1

	VALUES						CONCLUSION
x	1	.1	.01	.001	.0001	$\cdots$	As $x \to 0^+$ the value of $1/x$ increases without bound.
$1/x$	1	10	100	1000	10,000	$\cdots$	
x	−1	−.1	−.01	−.001	−.0001	$\cdots$	As $x \to 0^-$ the value of $1/x$ decreases without bound.
$1/x$	−1	−10	−100	−1000	−10,000	$\cdots$	
x	1	10	100	1000	10,000	$\cdots$	As $x \to +\infty$ the value of $1/x$ decreases toward zero.
$1/x$	1	.1	.01	.001	.0001	$\cdots$	
x	−1	−10	−100	−1000	−10,000	$\cdots$	As $x \to -\infty$ the value of $1/x$ increases toward zero.
$1/x$	−1	−.1	−.01	−.001	−.0001	$\cdots$	

The following theorem, parts of which are proved in Appendix G, will be our basic tool for finding limits algebraically.

2.2.2 THEOREM. *Let* lim *stand for one of the limits* $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^-}$, $\lim\limits_{x\to a^+}$, $\lim\limits_{x\to +\infty}$, *or* $\lim\limits_{x\to -\infty}$. *If* $L_1 = \lim f(x)$ *and* $L_2 = \lim g(x)$ *both exist, then*

(*a*) $\lim\,[f(x) + g(x)] = \lim f(x) + \lim g(x) = L_1 + L_2$

(*b*) $\lim\,[f(x) - g(x)] = \lim f(x) - \lim g(x) = L_1 - L_2$

(*c*) $\lim\,[f(x)g(x)] = \lim f(x) \lim g(x) = L_1 L_2$

(*d*) $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{L_1}{L_2}$ *if* $L_2 \neq 0$

(*e*) $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)} = \sqrt[n]{L_1}$ *provided* $L_1 \geq 0$ *if* n *is even.*

In words, this theorem states:

(*a*) *The limit of a sum is the sum of the limits.*

(*b*) *The limit of a difference is the difference of the limits.*

(*c*) *The limit of a product is the product of the limits.*

(*d*) *The limit of a quotient is the quotient of the limits provided the limit of the denominator is not zero.*

(*e*) *The limit of an nth root is the nth root of the limits.*

REMARK. Although results (*a*) and (*c*) are stated for two functions f and g, these results hold as well for any finite number of functions; that is, if the limits $\lim f_1(x)$,

$\lim f_2(x), \ldots, \lim f_n(x)$ all exist, then

$$\lim [f_1(x) + f_2(x) + \cdots + f_n(x)] = \lim f_1(x) + \lim f_2(x) + \cdots + \lim f_n(x) \tag{1}$$

$$\lim [f_1(x) f_2(x) \cdots f_n(x)] = \lim f_1(x) \lim f_2(x) \cdots \lim f_n(x) \tag{2}$$

In particular, if $f_1, f_2, \ldots, f_n$ are all the same function f, then (2) reduces to

$$\lim [f(x)]^n = [\lim f(x)]^n \tag{3}$$

It follows from this result that

$$\lim_{x \to a} x^n = [\lim_{x \to a} x]^n = a^n \tag{4}$$

and

$$\lim_{x \to +\infty} \frac{1}{x^n} = \left(\lim_{x \to +\infty} \frac{1}{x}\right)^n = 0 \qquad \lim_{x \to -\infty} \frac{1}{x^n} = \left(\lim_{x \to -\infty} \frac{1}{x}\right)^n = 0 \tag{5}$$

For example,

$$\lim_{x \to 3} x^4 = 3^4 = 81, \qquad \lim_{x \to +\infty} \frac{1}{x^4} = 0, \qquad \lim_{x \to -\infty} \frac{1}{x^4} = 0$$

Another useful result follows from part (c) of Theorem 2.2.2 in the special case where one of the factors is a constant k:

$$\lim kf(x) = \lim k \lim f(x) = k \lim f(x) \tag{6}$$

In words, the first and last expressions in (6) state:

A constant factor can be moved through a limit sign.

LIMITS OF POLYNOMIALS AS x → a

Example 1

Find $\lim_{x \to 5} (x^2 - 4x + 3)$ and justify each step.

Solution.

$$\begin{aligned} \lim_{x \to 5} (x^2 - 4x + 3) &= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3 && \boxed{\text{Theorem 2.2.2}(a), (b)} \\ &= \lim_{x \to 5} x^2 - 4 \lim_{x \to 5} x + \lim_{x \to 5} 3 && \boxed{\text{Equation (6)}} \\ &= 5^2 - 4(5) + 3 && \boxed{\text{Equation (4)}} \\ &= 8 \end{aligned}$$

◀

Our next result will show that the limit of a polynomial $p(x)$ at a point $x = a$ is the same as the value of the polynomial at that point. This greatly simplifies the computation of limits of polynomials by allowing us to evaluate the polynomial instead. Moreover, as discussed in the last section, this result also establishes that graphs of polynomials are continuous curves (see the discussion in the subsection of Section 2.1 entitled *A First Look at Continuity*).

2.2.3 THEOREM. *For any polynomial*

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

Proof.

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(c_0 + c_1x + \cdots + c_nx^n\right)\\ &= \lim_{x\to a} c_0 + \lim_{x\to a} c_1x + \cdots + \lim_{x\to a} c_nx^n\\ &= \lim_{x\to a} c_0 + c_1\lim_{x\to a} x + \cdots + c_n\lim_{x\to a} x^n\\ &= c_0 + c_1a + \cdots + c_na^n = p(a)\end{aligned}$$

■

Example 2

If we apply Theorem 2.2.3 to the problem in Example 1, we can bypass the intermediate steps and write immediately

$$\lim_{x\to 5}(x^2 - 4x + 3) = 5^2 - 4(5) + 3 = 8$$

◀

LIMITS OF x^n AS $x \to +\infty$ OR $x \to -\infty$

In Figure 2.2.2 we have graphed the polynomials of the form x^n for $n = 1, 2, 3$, and 4; and below each figure we have indicated the limits as $x \to +\infty$ and $x \to -\infty$. The results in the figure are special cases of the following general results:

$$\lim_{x\to +\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots \tag{7}$$

$$\lim_{x\to -\infty} x^n = \begin{cases} +\infty, & n = 2, 4, 6, \ldots \\ -\infty, & n = 1, 3, 5, \ldots \end{cases} \tag{8}$$

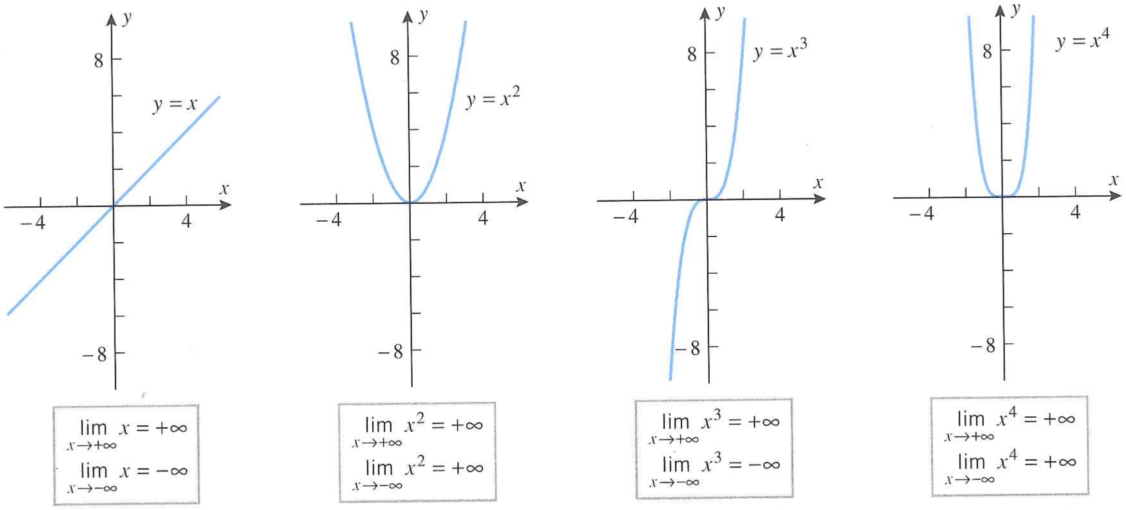


Figure 2.2.2

Multiplying x^n by a positive real number does not affect limits (7) and (8), but multiplying by a negative real number reverses the signs.

Example 3

$$\lim_{x\to +\infty} 2x^5 = +\infty, \quad \lim_{x\to -\infty} 2x^5 = -\infty$$

$$\lim_{x\to +\infty} -7x^6 = -\infty, \quad \lim_{x\to -\infty} -7x^6 = -\infty$$

◀

LIMITS OF POLYNOMIALS AS $x \to +\infty$ OR $x \to -\infty$

There is a useful principle about polynomials which, expressed informally, states that:

A polynomial behaves like its term of highest degree as $x \to +\infty$ or $x \to -\infty$.

Stated more precisely, if $c_n \neq 0$, then

$$\lim_{x \to +\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to +\infty} c_n x^n \tag{9}$$

$$\lim_{x \to -\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to -\infty} c_n x^n \tag{10}$$

We can motivate these results by factoring out the highest power of x from the polynomial and examining the limit of the factored expression. Thus,

$$c_0 + c_1 x + \cdots + c_n x^n = x^n \left(\frac{c_0}{x^n} + \frac{c_1}{x^{n-1}} + \cdots + c_n\right)$$

As $x \to +\infty$ or $x \to -\infty$, it follows from (5) that all of the terms with positive powers of x in the denominator approach 0, so (9) and (10) are certainly plausible.

Example 4

$$\lim_{x \to -\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x \to -\infty} 7x^5 = -\infty$$

$$\lim_{x \to -\infty} (-4x^8 + 17x^3 - 5x + 1) = \lim_{x \to -\infty} -4x^8 = -\infty$$ ◀

LIMITS OF RATIONAL FUNCTIONS AS $x \to a$

Recall that a rational function is the ratio of two polynomials. Theorem 2.2.3 and Theorem 2.2.2(*d*) can often be used in combination to compute limits of rational functions.

Example 5

Find $\displaystyle\lim_{x \to 2} \frac{5x^3 + 4}{x - 3}$.

Solution.

$$\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} = \frac{\displaystyle\lim_{x \to 2} (5x^3 + 4)}{\displaystyle\lim_{x \to 2} (x - 3)} = \frac{5 \cdot 2^3 + 4}{2 - 3} = -44$$ ◀

The method of the preceding example will not work if the limit of the denominator is zero, since Theorem 2.2.2(*d*) is not applicable in this situation. However, if the numerator and denominator *both* approach zero as x approaches a, then the numerator and denominator will have a common factor of $x - a$ and the limit can often be obtained by first canceling the common factors. The following example illustrates this technique.

Example 6

Find $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution. The numerator and denominator both have a limit of zero as x approaches 2, so they share a common factor of $x - 2$. The limit can be obtained as follows:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$ ◀

REMARK. Although correct, the second equality in the preceding computation needs some justification, since canceling the factor $x-2$ alters the function. However, as discussed in Example 5 of Section 1.2, the two functions are identical, except at $x = 2$ (Figure 1.2.9); and we know from our discussions in the last section that this difference has no effect on the limit as x approaches 2.

Example 7

Find

$$\text{(a)}\ \lim_{x\to 3}\frac{x^2-6x+9}{x-3} \qquad \text{(b)}\ \lim_{x\to -4}\frac{2x+8}{x^2+x-12}$$

Solution (a). The numerator and denominator both have a limit of zero as x approaches 3, so there is a common factor of $x-3$. We proceed as follows:

$$\lim_{x\to 3}\frac{x^2-6x+9}{x-3} = \lim_{x\to 3}\frac{(x-3)^2}{x-3} = \lim_{x\to 3}(x-3) = 0$$

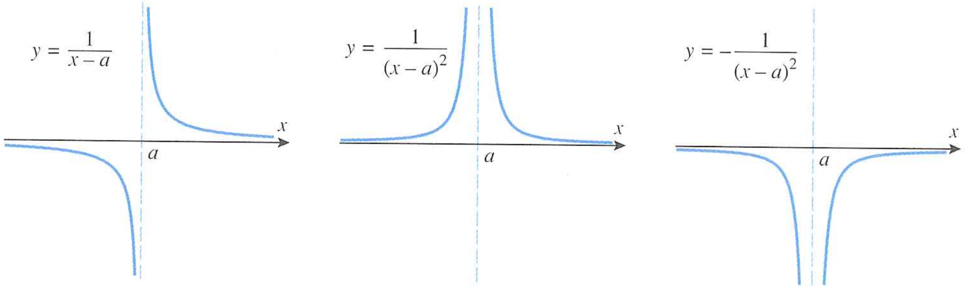
Solution (b). The numerator and denominator both have a limit of zero as x approaches -4, so there is a common factor of $x-(-4) = x+4$. We proceed as follows:

$$\lim_{x\to -4}\frac{2x+8}{x^2+x-12} = \lim_{x\to -4}\frac{2(x+4)}{(x+4)(x-3)} = \lim_{x\to -4}\frac{2}{x-3} = -\frac{2}{7}$$ ◀

If the limit of the denominator is zero, but the limit of the numerator is not, then there are three possibilities for the limit of the rational function as $x\to a$:

- The limit may be $+\infty$.
- The limit may be $-\infty$.
- The limit may be $+\infty$ from one side and $-\infty$ from the other.

Figure 2.2.3 illustrates this graphically for functions of the form $1/(x-a)$, $1/(x-a)^2$, and $-1/(x-a)^2$.



$$\lim_{x\to a^+}\frac{1}{x-a} = +\infty$$
$$\lim_{x\to a^-}\frac{1}{x-a} = -\infty$$

$$\lim_{x\to a}\frac{1}{(x-a)^2} = +\infty$$

$$\lim_{x\to a}-\frac{1}{(x-a)^2} = -\infty$$

Figure 2.2.3

Example 8

Find

$$\text{(a)}\ \lim_{x\to 4^+}\frac{2-x}{(x-4)(x+2)} \qquad \text{(b)}\ \lim_{x\to 4^-}\frac{2-x}{(x-4)(x+2)} \qquad \text{(c)}\ \lim_{x\to 4}\frac{2-x}{(x-4)(x+2)}$$

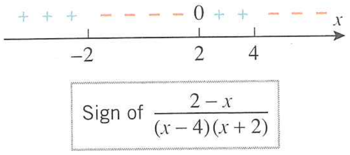


Figure 2.2.4

Solution. In all three parts the limit of the numerator is -2, and the limit of the denominator is 0, so the limit of the ratio does not exist. To be more specific than this, we need to analyze the sign of the ratio. The sign of the ratio, which is given in Figure 2.2.4, is determined by the signs of $2-x$, $x-4$, and $x+2$. (The method of test points, discussed in Appendix A, provides a simple way of finding the sign of the ratio here.) It follows from this figure that as x approaches 4 from the right, the ratio is always negative; and as x approaches 4 from the left, the ratio is eventually positive (after x exceeds 2). Thus,

$$\lim_{x\to 4^+}\frac{2-x}{(x-4)(x+2)}=-\infty \quad\text{and}\quad \lim_{x\to 4^-}\frac{2-x}{(x-4)(x+2)}=+\infty$$

Because the one-sided limits have opposite signs, all we can say about the two-sided limit is that it does not exist. ◀

LIMITS OF RATIONAL FUNCTIONS AS $x \to +\infty$ OR $x \to -\infty$

If we divide the numerator and denominator of a rational function by the highest power of x that occurs in the denominator, then all the powers of x in the denominator become constants or powers of $1/x$. The following examples show how this observation together with (5), (9), and (10) can be used to find limits of rational functions as $x\to+\infty$ or $x\to-\infty$.

Example 9

Find $\displaystyle\lim_{x\to+\infty}\frac{3x+5}{6x-8}$.

Solution. Divide the numerator and denominator by the highest power of x that occurs in the denominator; this is $x^1=x$. We obtain

$$\lim_{x\to+\infty}\frac{3x+5}{6x-8}=\lim_{x\to+\infty}\frac{3+5/x}{6-8/x}=\frac{\displaystyle\lim_{x\to+\infty}(3+5/x)}{\displaystyle\lim_{x\to+\infty}(6-8/x)}$$

$$=\frac{\displaystyle\lim_{x\to+\infty}3+\lim_{x\to+\infty}5/x}{\displaystyle\lim_{x\to+\infty}6-\lim_{x\to+\infty}8/x}=\frac{3+5\displaystyle\lim_{x\to+\infty}1/x}{6-8\displaystyle\lim_{x\to+\infty}1/x}$$

$$=\frac{3+(5\cdot 0)}{6-(8\cdot 0)}=\frac{1}{2}$$

◀

Example 10

Find

(a) $\displaystyle\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}$ (b) $\displaystyle\lim_{x\to-\infty}\frac{5x^3-2x^2+1}{3x+5}$

Solution (a). Divide the numerator and denominator by the highest power of x that occurs in the denominator, namely x^3. We obtain

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4/x-1/x^2}{2-5/x^3}=\frac{\displaystyle\lim_{x\to-\infty}(4/x-1/x^2)}{\displaystyle\lim_{x\to-\infty}(2-5/x^3)}$$

$$=\frac{(4\cdot 0)-0}{2-(5\cdot 0)}=\frac{0}{2}=0$$

Solution (b). Divide the numerator and denominator by x to obtain

$$\lim_{x\to-\infty}\frac{5x^3-2x^2+1}{3x+5}=\lim_{x\to-\infty}\frac{5x^2-2x+1/x}{3+5/x}=+\infty$$

where the final step is justified by the fact that

$$5x^2 - 2x \to +\infty, \quad 1/x \to 0, \quad \text{and} \quad 3 + 5/x \to 3$$

as $x \to -\infty$. ◀

A QUICK METHOD FOR FINDING LIMITS OF RATIONAL FUNCTIONS AS $x \to +\infty$ OR $x \to -\infty$

Since a polynomial behaves like its term of highest degree as $x \to +\infty$ or $x \to -\infty$, it follows that a rational function behaves like the ratio of the terms of highest degree in the numerator and denominator as $x \to +\infty$ or $x \to -\infty$; that is, if $c_n \neq 0$ and $d_n \neq 0$, then

$$\lim_{x \to +\infty} \frac{c_0 + c_1 x + \cdots + c_n x^n}{d_0 + d_1 x + \cdots + d_m x^m} = \lim_{x \to +\infty} \frac{c_n x^n}{d_m x^m} \tag{11}$$

and

$$\lim_{x \to -\infty} \frac{c_0 + c_1 x + \cdots + c_n x^n}{d_0 + d_1 x + \cdots + d_m x^m} = \lim_{x \to -\infty} \frac{c_n x^n}{d_m x^m} \tag{12}$$

Example 11

Use Formulas (11) and (12) to find

$$\text{(a)}\ \lim_{x \to +\infty} \frac{3x + 5}{6x - 8} \qquad \text{(b)}\ \lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5} \qquad \text{(c)}\ \lim_{x \to +\infty} \frac{3 - 2x^4}{x + 1}$$

Solution (a).

$$\lim_{x \to +\infty} \frac{3x + 5}{6x - 8} = \lim_{x \to +\infty} \frac{3x}{6x} = \lim_{x \to +\infty} \frac{1}{2} = \frac{1}{2}$$

which agrees with the result obtained in Example 9.

Solution (b).

$$\lim_{x \to -\infty} \frac{4x^2 - x}{2x^3 - 5} = \lim_{x \to -\infty} \frac{4x^2}{2x^3} = \lim_{x \to -\infty} \frac{2}{x} = 0$$

which agrees with the result obtained in Example 10.

Solution (c).

$$\lim_{x \to +\infty} \frac{3 - 2x^4}{x + 1} = \lim_{x \to +\infty} \frac{-2x^4}{x} = \lim_{x \to +\infty} -2x^3 = -\infty$$ ◀

REMARK. We emphasize that Formulas (11) and (12) are only applicable if $x \to +\infty$ or $x \to -\infty$; they do not apply to limits in which x approaches a *finite* number a.

LIMITS INVOLVING RADICALS

Example 12

Find $\displaystyle \lim_{x \to +\infty} \sqrt[3]{\frac{3x + 5}{6x - 8}}$.

Solution.

$$\lim_{x \to +\infty} \sqrt[3]{\frac{3x + 5}{6x - 8}} \underset{\boxed{\text{Theorem 2.2.2}(e)}}{=} \sqrt[3]{\lim_{x \to +\infty} \frac{3x + 5}{6x - 8}} \underset{\boxed{\text{Example 9}}}{=} \sqrt[3]{\frac{1}{2}}$$ ◀

Example 13

Find

$$\text{(a)}\ \lim_{x \to +\infty} \frac{\sqrt{x^2 + 2}}{3x - 6} \qquad \text{(b)}\ \lim_{x \to -\infty} \frac{\sqrt{x^2 + 2}}{3x - 6}$$

In both parts it would be helpful to manipulate the function so that the powers of x become powers of $1/x$. This can be achieved in both cases by dividing the numerator and denominator by $|x|$ and using the fact that $\sqrt{x^2} = |x|$.

Solution (a). As $x \to +\infty$, the values of x are eventually positive, so we can replace $|x|$ by x where helpful. We obtain

$$\begin{aligned}
\lim_{x\to+\infty} \frac{\sqrt{x^2+2}}{3x-6} &= \lim_{x\to+\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x\to+\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x} \\
&= \lim_{x\to+\infty} \frac{\sqrt{1+2/x^2}}{3-6/x} = \frac{\lim\limits_{x\to+\infty} \sqrt{1+2/x^2}}{\lim\limits_{x\to+\infty} (3-6/x)} \\
&= \frac{\sqrt{\lim\limits_{x\to+\infty} (1+2/x^2)}}{\lim\limits_{x\to+\infty} (3-6/x)} = \frac{\sqrt{\lim\limits_{x\to+\infty} 1 + 2\lim\limits_{x\to+\infty} 1/x^2}}{\lim\limits_{x\to+\infty} 3 - 6\lim\limits_{x\to+\infty} 1/x} \\
&= \frac{\sqrt{1+(2\cdot 0)}}{3-(6\cdot 0)} = \frac{1}{3}
\end{aligned}$$

Solution (b). As $x \to -\infty$, the values of x are eventually negative, so we can replace $|x|$ by $-x$ where helpful. We obtain

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2+2}}{3x-6} &= \lim_{x\to-\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x\to-\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/(-x)} \\
&= \lim_{x\to-\infty} \frac{\sqrt{1+2/x^2}}{(6/x)-3} = -\frac{1}{3}
\end{aligned}$$

◀

LIMITS OF FUNCTIONS DEFINED PIECEWISE

For functions that are defined piecewise, a two-sided limit at a point where the formula changes is best obtained by first finding the one-sided limits at the point.

Example 14

Find $\lim\limits_{x\to 3} f(x)$ for $f(x) = \begin{cases} x^2-5, & x \le 3 \\ \sqrt{x+13}, & x > 3. \end{cases}$

Solution. As x approaches 3 from the left, the formula for f is

$$f(x) = x^2 - 5$$

so that

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2-5) = 3^2 - 5 = 4$$

As x approaches 3 from the right, the formula for f is

$$f(x) = \sqrt{x+13}$$

so that

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x+13} = \sqrt{\lim_{x\to 3^+} (x+13)} = \sqrt{16} = 4$$

Since the one-sided limits are equal, we have

$$\lim_{x\to 3} f(x) = 4$$

◀

EXERCISE SET 2.2

1. Given that

$$\lim_{x\to a} f(x) = 2, \quad \lim_{x\to a} g(x) = -4, \quad \lim_{x\to a} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to a} [f(x) + 2g(x)]$ (b) $\lim_{x\to a} [h(x) - 3g(x) + 1]$

(c) $\lim_{x\to a} [f(x)g(x)]$ (d) $\lim_{x\to a} [g(x)]^2$

(e) $\lim_{x\to a} \sqrt[3]{6 + f(x)}$ (f) $\lim_{x\to a} \dfrac{2}{g(x)}$

(g) $\lim_{x\to a} \dfrac{3f(x) - 8g(x)}{h(x)}$ (h) $\lim_{x\to a} \dfrac{7g(x)}{2f(x) + g(x)}$

2. Use the graphs of f and g in the accompanying figure to find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to 2} [f(x) + g(x)]$ (b) $\lim_{x\to 0} [f(x) + g(x)]$

(c) $\lim_{x\to 0^+} [f(x) + g(x)]$ (d) $\lim_{x\to 0^-} [f(x) + g(x)]$

(e) $\lim_{x\to 2} \dfrac{f(x)}{1 + g(x)}$ (f) $\lim_{x\to 2} \dfrac{1 + g(x)}{f(x)}$

(g) $\lim_{x\to 0^+} \sqrt{f(x)}$ (h) $\lim_{x\to 0^-} \sqrt{f(x)}$

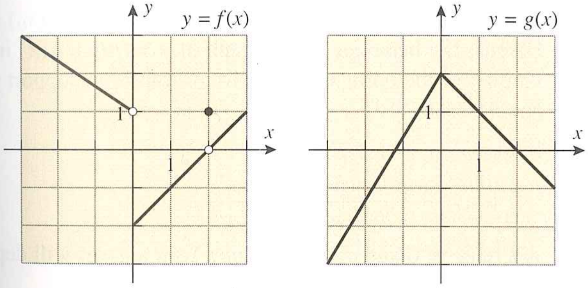


Figure Ex-2

3. In each part, find the limit by inspection.

(a) $\lim_{x\to 8} 7$ (b) $\lim_{x\to -\infty} (-3)$ (c) $\lim_{x\to 0^+} \pi$

(d) $\lim_{x\to -2} 3x$ (e) $\lim_{y\to 3^+} 12y$ (f) $\lim_{h\to +\infty} (-2h)$

4. In each part, find the stated limit of $f(x) = x/|x|$ by inspection.

(a) $\lim_{x\to 5} f(x)$ (b) $\lim_{x\to -5} f(x)$ (c) $\lim_{x\to +\infty} f(x)$

(d) $\lim_{x\to -\infty} f(x)$ (e) $\lim_{x\to 0^+} f(x)$ (f) $\lim_{x\to 0^-} f(x)$

Find the limits in Exercises 5–48.

5. $\lim_{y\to 2^-} \dfrac{(y-1)(y-2)}{y+1}$

6. $\lim_{x\to 3} \dfrac{x^2 - 2x}{x+1}$

7. $\lim_{x\to 4} \dfrac{x^2 - 16}{x - 4}$

8. $\lim_{x\to 0} \dfrac{6x - 9}{x^3 - 12x + 3}$

9. $\lim_{x\to 1^+} \dfrac{x^4 - 1}{x - 1}$

10. $\lim_{t\to -2} \dfrac{t^3 + 8}{t + 2}$

11. $\lim_{x\to -1} \dfrac{x^2 + 6x + 5}{x^2 - 3x - 4}$

12. $\lim_{x\to 2} \dfrac{x^2 - 4x + 4}{x^2 + x - 6}$

13. $\lim_{x\to +\infty} \dfrac{3x + 1}{2x - 5}$

14. $\lim_{t\to 1} \dfrac{t^3 + t^2 - 5t + 3}{t^3 - 3t + 2}$

15. $\lim_{y\to -\infty} \dfrac{3}{y + 4}$

16. $\lim_{x\to +\infty} \dfrac{1}{x - 12}$

17. $\lim_{x\to -\infty} \dfrac{x - 2}{x^2 + 2x + 1}$

18. $\lim_{x\to +\infty} \dfrac{5x^2 + 7}{3x^2 - x}$

19. $\lim_{x\to -\infty} \dfrac{\sqrt{5x^2 - 2}}{x + 3}$

20. $\lim_{s\to +\infty} \sqrt[3]{\dfrac{3s^7 - 4s^5}{2s^7 + 1}}$

21. $\lim_{y\to -\infty} \dfrac{2 - y}{\sqrt{7 + 6y^2}}$

22. $\lim_{x\to +\infty} \dfrac{\sqrt{5x^2 - 2}}{x + 3}$

23. $\lim_{x\to -\infty} \dfrac{\sqrt{3x^4 + x}}{x^2 - 8}$

24. $\lim_{y\to +\infty} \dfrac{2 - y}{\sqrt{7 + 6y^2}}$

25. $\lim_{x\to 3^+} \dfrac{x}{x - 3}$

26. $\lim_{x\to +\infty} \dfrac{\sqrt{3x^4 + x}}{x^2 - 8}$

27. $\lim_{x\to 3} \dfrac{x}{x - 3}$

28. $\lim_{x\to 3^-} \dfrac{x}{x - 3}$

29. $\lim_{x\to 2^-} \dfrac{x}{x^2 - 4}$

30. $\lim_{x\to 2^+} \dfrac{x}{x^2 - 4}$

31. $\lim_{y\to 6^+} \dfrac{y + 6}{y^2 - 36}$

32. $\lim_{x\to 2} \dfrac{x}{x^2 - 4}$

33. $\lim_{y\to 6} \dfrac{y + 6}{y^2 - 36}$

34. $\lim_{y\to 6^-} \dfrac{y + 6}{y^2 - 36}$

35. $\lim_{x\to 4^-} \dfrac{3 - x}{x^2 - 2x - 8}$

36. $\lim_{x\to 4^+} \dfrac{3 - x}{x^2 - 2x - 8}$

37. $\lim_{x\to +\infty} \dfrac{7 - 6x^5}{x + 3}$

38. $\lim_{x\to 4} \dfrac{3 - x}{x^2 - 2x - 8}$

39. $\lim_{t\to +\infty} \dfrac{6 - t^3}{7t^3 + 3}$

40. $\lim_{t\to -\infty} \dfrac{5 - 2t^3}{t^2 + 1}$

41. $\lim_{x\to 9} \dfrac{x - 9}{\sqrt{x} - 3}$

42. $\lim_{x\to 3^-} \dfrac{1}{|x - 3|}$

43. $\lim_{x\to +\infty} \sqrt{x}$

44. $\lim_{y\to 4} \dfrac{4 - y}{2 - \sqrt{y}}$

45. $\lim_{x\to -\infty} (3 - x)$

46. $\lim_{x\to -\infty} \sqrt{5 - x}$

47. $\lim_{x\to +\infty} (1 + 2x - 3x^5)$

48. $\lim_{x\to +\infty} (2x^3 - 100x + 5)$

49. Let

$$f(x) = \begin{cases} x - 1, & x \le 3 \\ 3x - 7, & x > 3 \end{cases}$$

Find

(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$ (c) $\lim_{x\to 3} f(x)$.

50. Let

$$g(t)=\begin{cases} t^2, & t\geq 0\\ t-2, & t<0\end{cases}$$

Find

(a) $\lim_{t\to 0^-} g(t)$ (b) $\lim_{t\to 0^+} g(t)$ (c) $\lim_{t\to 0} g(t)$.

51. Let $f(x)=\dfrac{x^3-1}{x-1}$.

(a) Find $\lim_{x\to 1} f(x)$.

(b) Sketch the graph of $y=f(x)$.

52. Let

$$f(x)=\begin{cases} \dfrac{x^2-9}{x+3}, & x\neq -3\\ k, & x=-3\end{cases}$$

(a) Find k so that $F(-3)=\lim_{x\to -3} F(x)$.

(b) With k assigned the value $\lim_{x\to -3} F(x)$, show that $F(x)$ can be expressed as a polynomial.

53. (a) Explain why the following calculation is incorrect.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right)=\lim_{x\to 0^+}\frac{1}{x}-\lim_{x\to 0^+}\frac{1}{x^2}$$
$$=+\infty-(+\infty)=0$$

(b) Show that $\lim_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)=-\infty$.

54. Find $\lim_{x\to 0^-}\left(\dfrac{1}{x}+\dfrac{1}{x^2}\right)$.

In Exercises 55 and 56, first rationalize the numerator, then find the limit.

55. $\lim_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}$

56. $\lim_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x}$

Find the limits in Exercises 57–60.

57. $\lim_{x\to +\infty}(\sqrt{x^2+3}-x)$

58. $\lim_{x\to +\infty}(\sqrt{x^2-3x}-x)$

59. $\lim_{x\to +\infty}(\sqrt{x^2+ax}-x)$

60. $\lim_{x\to +\infty}(\sqrt{x^2+ax}-\sqrt{x^2+bx})$

61. Discuss the limits of $p(x)=(1-x)^n$ as $x\to+\infty$ and $x\to-\infty$ for positive integer values of n.

62. Let $p(x)=(1-x)^n$ and $q(x)=(1-x)^m$. Discuss the limits of $p(x)/q(x)$ as $x\to+\infty$ and $x\to-\infty$ for positive integer values of m and n.

63. Let $p(x)$ be a polynomial of degree n. Discuss the limits of $p(x)/x^m$ as $x\to+\infty$ and $x\to-\infty$ for positive integer values of m.

64. In each part, find examples of polynomials $p(x)$ and $q(x)$ that satisfy the stated condition and such that $p(x)\to+\infty$ and $q(x)\to+\infty$ as $x\to+\infty$.

(a) $\lim_{x\to+\infty}\dfrac{p(x)}{q(x)}=1$ (b) $\lim_{x\to+\infty}\dfrac{p(x)}{q(x)}=0$

(c) $\lim_{x\to+\infty}\dfrac{p(x)}{q(x)}=+\infty$ (d) $\lim_{x\to+\infty}[p(x)-q(x)]=3$

65. Let $p(x)$ and $q(x)$ be polynomials, and suppose $q(x_0)=0$. Discuss the behavior of the graph of $y=p(x)/q(x)$ in the vicinity of the point $x=x_0$. Give examples to support your conclusions.

66. Find

$$\lim_{x\to+\infty}\frac{c_0+c_1x+\cdots+c_nx^n}{d_0+d_1x+\cdots+d_mx^m}$$

where $c_n\neq 0$ and $d_m\neq 0$. [*Hint:* Your answer will depend on whether $m<n$, $m=n$, or $m>n$.]

2.3 LIMITS (DISCUSSED MORE RIGOROUSLY)

Thus far, our discussion of limits has been based on our intuitive feeling of what it means for the values of a function to get closer and closer to a limiting value. However, this level of informality can only take us so far, so our goal in this section is to define limits precisely. From a purely mathematical point of view these definitions are needed to establish limits with certainty and to prove theorems about them. However, they will also provide us with a deeper understanding of the limit concept, making it possible for us to visualize some of the more subtle properties of functions.

DEFINITION OF A LIMIT

In earlier sections we interpreted the limit

$$\lim_{x\to a} f(x)=L$$

to mean that we can force the values of $f(x)$ closer and closer to L by making x closer and closer (but not equal) to a. Our goal here is to try to make the notion of a limit more precise by giving the informal phrase "closer and closer to" a precise mathematical meaning. However,

57. $\lim\limits_{x\to 2^+}(x+1)=3$

58. $\lim\limits_{x\to 1^-}(3x+2)=5$

59. $\lim\limits_{x\to 4^+}\sqrt{x-4}=0$

60. $\lim\limits_{x\to 0^-}\sqrt{-x}=0$

61. $\lim\limits_{x\to 2^+} f(x)=2$, where $f(x)=\begin{cases} x, & x>2 \\ 3x, & x\le 2 \end{cases}$

62. $\lim\limits_{x\to 2^-} f(x)=6$, where $f(x)=\begin{cases} x, & x>2 \\ 3x, & x\le 2 \end{cases}$

In Exercises 63 and 64, use the remark following Definitions 2.3.6 and 2.3.7 to prove that the stated limit is correct.

63. (a) $\lim\limits_{x\to 1^+}\dfrac{1}{1-x}=-\infty$ (b) $\lim\limits_{x\to 1^-}\dfrac{1}{1-x}=+\infty$

64. (a) $\lim\limits_{x\to 0^+}\dfrac{1}{x}=+\infty$ (b) $\lim\limits_{x\to 0^-}\dfrac{1}{x}=-\infty$

For Exercises 65 and 66, write out definitions of the four limits in (18), and use your definitions to prove that the stated limit is correct.

65. (a) $\lim\limits_{x\to +\infty}(x+1)=+\infty$ (b) $\lim\limits_{x\to -\infty}(x+1)=-\infty$

66. (a) $\lim\limits_{x\to +\infty}(x^2-3)=+\infty$ (b) $\lim\limits_{x\to -\infty}(x^3+5)=-\infty$

67. Prove the result in Example 3 under the assumption that $\delta\le 2$ rather than $\delta\le 1$.

68. (a) In Definition 2.3.3 there is a condition requiring that $f(x)$ be defined for all x in some open interval containing a, except possibly at a itself. What is the purpose of this requirement?

(b) Why is $\lim\limits_{x\to 0}\sqrt{x}=0$ an incorrect statement?

(c) Is $\lim\limits_{x\to 0.01}\sqrt{x}=0.1$ a correct statement?

69. Generate the graph of $f(x)=x^3-4x+5$ with a graphing utility, and use the graph to find a number δ such that $|f(x)-2|<0.05$ if $0<|x-1|<\delta$. [*Hint:* Show that the inequality $|f(x)-2|<0.05$ can be rewritten as $1.95<x^3-4x+5<2.05$, and estimate the values of x for which $x^3-4x+5=1.95$ and $x^3-4x+5=2.05$.]

70. Use the method of Exercise 69 to find a number δ such that $|\sqrt{5x+1}-4|<0.5$ if $0<|x-3|<\delta$.

2.4 CONTINUITY

A moving object cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of a moving object as a continuous curve, that is, a curve without gaps, breaks, or holes. Earlier, we discussed continuity from an intuitive viewpoint; in this section we will define this concept precisely and develop some fundamental properties of continuous curves.

DEFINITION OF CONTINUITY

Recall from Section 2.1 that the graph of a function f will have a hole or a break in it at a point c if any of the following situations occur:

- The function f is undefined at c (Figure 2.4.1a).
- The limit of $f(x)$ does not exist as x approaches c (Figures 2.4.1b, 2.4.1c).
- The value of the function and the value of the limit at c are different (Figure 2.4.1d).

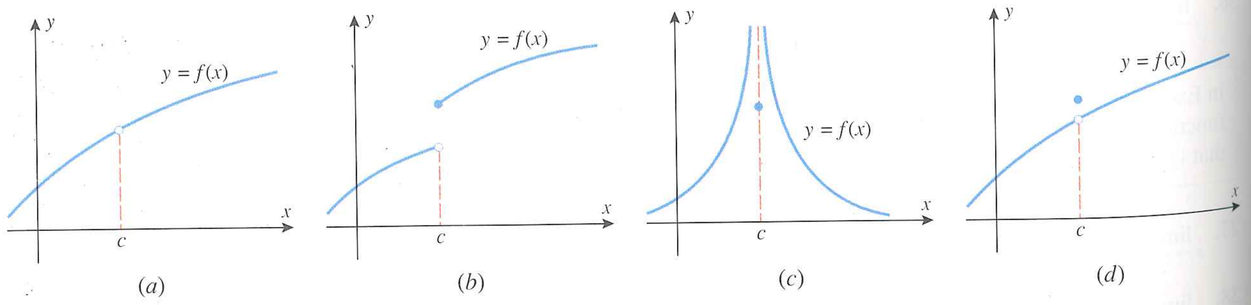


Figure 2.4.1

This suggests the following definition.

2.4.1 DEFINITION. A function f is said to be ***continuous at a point c*** if the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

If one or more of the conditions in this definition fails to hold, then we will say that f has a ***discontinuity*** at the point $x = c$. If f is continuous at each point of an open interval (a, b), then we will say that f is ***continuous on*** $(\boldsymbol{a}, \boldsymbol{b})$. This definition also applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the case where f is continuous on $(-\infty, +\infty)$, we will say that f is ***continuous everywhere***. If f is continuous on an open interval, but the particular interval is not important for the discussion, we will say that f is ***continuous*** (without referencing the interval).

REMARK. The first two conditions in Definition 2.4.1 are actually superfluous, since it is implicit in the third condition that $f(c)$ is defined and the limit exists (otherwise the equality would make no sense). We have included the first two conditions for emphasis and clarity, but, as a practical matter, you need only confirm that the third condition holds when you want to show that a function f is continuous at a point c.

Example 1

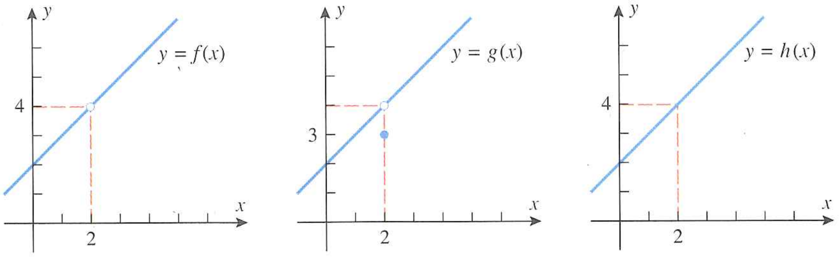
Determine whether the following functions are continuous at the point $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at the point $x = 2$, and hence all three have the same limit at $x = 2$, namely

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

The function f is undefined at $x = 2$, and hence is not continuous at that point (Figure 2.4.2*a*). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit at that point; hence, g is also not continuous at $x = 2$ (Figure 2.4.2*b*). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit at that



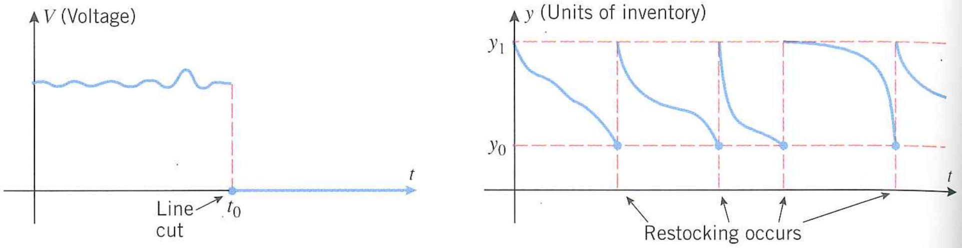
(*a*) (*b*) (*c*)

Figure 2.4.2

point; hence, h is continuous at $x = 2$ (Figure 2.4.2c). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀

CONTINUITY IN APPLICATIONS

In applications, discontinuities often signal the occurrence of important physical phenomena. For example, Figure 2.4.3a is a graph of voltage versus time for an underground cable that is accidentally cut by a work crew at time $t = t_0$ (the voltage drops to zero when the line is cut). Figure 2.4.3b shows the graph of inventory versus time for a company that restocks its warehouse to y_1 units when the inventory falls to y_0 units. The discontinuities occur at those times when restocking occurs.



(a) (b)

Figure 2.4.3

Given the possible physical significance of discontinuities, it is important to be able to identify points of discontinuity for specific functions, and to be able to make general statements about the continuity properties of entire families of functions. This is our next goal.

CONTINUITY OF POLYNOMIALS

The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an *arbitrary* point. For example, we showed in Theorem 2.2.3 that if $p(x)$ is a polynomial and a is any real number, then

$$\lim_{x \to a} p(x) = p(a)$$

Thus, we have the following result.

2.4.2 THEOREM. *Polynomials are continuous everywhere.*

Example 2

Show that $|x|$ is continuous everywhere (Figure 1.2.5).

Solution. We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

so $|x|$ is the same as the polynomial x on the interval $(0, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, 0)$. But polynomials are continuous functions, so $x = 0$ is the only possible point of discontinuity for $|x|$. At this point we have $|0| = 0$, so to prove the continuity at $x = 0$ we must show that

$$\lim_{x \to 0} |x| = 0 \tag{1}$$

Because the formula for $|x|$ changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \quad \text{and} \quad \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Thus, (1) holds and $|x|$ is continuous at $x = 0$. ◀

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

The following theorem, which is a consequence of Theorem 2.2.2, will enable us to reach conclusions about the continuity of functions that are obtained by adding, subtracting, multiplying, and dividing continuous functions.

> **2.4.3 THEOREM.** *If the functions f and g are continuous at c, then*
> (*a*) *$f + g$ is continuous at c.*
> (*b*) *$f - g$ is continuous at c.*
> (*c*) *fg is continuous at c.*
> (*d*) *f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.*

We will prove part (*d*). The remaining proofs are similar and will be omitted.

Proof. First, consider the case where $g(c) = 0$. In this case $f(c)/g(c)$ is undefined, so the function f/g has a discontinuity at c.

Next, consider the case where $g(c) \neq 0$. To prove that f/g is continuous at c, we must show that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \tag{2}$$

Since f and g are continuous at c,

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c)$$

Thus, by Theorem 2.2.2(*d*)

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)}$$

which proves (2). ■

CONTINUITY OF RATIONAL FUNCTIONS

Since polynomials are continuous functions, and since rational functions are ratios of polynomials, part (*d*) of Theorem 2.4.3 yields the following result.

> **2.4.4 THEOREM.** *A rational function is continuous everywhere except at the points where the denominator is zero.*

Example 3

For what values of x is there a hole or a gap in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. The function being graphed is a rational function, and hence is continuous everywhere except at the points where the denominator is zero. Solving the equation

$$x^2 - 5x + 6 = 0$$

yields two points of discontinuity, $x = 2$ and $x = 3$. ◀

FOR THE READER. If you use a graphing utility to generate the graph of the equation in this example, then there is a good chance that you will see the discontinuity at $x = 2$ but not at $x = 3$. Try it, and explain what you think is happening.

CONTINUITY OF COMPOSITIONS

The following theorem, whose proof is given in Appendix G, will be useful for calculating limits of compositions of functions.

2.4.5 THEOREM. *Let* lim *stand for one of the limits* $\lim\limits_{x\to c}$, $\lim\limits_{x\to c^-}$, $\lim\limits_{x\to c^+}$, $\lim\limits_{x\to +\infty}$, *or* $\lim\limits_{x\to -\infty}$. *If* $\lim g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim f(g(x)) = f(L)$. *That is,* $\lim f(g(x)) = f(\lim g(x))$.

In words, this theorem states:

> *A limit symbol can be moved through a function sign provided the limit of the expression inside the function sign exists and the function is continuous at this limit.*

Example 4

Suppose that $\lim g(x)$ exists, where lim stands for any of the limits in Theorem 2.4.5. We know from Example 2 that the function $|x|$ is continuous everywhere; thus, it follows that

$$\lim |g(x)| = |\lim g(x)| \tag{3}$$

that is, a limit symbol can be moved through an absolute value sign, provided the limit of the expression inside the absolute value signs exists. For example,

$$\lim_{x\to 3} |5 - x^2| = \left|\lim_{x\to 3} (5 - x^2)\right| = |-4| = 4$$ ◀

The following theorem is concerned with the continuity of compositions of functions; the first part deals with continuity at a specific point, and the second part with continuity everywhere.

2.4.6 THEOREM.

(*a*) *If the function* g *is continuous at the point* c, *and the function* f *is continuous at the point* $g(c)$, *then the composition* $f \circ g$ *is continuous at* c.

(*b*) *If the function* g *is continuous everywhere and the function* f *is continuous everywhere, then the composition* $f \circ g$ *is continuous everywhere.*

Proof. We will prove part (*a*) only; the proof of part (*b*) can be obtained by applying part (*a*) at an arbitrary point c. To prove that $f \circ g$ is continuous at c, we must show that the value of $f \circ g$ and the value of its limit are the same at $x = c$. But this is so, since we can write

$$\lim_{x\to c} (f\circ g)(x) = \lim_{x\to c} f(g(x)) \underset{\text{Theorem 2.4.5}}{=} f(\lim_{x\to c} g(x)) \underset{g\text{ is continuous at }c.}{=} f(g(c)) = (f\circ g)(c)$$ ■

We know from Example 2 that the function $|x|$ is continuous everywhere. Thus, if $g(x)$ is continuous at the point c, then by part (*a*) of Theorem 2.4.6, the function $|g(x)|$ must also be continuous at the point c; and, more generally, if $g(x)$ is continuous everywhere, then so is $|g(x)|$. Stated informally:

> *The absolute value of a continuous function is continuous.*

For example, the polynomial $g(x) = 4 - x^2$ is continuous everywhere, so we can conclude that the function $|4 - x^2|$ is also continuous everywhere (Figure 2.4.4).

FOR THE READER. Can the absolute value of a function that is not continuous be continuous? Justify your answer.

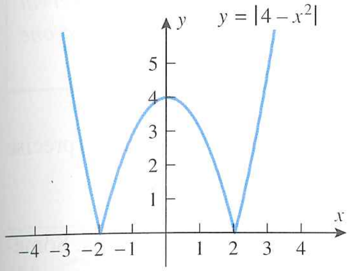


Figure 2.4.4

CONTINUITY FROM THE LEFT AND RIGHT

Because Definition 2.4.1 involves a two-sided limit, that definition does not generally apply at the endpoints of a closed interval $[a, b]$ or at the endpoint of an interval of the form $[a, b)$, $(a, b]$, $(-\infty, b]$, or $[a, +\infty)$. To remedy this problem, we will agree that a function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that point. For example, the function graphed in Figure 2.4.5 is continuous at the right endpoint of the interval $[a, b]$ because

$$\lim_{x \to b^-} f(x) = f(b)$$

but it is not continuous at the left endpoint because

$$\lim_{x \to a^+} f(x) \neq f(a)$$

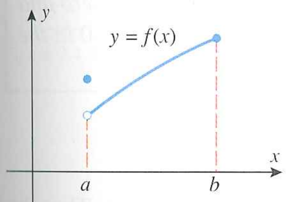


Figure 2.4.5

In general, we will say a function f is ***continuous from the left*** at a point c if

$$\lim_{x \to c^-} f(x) = f(c)$$

and is ***continuous from the right*** at a point c if

$$\lim_{x \to c^+} f(x) = f(c)$$

Using this terminology we define continuity on a closed interval as follows.

2.4.7 DEFINITION. A function f is said to be ***continuous on a closed interval*** $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

FOR THE READER. We leave it for you to modify this definition appropriately so that it applies to intervals of the form $[a, +\infty)$, $(-\infty, b]$, $(a, b]$, and $[a, b)$.

Example 5

What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution. Because the natural domain of this function is the closed interval $[-3, 3]$, we will need to investigate the continuity of f on the open interval $(-3, 3)$ and at the two endpoints. If c is any point in the interval $(-3, 3)$, then it follows from Theorem 2.2.2(*e*) that

$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

which proves f is continuous at each point of the interval $(-3, 3)$. The function f is also continuous at the endpoints since

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$

Thus, f is continuous on the closed interval $[-3, 3]$. ◀

THE INTERMEDIATE-VALUE THEOREM

Figure 2.4.6 shows the graph of a function that is continuous on the closed interval $[a, b]$. The figure suggests that if we draw any horizontal line $y = k$, where k is between $f(a)$ and $f(b)$, then that line will cross the curve $y = f(x)$ line at least once over the $[a, b]$. Stated in numerical terms, if f is continuous on $[a, b]$, then the function f must take on every value k between $f(a)$ and $f(b)$ at least once as x varies from a to b. For example, the polynomial $p(x) = x^5 - x + 3$ has a value of 3 at $x = 1$ and a value of 33 at $x = 2$. Thus, it follows from the continuity of p that the equation $x^5 - x + 3 = k$ has at least one solution in the interval $[1, 2]$ for every value of k between 3 and 33. This idea is stated more precisely in the following theorem.

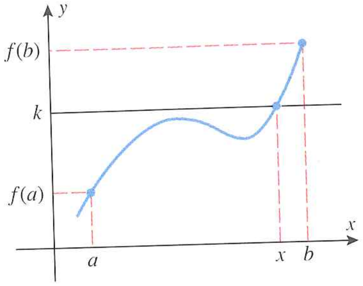


Figure 2.4.6

2.4.8 THEOREM (*Intermediate-Value Theorem*). *If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.*

Although this theorem is intuitively obvious, its proof depends on a mathematically precise development of the real number system, which is beyond the scope of this text.

APPROXIMATING ROOTS USING THE INTERMEDIATE-VALUE THEOREM

A variety of problems can be reduced to solving an equation $f(x) = 0$ for its roots. Sometimes it is possible to solve for the roots exactly using algebra, but often this is not possible and one must settle for decimal approximations of the roots. One procedure for approximating roots is based on the following consequence of the Intermediate-Value Theorem.

2.4.9 THEOREM. *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*

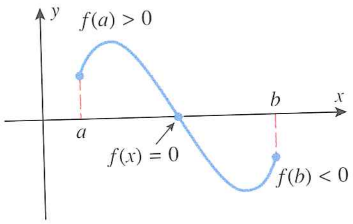


Figure 2.4.7

This result, which is illustrated in Figure 2.4.7, can be proved as follows.

Proof. Since $f(a)$ and $f(b)$ have opposite signs, 0 is between $f(a)$ and $f(b)$. Thus, by the Intermediate-Value Theorem there is at least one number x in the interval $[a, b]$ such that $f(x) = 0$. However, $f(a)$ and $f(b)$ are nonzero, so x must lie in the interval (a, b), which completes the proof. ■

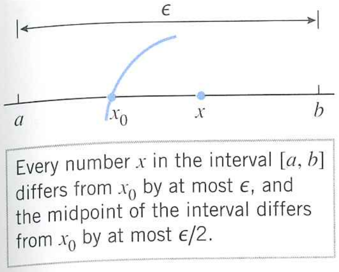
Before we illustrate how this theorem can be used to approximate roots, it will be helpful to discuss some standard terminology for describing errors in approximations. If x is an approximation to a quantity x_0, then we call

$$\epsilon = |x - x_0|$$

the ***absolute error*** or (less precisely) the ***error*** in the approximation. The following terminology is used to describe the size of such errors:

Table 2.4.1

ERROR	DESCRIPTION
$\|x - x_0\| \le 0.1$	x approximates x_0 with an error of at most 0.1.
$\|x - x_0\| \le 0.01$	x approximates x_0 with an error of at most 0.01.
$\|x - x_0\| \le 0.001$	x approximates x_0 with an error of at most 0.001.
$\|x - x_0\| \le 0.0001$	x approximates x_0 with an error of at most 0.0001.
$\|x - x_0\| \le 0.5$	x approximates x_0 to the nearest integer.
$\|x - x_0\| \le 0.05$	x approximates x_0 to 1 decimal place (i.e., to the nearest tenth).
$\|x - x_0\| \le 0.005$	x approximates x_0 to 2 decimal places (i.e., to the nearest hundredth).
$\|x - x_0\| \le 0.0005$	x approximates x_0 to 3 decimal places (i.e., to the nearest thousandth).



Every number x in the interval $[a, b]$ differs from x_0 by at most ϵ, and the midpoint of the interval differs from x_0 by at most $\epsilon/2$.

Figure 2.4.8

We will also need the following result, which should be evident geometrically from Figure 2.4.8.

2.4.10 APPROXIMATION PRINCIPLE. Suppose that the equation $f(x) = 0$ has a root x_0 in the interval $[a, b]$ and that this interval has length $\epsilon = b - a$. Then any number x in the interval $[a, b]$ approximates x_0 with an error of at most ϵ, and the midpoint of the interval approximates x_0 with an error of at most $\epsilon/2$.

Example 6

The equation

$$x^3 - x - 1 = 0$$

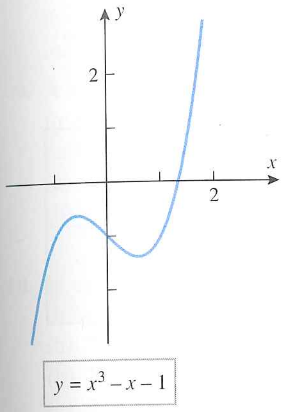


Figure 2.4.9

cannot be solved algebraically very easily because the left side has no simple factors. However, if we graph $p(x) = x^3 - x - 1$ with a graphing utility (Figure 2.4.9), then we are led to conjecture that there is one real root and that this root lies inside the interval [1, 2]. The existence of a root in this interval is also confirmed by Theorem 2.4.8, since $p(1) = -1$ and $p(2) = 5$ have opposite signs. Approximate this root to two decimal-place accuracy.

Solution. Our objective is to approximate the unknown root x_0 with an error of at most 0.005. It follows from the Approximation Principle (2.4.10) that if we can find an interval of length 0.01 that contains the root, then the midpoint of that interval will approximate the root with an error of at most $0.01/2 = 0.005$, which will achieve the desired accuracy.

We know that the root x_0 lies in the interval [1, 2]. However, this interval has length 1, which is too large. We can pinpoint the location of the root more precisely by dividing the interval [1, 2] into 10 equal parts and evaluating p at the points of subdivision using a calculating utility (Table 2.4.2). In this table $p(1.3)$ and $p(1.4)$ have opposite signs, so we know that the root lies in the interval [1.3, 1.4]. This interval has length 0.1, which is still too large, so we repeat the process by dividing the interval [1.3, 1.4] into 10 parts and evaluating p at the points of subdivision; this yields Table 2.4.3, which tells us that the root is inside the interval [1.32, 1.33]. Since this interval has length 0.01, its midpoint 1.325 will approximate the root with an error of at most 0.005. Thus, $x_0 \approx 1.325$ to two decimal-place accuracy. ◀

Table 2.4.2

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$f(x)$	−1	−0.77	−0.47	−0.10	0.34	0.88	1.50	2.21	3.03	3.96	5

Table 2.4.3

x	1.3	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.4
$f(x)$	−0.103	−0.062	−0.020	0.023	0.066	0.110	0.155	0.201	0.248	0.296	0.344

APPROXIMATING ROOTS BY ZOOMING WITH A GRAPHING UTILITY

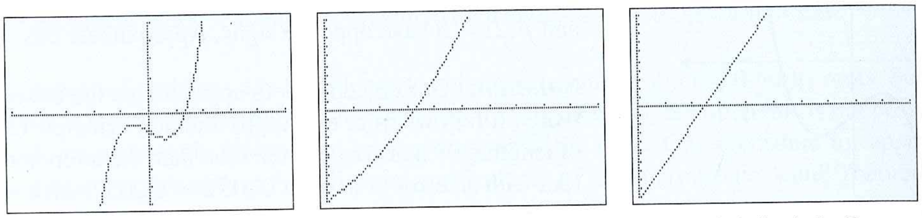
The method illustrated in Example 6 can also be implemented with a graphing utility as follows.

Step 1. Figure 2.4.10*a* shows the graph of f in the window $[-5, 5] \times [-5, 5]$ with xScl $= 1$ and yScl $= 1$. That graph places the root between $x = 1$ and $x = 2$.

Step 2. Since we know that the root lies between $x = 1$ and $x = 2$, we will zoom in by regraphing f over an x-interval that extends between these points and in which $x\text{Scl} = .1$. The y-interval and $y\text{Scl}$ are not critical, as long as the y-interval extends above and below the x-axis. Figure 2.4.10*b* shows the graph of f in the window $[1, 2] \times [-1, 1]$ with $x\text{Scl} = .1$ and $y\text{Scl} = .1$. That graph places the root between $x = 1.3$ and $x = 1.4$.

Step 4. Since we know that the root lies between $x = 1.3$ and $x = 1.4$, we will zoom in again by regraphing f over an x-interval that extends between these points and in which $x\text{Scl} = .01$. Figure 2.4.10*c* shows the graph of f in the window $[1.3, 1.4] \times [-.1, .1]$ with $x\text{Scl} = .01$ and $y\text{Scl} = .01$. That graph places the root between $x = 1.32$ and $x = 1.33$.

Step 5. Since the interval in Step 3 has length .01, its midpoint 1.325 approximates the root with an error of at most 0.005, so $x_0 \approx 1.325$ to two decimal-place accuracy.



$[-5, 5] \times [-5, 5]$
$x\text{Scl} = 1$, $y\text{Scl} = 1$
(*a*)

$[1, 2] \times [-1, 1]$
$x\text{Scl} = .1$, $y\text{Scl} = .1$
(*b*)

$[1.3, 1.4] \times [-.1, .1]$
$x\text{Scl} = .01$, $y\text{Scl} = .01$
(*c*)

Figure 2.4.10

REMARK. To say that x approximates x_0 to n decimal places does *not* mean that the first n decimal places of x and x_0 will be the same when the numbers are rounded to n decimal places. For example, $x = 1.084$ approximates $x_0 = 1.087$ to two decimal places because $|x - x_0| = 0.003(<0.005)$. However, if we round these values to two decimal places, then we obtain $x \approx 1.08$ and $x_0 \approx 1.09$. Thus, if you approximate a number to n decimal places, then you should display that approximation to at least $n + 1$ decimal places to preserve the accuracy.

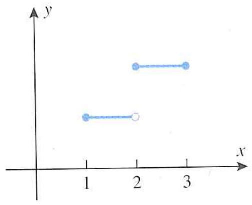
FOR THE READER. Use a graphing or calculating utility to show that the root x_0 in Example 6 can be approximated as $x_0 \approx 1.3245$ to three decimal-place accuracy.

EXERCISE SET 2.4

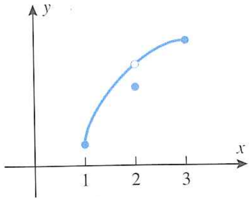
Graphing Calculator

In Exercises 1–4, let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?
(a) $[1, 3]$ (b) $(1, 3)$ (c) $[1, 2]$
(d) $(1, 2)$ (e) $[2, 3]$ (f) $(2, 3)$
On those intervals where f is not continuous, state where the discontinuities occur.

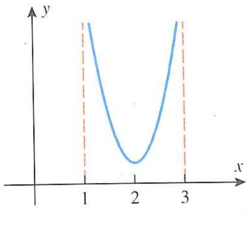
1.



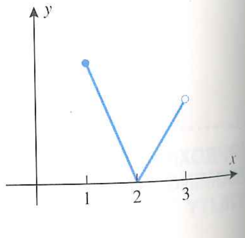
2.



3.



4.



5. Suppose that f and g are continuous functions such that $f(2) = 1$ and $\lim_{x \to 2} [f(x) + 4g(x)] = 13$. Find
(a) $g(2)$ (b) $\lim_{x \to 2} g(x)$.

6. Suppose that f and g are continuous functions such that $\lim_{x \to 3} g(x) = 5$ and $f(3) = -2$. Find $\lim_{x \to 3} [f(x)/g(x)]$.

7. In each part sketch the graph of a function f that satisfies the stated conditions.

(a) f is continuous everywhere except at $x = 3$, at which point it is continuous from the right.

(b) f has a two-sided limit at $x = 3$, but it is not continuous at that point.

(c) f is not continuous at $x = 3$, but if its value at $x = 3$ is changed from $f(3) = 1$ to $f(3) = 0$, it becomes continuous at $x = 3$.

(d) f is continuous on the interval $[0, 3)$ and is defined on the closed interval $[0, 3]$; but f is not continuous on the interval $[0, 3]$.

8. Find formulas for some functions that are continuous on the intervals $(-\infty, 0)$ and $(0, +\infty)$, but are not continuous on the interval $(-\infty, +\infty)$.

9. A student parking lot at a university charges \$2.00 for the first half hour (or any part) and \$1.00 for each subsequent half hour (or any part) up to a daily maximum of \$10.00.

(a) Sketch a graph of cost as a function of the time parked.

(b) Discuss the significance of the discontinuities in the graph to a student who parks there.

10. In each part determine whether the function is continuous or not, and explain your reasoning.

(a) The Earth's population as a function of time

(b) Your exact height as a function of time

(c) The cost of a taxi ride in your city as a function of the distance traveled

(d) The volume of a melting ice cube as a function of time

In Exercises 11–22, find the points of discontinuity, if any.

11. $f(x) = x^3 - 2x + 3$

12. $f(x) = (x - 5)^{17}$

13. $f(x) = \dfrac{x}{x^2 + 1}$

14. $f(x) = \dfrac{x}{x^2 - 1}$

15. $f(x) = \dfrac{x - 4}{x^2 - 16}$

16. $f(x) = \dfrac{3x + 1}{x^2 + 7x - 2}$

17. $f(x) = \dfrac{x}{|x| - 3}$

18. $f(x) = \dfrac{5}{x} + \dfrac{2x}{x + 4}$

19. $f(x) = |x^3 - 2x^2|$

20. $f(x) = \dfrac{x + 3}{|x^2 + 3x|}$

21. $f(x) = \begin{cases} 2x + 3, & x \le 4 \\ 7 + \dfrac{16}{x}, & x > 4 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{3}{x - 1}, & x \ne 1 \\ 3, & x = 1 \end{cases}$

23. Find a value for the constant k, if possible, that will make the function continuous.

(a) $f(x) = \begin{cases} 7x - 2, & x \le 1 \\ kx^2, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$

24. On which of the following intervals is

$$f(x) = \frac{1}{\sqrt{x - 2}}$$

continuous?

(a) $[2, +\infty)$ (b) $(-\infty, +\infty)$

(c) $(2, +\infty)$ (d) $[1, 2)$

A function f is said to have a ***removable discontinuity*** at $x = c$ if $\lim_{x \to c} f(x)$ exists, but

$$f(c) \ne \lim_{x \to c} f(x)$$

either because $f(c)$ is undefined or the value of $f(c)$ differs from the value of the limit. This terminology will be needed in Exercises 25–28.

25. (a) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is undefined.

(b) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is defined.

26. (a) The terminology *removable discontinuity* is appropriate because a removable discontinuity of a function f at a point $x = c$ can be "removed" by redefining the value of f appropriately at $x = c$. What value for $f(c)$ removes the discontinuity?

(b) Show that the following functions have removable discontinuities at $x = 1$, and sketch their graphs.

$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x = 1 \\ 1, & x < 1 \end{cases}$$

(c) What values should be assigned to $f(1)$ and $g(1)$ to remove the discontinuities?

In Exercises 27 and 28, find the points of discontinuity, and determine whether the discontinuities are removable.

27. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$

(c) $f(x) = \dfrac{x - 2}{|x| - 2}$

28. (a) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$

(b) $f(x) = \begin{cases} 2x - 3, & x \le 2 \\ x^2, & x > 2 \end{cases}$

(c) $f(x) = \begin{cases} 3x^2 + 5, & x \ne 1 \\ 6, & x = 1 \end{cases}$

29. (a) Use a graphing utility to generate the graph of the function $f(x) = (x + 3)/(2x^2 + 5x - 3)$, and then use the graph to make a conjecture about the number and location of all discontinuities.

(b) Check your conjecture by factoring the denominator.

30. (a) Use a graphing utility to generate the graph of the function $f(x) = x/(x^3 - x + 2)$, and then use the graph to make a conjecture about the number and location of all discontinuities.

(b) Use the Intermediate-Value Theorem to approximate the location of all points of discontinuity to two decimal places.

31. Prove that $f(x) = x^{3/5}$ is continuous everywhere, carefully justifying each step.

32. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere, carefully justifying each step.

33. Let f and g be discontinuous at c. Give examples to show that

(a) $f + g$ can be continuous or discontinuous at c

(b) fg can be continuous or discontinuous at c.

34. Prove Theorem 2.4.4.

35. Prove:

(a) part (a) of Theorem 2.4.3

(b) part (b) of Theorem 2.4.3

(c) part (c) of Theorem 2.4.3.

36. Prove: If f and g are continuous on $[a, b]$, and $f(a) > g(a)$, $f(b) < g(b)$, then there is at least one solution of the equation $f(x) = g(x)$ in (a, b). [*Hint:* Consider $f(x) - g(x)$.]

37. Give an example of a function f that is defined at every point in a closed interval, and whose values at the endpoints have opposite signs, but for which the equation $f(x) = 0$ has no solution in the interval.

38. Use the Intermediate-Value Theorem to show that there is a square with a diagonal length that is between r and $2r$ and an area that is half the area of a circle of radius r.

39. Use the Intermediate-Value Theorem to show that there is a right circular cylinder of height h and radius less than r whose volume is equal to that of a right circular cone of height h and radius r.

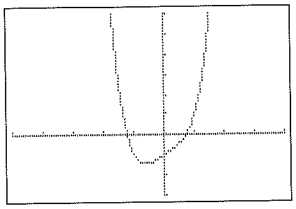
In Exercises 40 and 41, show that the equation has at least one solution in the given interval.

40. $x^3 - 4x + 1 = 0$; $[1, 2]$

41. $x^3 + x^2 - 2x = 1$; $[-1, 1]$

42. Prove: If $p(x)$ is a polynomial of odd degree, then the equation $p(x) = 0$ has at least one real solution.

43. The accompanying figure shows the graph of $y = x^4 + x - 1$. Use the method of Example 6 to approximate the x-intercepts with an error of at most 0.05.

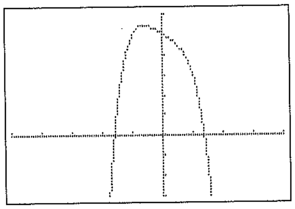


$[-5, 4] \times [-3, 6]$
$x\text{Scl} = 1, y\text{Scl} = 1$

Figure Ex-43

44. Use a graphing utility to solve the problem in Exercise 4 by zooming.

45. The accompanying figure shows the graph of $y = 5-x-x$ Use the method of Example 6 to approximate the roots the equation $5 - x - x^4 = 0$ to two decimal-place accurac



$[-5, 4] \times [-3, 6]$
$x\text{Scl} = 1, y\text{Scl} = 1$

Figure Ex-45

46. Use a graphing utility to solve the problem in Exercise by zooming.

47. Use the fact that $\sqrt{5}$ is a solution of $x^2 - 5 = 0$ to appro mate $\sqrt{5}$ with an error of at most 0.005.

48. Prove that if a and b are positive, then the equation

$$\frac{a}{x-1} + \frac{b}{x-3} = 0$$

has at least one solution in the interval $(1, 3)$.

49. A sphere of unknown radius x consists of a spherical c and a coating that is 1 cm thick (see the accompanying f ure). Given that the volume of the coating and the volume the core are the same, approximate the radius of the sph to three decimal-place accuracy.

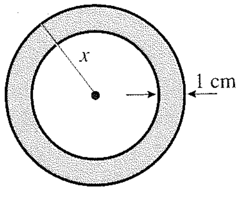


Figure Ex-49

50. A monk begins walking up a mountain road at 12:00 noon and reaches the top at 12:00 midnight. He meditates and rests until 12:00 noon the next day, at which time he begins walking down the same road, reaching the bottom at 12:00 midnight. Show that there is at least one point on the road that he reaches at the same time of day on the way up as on the way down.

51. Let f be defined at c. Prove that f is continuous at c if, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ if $|x - c| < \delta$.

2.5 LIMITS AND CONTINUITY OF TRIGONOMETRIC FUNCTIONS

In this section we will investigate the continuity properties of the trigonometric functions, and we will discuss some important limits involving these functions.

CONTINUITY OF TRIGONOMETRIC FUNCTIONS

Before we begin, recall that in the expressions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ it is understood that x is in radian measure.

In trigonometry, the graphs of $\sin x$ and $\cos x$ are drawn as continuous curves (Figure 2.5.1). To actually prove that these functions are continuous everywhere, we must show that the following equalities hold for every real number c:

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c \tag{1–2}$$

Although we will not formally prove these results, we can make them plausible by considering the behavior of the point $P(\cos x, \sin x)$ as it moves around the unit circle. For this purpose, view c as a fixed angle in radian measure, and let $Q(\cos c, \sin c)$ be the corresponding point on the unit circle. As $x \to c$ (i.e., as the angle x approaches the angle c), the point P moves along the circle toward Q, and this implies that the coordinates of P approach the corresponding coordinates of Q; that is, $\cos x \to \cos c$, and $\sin x \to \sin c$ (Figure 2.5.2).

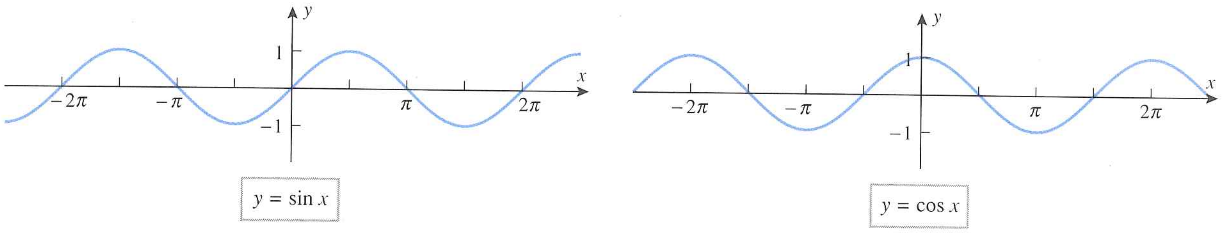


Figure 2.5.1

Formulas (1) and (2) can be used to find limits of the remaining trigonometric functions by expressing them in terms of $\sin x$ and $\cos x$; for example, if $\cos c \neq 0$, then

$$\lim_{x \to c} \tan x = \lim_{x \to c} \frac{\sin x}{\cos x} = \frac{\sin c}{\cos c} = \tan c$$

Thus, we are led to the following theorem.

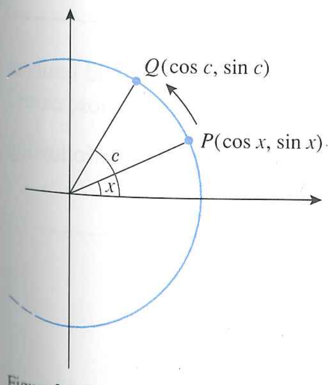


Figure 2.5.2

2.5.1 THEOREM. *If c is any number in the natural domain of the stated trigonometric function, then*

$$\lim_{x \to c} \sin x = \sin c \qquad \lim_{x \to c} \cos x = \cos c \qquad \lim_{x \to c} \tan x = \tan c$$

$$\lim_{x \to c} \csc x = \csc c \qquad \lim_{x \to c} \sec x = \sec c \qquad \lim_{x \to c} \cot x = \cot c$$

It follows from this theorem, for example, that $\sin x$ and $\cos x$ are continuous everywhere and that $\tan x$ is continuous, except at the points where it is undefined.

Example 1

Find the limit

$$\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right)$$

Solution. Recall from the last section that a limit symbol can be moved through a function sign if the function is continuous and the limit of the expression inside the function sign exists. Thus,

$$\lim_{x\to 1} \cos\left(\frac{x^2-1}{x-1}\right) = \lim_{x\to 1} \cos(x+1) = \cos(\lim_{x\to 1}(x+1)) = \cos 2$$ ◀

OBTAINING LIMITS BY SQUEEZING

In Section 2.1 we used the numerical evidence in Table 2.1.2 to *conjecture* that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad (3)$$

However, it is not a simple matter to establish this limit with certainty. The difficulty is that the numerator and denominator both approach zero as $x \to 0$; such limits are called ***indeterminate forms of type 0/0***. Sometimes indeterminate forms of this type can be established by manipulating the ratio algebraically (as in Example 7 of Section 2.2); but in this case no simple algebraic manipulation will work, so we must look for other methods.

The problem with indeterminate forms of type $0/0$ is that there are two conflicting influences at work: as the numerator approaches 0 it drives the magnitude of the ratio toward 0, and as the denominator approaches 0 it drives the magnitude of the ratio toward $\pm\infty$ (depending on the sign of the expression). The limiting behavior of the ratio is determined by the precise way in which these influences offset each other. Later in this text we will discuss general methods for attacking indeterminate forms, but for the limit in (3) we can use a method called *squeezing*.

In the method of squeezing one proves that a function f has a limit L at a point c by trapping the function between two other functions, g and h, whose limits at c are known to be L (Figure 2.5.3). This is the idea behind the following theorem, which we state without proof.

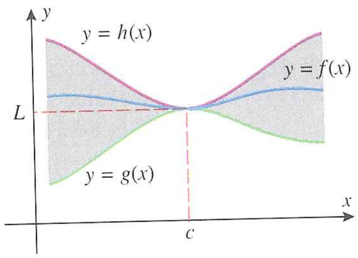


Figure 2.5.3

2.5.2 THEOREM (*The Squeezing Theorem*). *Let f, g, and h be functions satisfying*

$$g(x) \le f(x) \le h(x)$$

for all x in some open interval containing the point c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

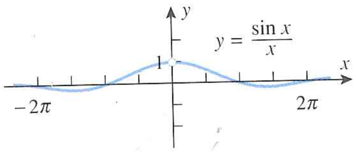
$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then f also has this limit as x approaches c, that is,

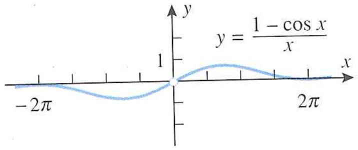
$$\lim_{x\to c} f(x) = L$$

FOR THE READER. The Squeezing Theorem also holds for one-sided limits and limits at $+\infty$ and $-\infty$. How do you think the hypotheses of the theorem would change in those cases?

The usefulness of the Squeezing Theorem will be evident in our proof of the following theorem (Figure 2.5.4).



$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$



$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Figure 2.5.4

2.5.3 THEOREM.

(*a*) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ (*b*) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

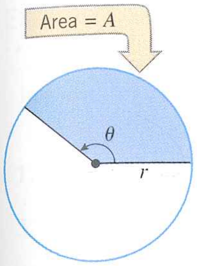


Figure 2.5.5

However, before giving the proof, it will be helpful to review the formula for the area A of a sector with radius r and a central angle of θ radians (Figure 2.5.5). The area of the sector can be derived by setting up the following proportion to the area of the entire circle:

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi} \qquad \left[\frac{\text{area of the sector}}{\text{area of the circle}} = \frac{\text{central angle of the sector}}{\text{central angle of the circle}}\right]$$

From this we obtain the formula

$$A = \tfrac{1}{2}r^2\theta \tag{4}$$

Now we are ready for the proof of Theorem 2.5.3.

Proof (a). In this proof we will interpret x as an angle in radian measure, and we will assume to start that $0 < x < \pi/2$. It follows from Formula (4) that the area of a sector of radius 1 and central angle x is $x/2$. Moreover, it is suggested by Figure 2.5.6 that the area of this sector lies between the areas of two triangles, one with area $(\tan x)/2$ and one with area $(\sin x)/2$. Thus,

$$\frac{\tan x}{2} \geq \frac{x}{2} \geq \frac{\sin x}{2}$$

Multiplying through by $2/(\sin x)$ yields

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1$$

and then taking reciprocals and reversing the inequalities yields

$$\cos x \leq \frac{\sin x}{x} \leq 1 \tag{5}$$

Moreover, these inequalities also hold for $-\pi/2 < x < 0$, since replacing x by $-x$ in (5) and using the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ leaves the inequalities unchanged (verify). Finally, since the functions $\cos x$ and 1 both have limits of 1 as $x \to 0$, it follows from the Squeezing Theorem that $(\sin x)/x$ also has a limit of 1 as $x \to 0$.

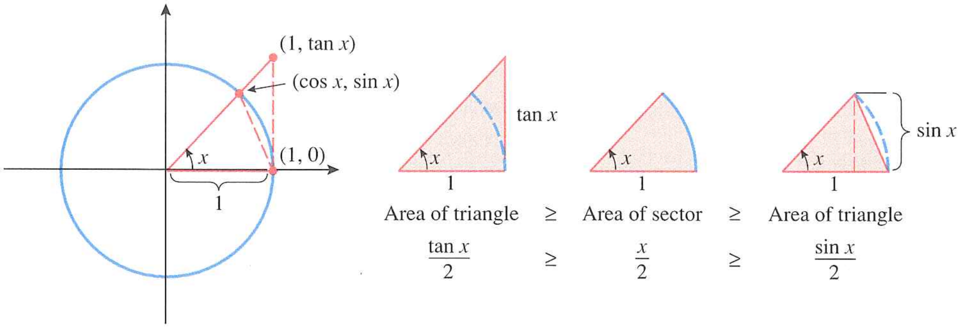


Figure 2.5.6

Proof (b). For this proof we will use the limit in part (*a*), the continuity of the sine function, and the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. We obtain

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\left[\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0}\frac{\sin^2 x}{(1+\cos x)x}$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right) = (1)\left(\frac{0}{1+1}\right) = 0$$

■

Example 2

Find

$$\text{(a)}\ \lim_{x\to 0}\frac{\tan x}{x}\qquad \text{(b)}\ \lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}\qquad \text{(c)}\ \lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$$

Solution (a).

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)=(1)(1)=1$$

Solution (b). The trick is to multiply and divide by 2, which will make the denominator the same as the argument of the sine function [just as in Theorem 2.5.3(*a*)]:

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=\lim_{\theta\to 0}2\cdot\frac{\sin 2\theta}{2\theta}=2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}$$

Now make the substitution $x=2\theta$, and use the fact that $x\to 0$ as $\theta\to 0$. This yields

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}=2\lim_{x\to 0}\frac{\sin x}{x}=2(1)=2$$

Solution (c).

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim_{x\to 0}\frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 5x}{x}}=\lim_{x\to 0}\frac{3\cdot\dfrac{\sin 3x}{3x}}{5\cdot\dfrac{\sin 5x}{5x}}=\frac{3\cdot 1}{5\cdot 1}=\frac{3}{5}$$

◀

FOR THE READER. Use a graphing utility to confirm the limits in the last example graphically, and if you have a CAS, then use it to obtain the limits.

Example 3

Make conjectures about the limits

$$\text{(a)}\ \lim_{x\to 0}\sin\left(\frac{1}{x}\right)\qquad \text{(b)}\ \lim_{x\to 0}x\sin\left(\frac{1}{x}\right)$$

and confirm your conclusions by generating the graphs of the functions near $x=0$ using a graphing utility.

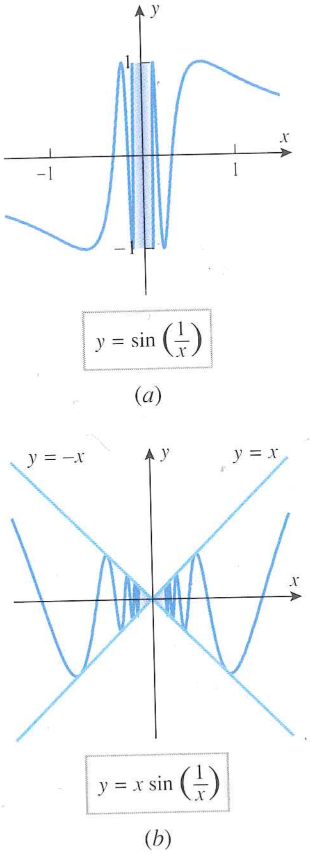
Solution (a). Since $1/x\to+\infty$ as $x\to 0^+$, we can view $\sin(1/x)$ as the sine of an angle that increases indefinitely as $x\to 0^+$. As this angle increases, the function $\sin(1/x)$ keeps oscillating between -1 and 1 without approaching a limit. Similarly, there is no limit from the left since $1/x\to-\infty$ as $x\to 0^-$. These conclusions are consistent with the graph of $y=\sin(1/x)$ shown in Figure 2.5.7*a*. Observe that the oscillations become more and more rapid as x approaches 0 because $1/x$ increases (or decreases) more and more rapidly as x approaches 0.

Solution (b). The values of $x\sin(1/x)$ oscillate between x and $-x$, both of which approach 0 as x approaches 0. Thus, the Squeezing Theorem suggests that $x\sin(1/x)\to 0$ as $x\to 0$. This is consistent with Figure 2.5.7*b*. ◀

REMARK. It follows from part (b) of this example that the function

$$f(x)=\begin{cases}x\sin(1/x), & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous at $x=0$, since the value of the function and the value of the limit are the same at that point. This shows that the behavior of a function can be very complex at a point of continuity.



(*a*)

(*b*)

Figure 2.5.7

EXERCISE SET 2.5 Graphing Calculator C CAS

In Exercises 1–10, find the points of discontinuity, if any.

1. $f(x) = \sin(x^2 - 2)$
2. $f(x) = \cos\left(\dfrac{x}{x-\pi}\right)$
3. $f(x) = \cot x$
4. $f(x) = \sec x$
5. $f(x) = \csc x$
6. $f(x) = \dfrac{1}{1+\sin^2 x}$
7. $f(x) = |\cos x|$
8. $f(x) = \sqrt{2+\tan^2 x}$
9. $f(x) = \dfrac{1}{1-2\sin x}$
10. $f(x) = \dfrac{3}{5+2\cos x}$

11. Use Theorem 2.4.6 to show that the following functions are continuous everywhere by expressing them as compositions of simpler functions that are known to be continuous.
(a) $\sin(x^3+7x+1)$ (b) $|\sin x|$
(c) $\cos^3(x+1)$ (d) $\sqrt{3+\sin 2x}$
(e) $\sin(\sin x)$ (f) $\cos^5 x - 2\cos^3 x + 1$

12. (a) Prove that if $g(x)$ is continuous everywhere, then so are $\sin(g(x))$, $\cos(g(x))$, $g(\sin(x))$, and $g(\cos(x))$.
(b) Illustrate the result in part (a) with some of your own choices for g.

Find the limits in Exercises 13–35.

13. $\lim_{x\to+\infty} \cos\left(\dfrac{1}{x}\right)$
14. $\lim_{x\to+\infty} \sin\left(\dfrac{2}{x}\right)$
15. $\lim_{x\to+\infty} \sin\left(\dfrac{\pi x}{2-3x}\right)$
16. $\lim_{h\to 0} \dfrac{\sin h}{2h}$
17. $\lim_{\theta\to 0} \dfrac{\sin 3\theta}{\theta}$
18. $\lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta^2}$
19. $\lim_{x\to 0^-} \dfrac{\sin x}{|x|}$
20. $\lim_{x\to 0} \dfrac{\sin^2 x}{3x^2}$
21. $\lim_{x\to 0^+} \dfrac{\sin x}{5\sqrt{x}}$
22. $\lim_{x\to 0} \dfrac{\sin 6x}{\sin 8x}$
23. $\lim_{x\to 0} \dfrac{\tan 7x}{\sin 3x}$
24. $\lim_{\theta\to 0} \dfrac{\sin^2\theta}{\theta}$
25. $\lim_{h\to 0} \dfrac{h}{\tan h}$
26. $\lim_{h\to 0} \dfrac{\sin h}{1-\cos h}$
27. $\lim_{\theta\to 0} \dfrac{\theta^2}{1-\cos\theta}$
28. $\lim_{x\to 0} \dfrac{x}{\cos\left(\frac{1}{2}\pi - x\right)}$
29. $\lim_{\theta\to 0} \dfrac{\theta}{\cos\theta}$
30. $\lim_{t\to 0} \dfrac{t^2}{1-\cos^2 t}$
31. $\lim_{h\to 0} \dfrac{1-\cos 5h}{\cos 7h - 1}$
32. $\lim_{x\to 0^+} \sin\left(\dfrac{1}{x}\right)$
33. $\lim_{x\to 0^+} \cos\left(\dfrac{1}{x}\right)$
34. $\lim_{x\to 0} \dfrac{x^2 - 3\sin x}{x}$
35. $\lim_{x\to 0} \dfrac{2x+\sin x}{x}$

36. Find a value for the constant k that makes

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

continuous at $x = 0$.

37. Find a nonzero value for the constant k that makes

$$f(x) = \begin{cases} \dfrac{\tan kx}{x}, & x < 0 \\ 3x + 2k^2, & x \geq 0 \end{cases}$$

continuous at $x = 0$.

38. Is

$$f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

continuous at $x = 0$?

39. In each part, find the limit by making the indicated substitution.

(a) $\lim_{x\to+\infty} x\sin\dfrac{1}{x}$. $\left[\textit{Hint: } \text{Let } t = \dfrac{1}{x}.\right]$

(b) $\lim_{x\to-\infty} x\left(1-\cos\dfrac{1}{x}\right)$. $\left[\textit{Hint: } \text{Let } t = \dfrac{1}{x}.\right]$

(c) $\lim_{x\to\pi} \dfrac{\pi - x}{\sin x}$. [*Hint:* Let $t = \pi - x$.]

40. Find $\lim_{x\to 2} \dfrac{\cos(\pi/x)}{x-2}$. $\left[\textit{Hint: } \text{Let } t = \dfrac{\pi}{2} - \dfrac{\pi}{x}.\right]$

41. Find $\lim_{x\to 1} \dfrac{\sin(\pi x)}{x-1}$.

42. Find $\lim_{x\to\pi/4} \dfrac{\tan x - 1}{x - \pi/4}$.

43. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x\cos\frac{50\pi}{x} = 0$$

and illustrate the principle involved by using a graphing utility to graph $y = x$, $y = -x$, and $y = x\cos(50\pi/x)$ on the same screen over the x-interval from -1 to 1.

44. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x^2 \sin\left(\frac{50\pi}{\sqrt[3]{x}}\right) = 0$$

and illustrate the principle involved by using a graphing utility to graph $y = x^2$, $y = -x^2$, and $y = x^2\sin(50\pi/\sqrt[3]{x})$ on the same screen over the x-interval from -0.5 to 0.5.

45. Sketch the graphs of $y = 1 - x^2$, $y = \cos x$, and $y = f(x)$, where f is any continuous function that satisfies the inequalities

$$1 - x^2 \leq f(x) \leq \cos x$$

for all x in the interval $(-\pi/2, \pi/2)$. What can you say about the limit of $f(x)$ as $x \to 0$? Explain your reasoning.

46. Sketch the graphs of $y = 1/x$, $y = -1/x$, and $y = f(x)$ in one coordinate system, where f is any continuous function that satisfies the inequalities

$$-\frac{1}{x} \le f(x) \le \frac{1}{x}$$

for all x in the interval $[1, +\infty)$. What can you say about the limit of $f(x)$ as $x \to +\infty$? Explain your reasoning.

47. Find formulas for functions g and h such that $g(x) \to 0$ and $h(x) \to 0$ as $x \to +\infty$ and such that

$$g(x) \le \frac{\sin x}{x} \le h(x)$$

for positive values of x. What can you say about the limit

$$\lim_{x \to +\infty} \frac{\sin x}{x}?$$

Explain your reasoning.

48. Draw pictures analogous to Figure 2.5.3 that illustrate the Squeezing Theorem for limits of the form $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

> Recall that unless stated otherwise the variable x in trigonometric functions such as $\sin x$ and $\cos x$ are assumed to be in radian measure. The limits in Theorem 2.5.3 are based on that assumption. Exercises 49 and 50 explore what happens to those limits if degree measure is used for x.

49. (a) Show that if x is in degrees, then

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$$

(b) Confirm that the limit in part (a) is consistent with the results produced by your calculating utility by setting the utility to degree measure and calculating $(\sin x)/x$ for some values of x that get closer and closer to 0.

50. What is the limit of $(1 - \cos x)/x$ as $x \to 0$ if x is in degrees?

51. It follows from part (*a*) of Theorem 2.5.3 that if θ is small (near zero) and measured in radians, then one should expect the approximation

$$\sin\theta \approx \theta$$

to be good.

(a) Find sin 10° using a calculating utility.

(b) Find sin 10° using the approximation above.

52. (a) Use the approximation of $\sin\theta$ that is given in Exercise 51 together with the identity $\cos 2\alpha = 1 - 2\sin^2\alpha$ with $\alpha = \theta/2$ to show that if θ is small (near zero) and measured in radians, then one should expect the approximation

$$\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

to be good.

(b) Find cos 10° using a calculating utility.

(c) Find cos 10° using the approximation above.

53. It follows from part (a) of Example 2 that if θ is small (near zero) and measured in radians, then one should expect the approximation

$$\tan\theta \approx \theta$$

to be good.

(a) Find tan 5° using a calculating utility.

(b) Find tan 5° using the approximation above.

54. Referring to the accompanying figure, suppose that the angle of elevation of the top of a building, as measured from a point L feet from its base, is found to be α degrees.

(a) Use the relationship $h = L\tan\alpha$ to calculate the height of a building for which $L = 500$ ft and $\alpha = 6°$.

(b) Show that if L is large compared to the building height h, then one should expect good results in approximating h by $h \approx \pi L\alpha/180$.

(c) Use the result in part (b) to approximate the building height h in part (a).

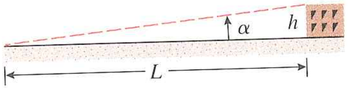


Figure Ex-54

55. (a) Use the Intermediate-Value Theorem to show that the equation $x = \cos x$ has at least one solution in the interval $[0, \pi/2]$.

(b) Show graphically that there is exactly one solution in the interval.

(c) Approximate the solution to three decimal places.

56. (a) Use the Intermediate-Value Theorem to show that the equation $x + \sin x = 1$ has at least one solution in the interval $[0, \pi/6]$.

(b) Show graphically that there is exactly one solution in the interval.

(c) Approximate the solution to three decimal places.

57. In the study of falling objects near the surface of the Earth, the ***acceleration g due to gravity*** is commonly taken to be 9.8 m/s^2 or 32 ft/s^2. However, the elliptical shape of the Earth and other factors cause variations in this constant that are latitude dependent. The following formula, known as the Geodetic Reference Formula of 1967, is commonly used to predict the value of g at a latitude of ϕ degrees (either north or south of the equator):

$$g = 9.7803185(1.0 + 0.005278895\sin^2\phi - 0.000023462\sin^4\phi)\ \text{m/s}^2$$

(a) Observe that g is an even function of ϕ. What does this suggest about the shape of the Earth, as modeled by the Geodetic Reference Formula?

(b) Show that $g = 9.8$ m/s^2 somewhere between latitudes of 38° and 39°.

58. Let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

(a) Make a conjecture about the limit of $f(x)$ as $x \to 0$.

(b) Make a conjecture about the limit of $xf(x)$ as $x \to 0$.

(c) Prove your conjectures.

SUPPLEMENTARY EXERCISES

1. For the function f graphed in the accompanying figure, find the limit if it exists.

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$ (c) $\lim_{x\to 3} f(x)$

(d) $\lim_{x\to 4} f(x)$ (e) $\lim_{x\to +\infty} f(x)$ (f) $\lim_{x\to -\infty} f(x)$

(g) $\lim_{x\to 3^+} f(x)$ (h) $\lim_{x\to 3^-} f(x)$ (i) $\lim_{x\to 0} f(x)$

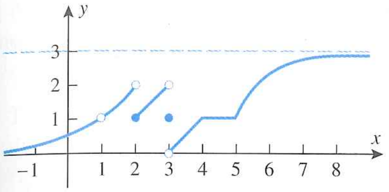


Figure Ex-1

2. (a) Find a formula for a rational function that has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.

(b) Check your work by using a graphing utility to graph the function.

3. (a) Write a paragraph or two that describes how the limit of a function can fail to exist at a point $x = a$. Accompany your description with some specific examples.

(b) Write a paragraph or two that describes how the limit of a function can fail to exist as $x \to +\infty$ or $x \to -\infty$. Also, accompany your description with some specific examples.

(c) Write a paragraph or two that describes how a function can fail to be continuous at a point $x = a$. Accompany your description with some specific examples.

4. Show that the Intermediate-Value Theorem is false if f is not continuous on the interval $[a, b]$.

5. In each part, evaluate the function for the stated values of x, and make a conjecture about the value of the limit. Confirm your conjecture by finding the limit algebraically.

(a) $f(x) = \dfrac{x-2}{x^2-4}$; $\lim_{x\to 2^+} f(x)$; $x = 2.5, 2.1, 2.01, 2.001, 2.0001, 2.00001$

(b) $f(x) = \dfrac{\tan 4x}{x}$; $\lim_{x\to 0} f(x)$; $x = \pm 1.0, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001, \pm 0.00001$

6. In each part, find the horizontal asymptotes, if any.

(a) $y = \dfrac{2x-7}{x^2-4x}$ (b) $y = \dfrac{x^3-x^2+10}{3x^2-4x}$

(c) $y = \dfrac{2x^2-6}{x^2+5x}$

7. (a) Approximate the value for the limit

$$\lim_{x\to 0} \frac{3^x - 2^x}{x}$$

to three decimal places by constructing an appropriate table of values.

(b) Confirm your approximation using graphical evidence.

8. According to Ohm's law, when a voltage of V volts is applied across a resistor with a resistance of R ohms, a current of $I = V/R$ amperes flows through the resistor.

(a) How much current flows if a voltage of 3.0 volts is applied across a resistance of 7.5 ohms?

(b) If the resistance varies by ± 0.1 ohm, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?

(c) If temperature variations cause the resistance to vary by $\pm\delta$ from its value of 7.5 ohms, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?

(d) If the current is not allowed to vary by more than $\epsilon = \pm 0.001$ ampere at a voltage of 3.0 volts, what variation of $\pm\delta$ from the value of 7.5 ohms is allowable?

(e) Certain alloys become ***superconductors*** as their temperature approaches absolute zero ($-273°$C), meaning that their resistance approaches zero. If the voltage remains constant, what happens to the current in a superconductor as $R \to 0^+$?

9. Suppose that f is continuous on the interval $[0, 1]$ and that $0 \le f(x) \le 1$ for all x in this interval.

(a) Sketch the graph of $y = x$ together with a possible graph for f over the interval $[0, 1]$.

(b) Use the Intermediate-Value Theorem to help prove that there is at least one number c in the interval $[0, 1]$ such that $f(c) = c$.

10. Use algebraic methods to find

(a) $\lim_{\theta\to 0} \tan\left(\dfrac{1-\cos\theta}{\theta}\right)$ (b) $\lim_{t\to 1} \dfrac{t-1}{\sqrt{t}-1}$

(c) $\lim_{x\to +\infty} \dfrac{(2x-1)^5}{(3x^2+2x-7)(x^3-9x)}$

(d) $\lim_{\theta\to 0} \cos\left(\dfrac{\sin(\theta+\pi)}{2\theta}\right)$.

11. Suppose that f is continuous on the interval $[0, 1]$, that $f(0) = 2$, and that f has no zeros in the interval. Prove that $f(x) > 0$ for all x in $[0, 1]$.

12. Suppose that

$$f(x) = \begin{cases} -x^4 + 3, & x \le 2 \\ x^2 + 9, & x > 2 \end{cases}$$

Is f continuous everywhere? Justify your conclusion.

13. Show that the equation $x^4 + 5x^3 + 5x - 1 = 0$ has at least two real solutions in the interval $[-6, 2]$.

14. Use the Intermediate-Value Theorem to approximate $\sqrt{11}$ to three decimal places, and check your answer by finding the root directly with a calculating utility.

15. Suppose that f is continuous and that $f(x_0) > 0$. Give either a δ-ϵ proof or a convincing verbal argument to show that there must be an open interval containing x_0 on which $f(x) > 0$.

16. Sketch the graph of $f(x) = |x^2 - 4|/(x^2 - 4)$.

17. In each part, approximate the points of discontinuity of f to three decimal places.

(a) $f(x) = \dfrac{\sqrt{x+1}}{x^2 + 2x - 5}$

(b) $f(x) = \dfrac{x+3}{|2\sin x - x|}$

18. In Example 3 of Section 2.5 we used the Squeezing Theorem to prove that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Why couldn't we have obtained the same result by writing

$$\begin{aligned}\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) &= \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right)\\ &= 0 \cdot \lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0?\end{aligned}$$

In Exercises 19 and 20, find $\lim_{x\to a} f(x)$, if it exists, for $a = 0,\ 5^+,\ -5^-,\ -5,\ 5,\ -\infty,\ +\infty$

19. (a) $f(x) = \sqrt{5-x}$ (b) $f(x) = (x^2 - 25)/(x-5)$

20. (a) $f(x) = (x+5)/(x^2 - 25)$

(b) $f(x) = \begin{cases} (x-5)/|x-5|, & x \neq 5 \\ 0, & x = 5 \end{cases}$

In Exercises 21–28, find the indicated limit, if it exists.

21. $\lim_{x\to 0} \dfrac{\tan ax}{\sin bx}$ $(a \neq 0, b \neq 0)$

22. $\lim_{x\to 0} \dfrac{\sin 3x}{\tan 3x}$

23. $\lim_{\theta\to 0} \dfrac{\sin 2\theta}{\theta^2}$

24. $\lim_{x\to 0} \dfrac{x \sin x}{1 - \cos x}$

25. $\lim_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}}$

26. $\lim_{x\to 0} \dfrac{\sin^2(kx)}{x^2},\quad k \neq 0$

27. $\lim_{x\to 0} \dfrac{3x - \sin(kx)}{x},\quad k \neq 0$

28. $\lim_{x\to +\infty} \dfrac{2x + x\sin 3x}{5x^2 - 2x + 1}$

29. The author's dictionary describes a continuous function as "one whose value at each point is closely approached by its values at neighboring points."

(a) How would you explain the meaning of the terms "neighboring points" and "closely approached" to a nonmathematician?

(b) Write a paragraph that explains why the dictionary definition is consistent with the definition given in the text.

30. (a) Show by rationalizing the numerator that

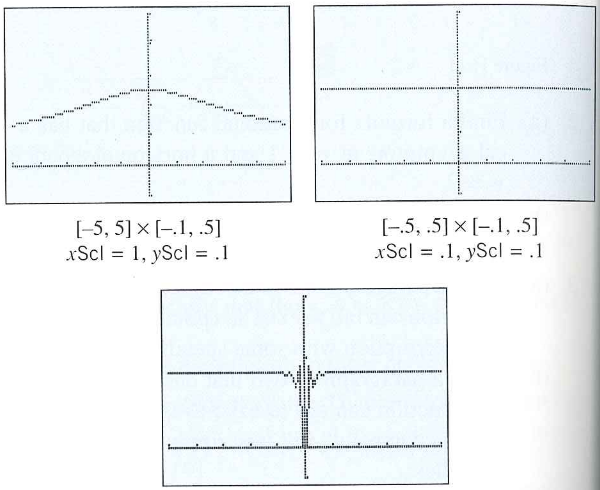
$$\lim_{x\to 0} \frac{\sqrt{x^2+4} - 2}{x^2} = \frac{1}{4}$$

(b) Evaluate $f(x)$ for

$$x = \pm 1.0, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001, \pm 0.00001$$

and explain why the values are not getting closer and closer to the limit.

(c) The accompanying figure shows the graph of f generated with a graphing utility and zooming in on the origin. Explain what is happening.



$[-5, 5] \times [-.1, .5]$
xScl = 1, yScl = .1

$[-.5, .5] \times [-.1, .5]$
xScl = .1, yScl = .1

$[-5 \times 10^{-6}, 5 \times 10^{-6}] \times [-.1, .5]$
xScl = 10^{-6}, yScl = .1

Figure Ex-30

In Exercises 31–36, approximate the limit of the function by looking at its graph and calculating values for some appropriate choices of x. Compare your answer with the value produced by a CAS.

C **31.** $\lim_{x\to 0} (1+x)^{1/x}$

C **32.** $\lim_{x\to 3} \dfrac{2^x - 8}{x - 3}$

C **33.** $\lim_{x\to 1} \dfrac{\sin x - \sin 1}{x - 1}$

C **34.** $\lim_{x\to 0^+} x^{-2}(1.001)^{-1/x}$

C **35.** $\lim_{x\to +\infty} \left(\sqrt{x + \sqrt{x}} - \sqrt{x}\right)$

C **36.** $\lim_{x\to +\infty} (3^x + 5^x)^{1/x}$

37. The limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

ensures that there is a number δ such that

$$\left|\frac{\sin x}{x} - 1\right| < 0.001$$

if $0 < |x| < \delta$. Estimate the largest such δ.

38. If \$1000 is invested in an account that pays 7% interest compounded n times each year, then in 10 years there will be $1000(1 + 0.07/n)^{10n}$ dollars in the account. How much money will be in the account in 10 years if the interest is compounded quarterly ($n = 4$)? Monthly ($n = 12$)? Daily ($n = 365$)? How much money will be in the account in 10 years if the interest is compounded *continuously*, that is, as $n \to +\infty$?

39. There are various numerical methods other than the method discussed in Section 2.4 to obtain approximate solutions of equations of the form $f(x) = 0$. One such method requires that the equation be expressed in the form $x = g(x)$, so that a solution $x = c$ can be interpreted as the value of x where the line $y = x$ intersects the curve $y = g(x)$, as shown in the accompanying figure. If x_1 is an initial estimate of c and the graph of $y = g(x)$ is not too steep in the vicinity of c, then a better approximation can be obtained from $x_2 = g(x_1)$ (see the figure). An even better approximation is obtained from $x_3 = g(x_2)$, and so forth. The formula $x_{n+1} = g(x_n)$ for $n = 1, 2, 3, \ldots$ generates successive approximations $x_2, x_3, x_4, \ldots$ that get closer and closer to c.

(a) The equation $x^3 - x - 1 = 0$ has only one real solution. Show that this equation can be written as

$$x = g(x) = \sqrt[3]{x + 1}$$

(b) Graph $y = x$ and $y = g(x)$ in the same coordinate system for $-1 \le x \le 3$.

(c) Starting with an arbitrary point x_1, make a sketch that shows the location of the successive iterates

$$x_2 = g(x_1), \quad x_3 = g(x_2), \ldots$$

(d) Use $x_1 = 1$ and calculate $x_2, x_3, \ldots$, continuing until you obtain two consecutive values that differ by less than 10^{-4}. Experiment with other starting values such as $x_1 = 2$ or $x_1 = 1.5$.

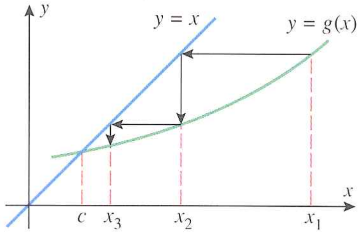


Figure Ex-39

40. The method described in Exercise 39 will not always work.

(a) The equation $x^3 - x - 1 = 0$ can be expressed as $x = g(x) = x^3 - 1$. Graph $y = x$ and $y = g(x)$ in the same coordinate system. Starting with an arbitrary point x_1, make a sketch illustrating the location of the successive iterates $x_2 = g(x_1)$, $x_3 = g(x_2), \ldots$.

(b) Use $x_1 = 1$ and calculate the successive iterates x_n for $n = 2, 3, 4, 5, 6$.

In Exercises 41 and 42, use the method of Exercise 39 to approximate the roots of the equation.

41. $x^5 - x - 2 = 0$

42. $x - \cos x = 0$

EXPANDING THE CALCULUS HORIZON

For additional material relating to this chapter, visit the Anton Website at http://www.wiley.com/college/anton