René Descartes

1

FUNCTIONS

One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a *function*, which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus, and we will examine some familiar ideas involving lines, polynomials, and trigonometric functions from viewpoints that may be new. We will also discuss ideas relating to the use of graphing utilities such as graphing calculators and graphing software for computers. Before you start reading, you may want to scan through the appendices, since they contain various kinds of precalculus material that may be helpful if you need to review some of those ideas.

1.1 FUNCTIONS AND THE ANALYSIS OF GRAPHICAL INFORMATION

In this section we will define and develop the concept of a function. Functions are used by mathematicians and scientists to describe the relationships between variable quantities and hence play a central role in calculus and its applications.

SCATTER PLOTS AND TABULAR DATA

Many scientific laws are discovered by collecting, organizing, and analyzing experimental data. Since graphs play a major role in studying data, we will begin by discussing the kinds of information that a graph can convey.

To start, we will focus on paired data. For example, Table 1.1.1 shows the top qualifying speed by year in the Indianapolis 500 auto race from 1975 to 1994. This table pairs up each year t between 1975 and 1994 with the top qualifying speed S for that year. This paired data can be represented graphically in a number of ways:

Table 1.1.1

INDIANAPOLIS 500 QUALIFYING SPEEDS

YEAR t	SPEED S (mi/h)
1975	193.976
1976	188.957
1977	198.884
1978	202.156
1979	193.736
1980	192.256
1981	200.546
1982	207.004
1983	207.395
1984	210.029
1985	212.583
1986	216.828
1987	215.390
1988	219.198
1989	223.885
1990	225.301
1991	224.113
1992	232.482
1993	223.967
1994	228.011

- One possibility is to plot the paired data points in a rectangular tS-coordinate system (t horizontal and S vertical), in which case we obtain a ***scatter plot*** of S versus t (Figure 1.1.1a).
- A second possibility is to enhance the scatter plot visually by joining successive points with straight-line segments, in which case we obtain a ***line graph*** (Figure 1.1.1b).
- A third possibility is to represent the paired data by a ***bar graph*** (Figure 1.1.1c).

All three graphical representations reveal an upward trend in the data, as one would expect with improvements in automotive technology.

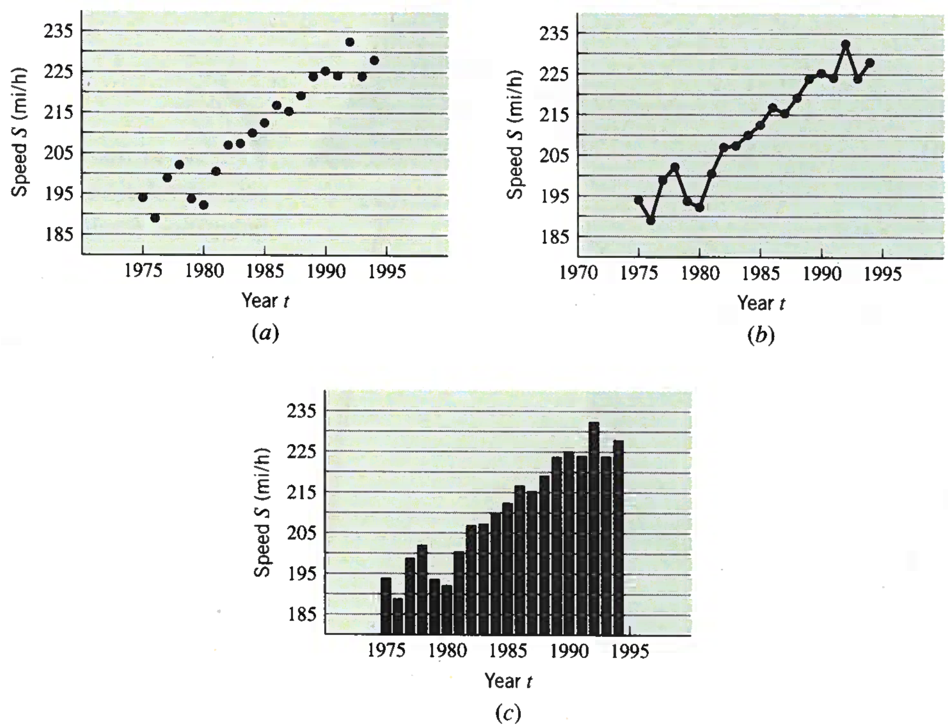
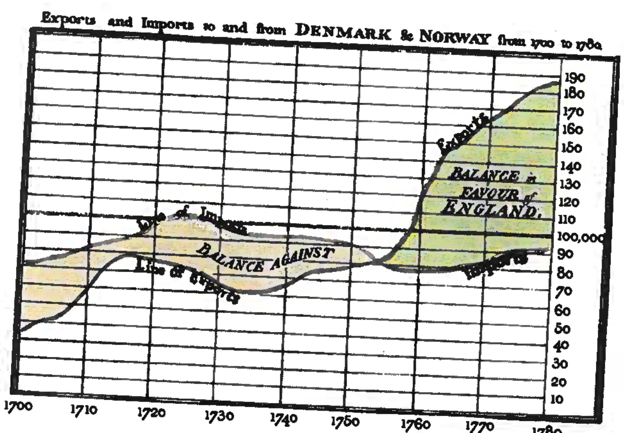


Figure 1.1.1

EXTRACTING INFORMATION FROM GRAPHS

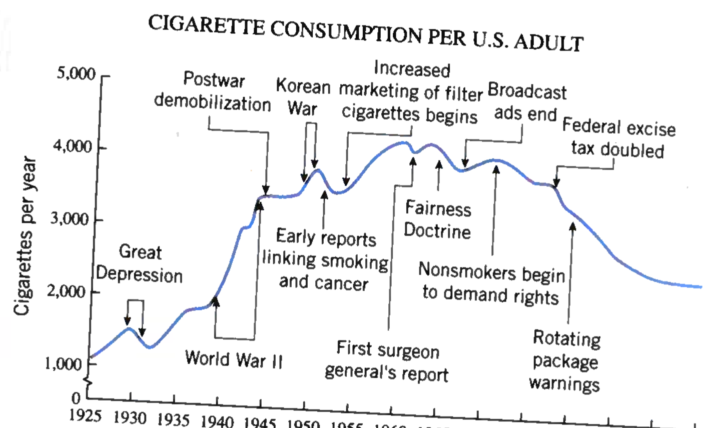
One of the first books to use graphs for representing numerical data was *The Commercial and Political Atlas*, published in 1786 by the Scottish political economist William Playfair (1759–1823). Figure 1.1.2a shows an engraving from that work that compares exports and imports by England to Denmark and Norway (combined). In spite of its antiquity, the engraving is modern in spirit and provides a wealth of information. You should be able to extract the following information from Playfair's graphs:

- In the year 1700 imports were valued at about 70,000 pounds and exports at about 35,000 pounds.
- During the period from 1700 to about 1754 imports exceeded exports (a trade deficit for England).
- In the year 1754 the imports and exports were equal (a trade balance in today's economic terminology).
- From 1754 to 1780 exports exceeded imports (a trade surplus for England). The greatest surplus occurred in 1780, at which time exports exceeded imports by about 95,000 pounds.
- During the period from 1700 to 1725 imports were rising. They peaked in 1725, and then slowly fell until about 1760, at which time they bottomed out and began to rise again slowly until 1780.
- During the period from 1760 to 1780 exports and imports were both rising, but exports were rising more rapidly than imports, resulting in an ever-widening trade surplus for England.



Playfair's Graph of 1786: The horizontal scale is in years from 1700 to 1780 and the vertical scale is in units of 1,000 pounds sterling from 0 to 200.

(*a*)



Source: U.S. Department of Health and Human Services.

(*b*)

Figure 1.1.2

Figure 1.1.2*b* is a more contemporary graph; it describes the per capita consumption of cigarettes in the United States between 1925 and 1995.

FOR THE READER. Use the graph in Figure 1.1.2*b* to provide reasonable answers to the following questions:

- When did the maximum annual cigarette consumption per adult occur and how many were consumed?
- What factors are likely to cause sharp decreases in cigarette consumption?
- What factors are likely to cause sharp increases in cigarette consumption?
- What were the long- and short-term effects of the first surgeon general's report on the health risks of smoking?

GRAPHS OF EQUATIONS

Graphs can be used to describe mathematical equations as well as physical data. For example, consider the equation

$$y = x\sqrt{9 - x^2} \tag{1}$$

For each value of x in the interval $-3 \leq x \leq 3$, this equation produces a corresponding real value of y, which is obtained by substituting the value of x into the right side of the equation. Some typical values are shown in Table 1.1.2.

Table 1.1.2

x	-3	-2	-1	0	1	2	3
y	0	$-2\sqrt{5} \approx -4.47214$	$-2\sqrt{2} \approx -2.82843$	0	$2\sqrt{2} \approx 2.82843$	$2\sqrt{5} \approx 4.47214$	0

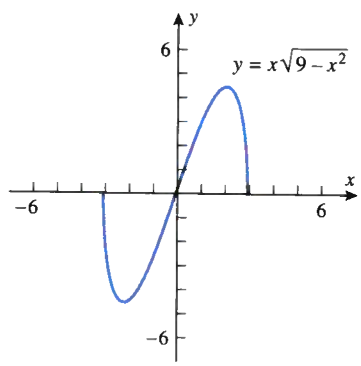


Figure 1.1.3

The set of *all* points in the xy-plane whose coordinates satisfy an equation in x and y is called the ***graph*** of that equation in the xy-plane. Figure 1.1.3 shows the graph of Equation (1) in the xy-plane. Notice that the graph extends only over the interval $[-3, 3]$. This is because values of x outside of this interval produce complex values of y, and in these cases the ordered pairs (x, y) do not correspond to points in the xy-plane. For example, if $x = 8$, then the corresponding value of y is $y = 8\sqrt{-55} = 8\sqrt{55}\,i$, and the ordered pair $(8, 8\sqrt{55}\,i)$ is not a point in the xy-plane.

Example 1

Figure 1.1.4 shows the graph of an unspecified equation that was used to obtain the values that appear in the shaded parts of the accompanying tables. Examine the graph and confirm that the values in the tables are reasonable approximations. ◀

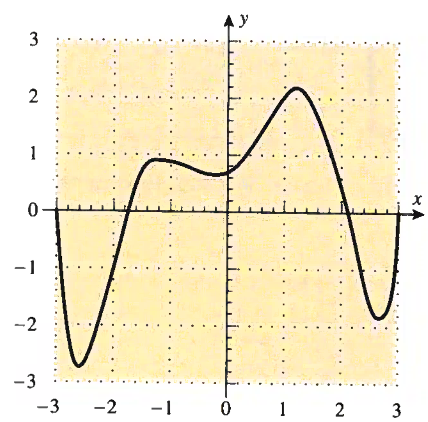


Figure 1.1.4

x	y
-3	0
-2	-1
-1	0.9
0	0.7
1	2
2	0.4
3	0

x	y
None	-3
$-2.3, -2.8$	-2
$-2, -2.9, 2.4, 2.9$	-1
$-3, -1.7, 2.1, 3$	0
0.3, 1.8	1
1, 1.4	2
None	3

FUNCTIONS

Tables, graphs, and equations provide three methods for describing how one quantity depends on another—numerical, visual, and algebraic. The fundamental importance of this idea was recognized by Leibniz in 1673 when he coined the term *function* to describe the dependence of one quantity on another. The following examples illustrate how this term is used:

- The area A of a circle depends on its radius r by the equation $A = \pi r^2$, so we say that *A is a function of r*.

- The velocity v of a ball falling freely in the Earth's gravitational field increases with time t until it hits the ground, so we say that *v is a function of t*.
- In a bacteria culture, the number n of bacteria present after 1 hour of growth depends on the number n_0 of bacteria present initially, so we say that *n is a function of n_0*.

This idea is captured in the following definition.

1.1.1 DEFINITION. If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that ***y is a function of x***.

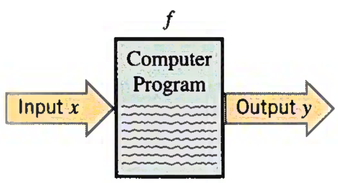


Figure 1.1.5

In the mid-eighteenth century the Swiss mathematician Leonhard Euler* (pronounced "oiler") conceived the idea of denoting functions by letters of the alphabet, thereby making it possible to describe functions without stating specific formulas, graphs, or tables. To understand Euler's idea, think of a function as a computer program that takes an *input* x, operates on it in some way, and produces exactly one *output* y. The computer program is an object in its own right, so we can give it a name, say f. Thus, the function f (the computer program) associates a unique output y with each input x (Figure 1.1.5). This suggests the following definition.

1.1.2 DEFINITION. A ***function*** f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$ (read "f of x").

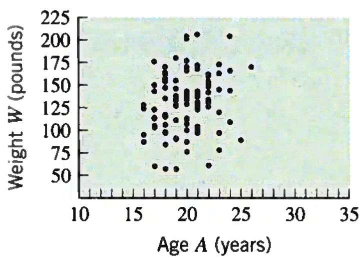


Figure 1.1.6

REMARK. In this definition the term *unique* means "exactly one." Thus, a function cannot assign two different outputs to the same input. For example, Figure 1.1.6 shows a scatter plot of weight versus age for a random sample of 100 college students. This scatter plot does not describe the weight W as a function of the age A because there are some values of A with more than one corresponding value of W. This is to be expected, since two people with the same age need not have the same weight. In contrast, Table 1.1.1 describes S as a function of t because there is only one top qualifying speed in a given year; similarly, Equation (1) describes y as a function of x because each input x in the interval $-3 \le x \le 3$ produces exactly one output $y = x\sqrt{9 - x^2}$.

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* LEONHARD EULER (1707–1783). Euler was probably the most prolific mathematician who ever lived. It has been said that "Euler wrote mathematics as effortlessly as most men breathe." He was born in Basel, Switzerland, and was the son of a Protestant minister who had himself studied mathematics. Euler's genius developed early. He attended the University of Basel, where by age 16 he obtained both a Bachelor of Arts degree and a Master's degree in philosophy. While at Basel, Euler had the good fortune to be tutored one day a week in mathematics by a distinguished mathematician, Johann Bernoulli. At the urging of his father, Euler then began to study theology. The lure of mathematics was too great, however, and by age 18 Euler had begun to do mathematical research. Nevertheless, the influence of his father and his theological studies remained, and throughout his life Euler was a deeply religious, unaffected person. At various times Euler taught at St. Petersburg Academy of Sciences (in Russia), the University of Basel, and the Berlin Academy of Sciences. Euler's energy and capacity for work were virtually boundless. His collected works form more than 100 quarto-sized volumes and it is believed that much of his work has been lost. What is particularly astonishing is that Euler was blind for the last 17 years of his life, and this was one of his most productive periods! Euler's flawless memory was phenomenal. Early in his life he memorized the entire *Aeneid* by Virgil and at age 70 could not only recite the entire work, but could also state the first and last sentence on each page of the book from which he memorized the work. His ability to solve problems in his head was beyond belief. He worked out in his head major problems of lunar motion that baffled Isaac Newton and once did a complicated calculation in his head to settle an argument between two students whose computations differed in the fiftieth decimal place.

Following the development of calculus by Leibniz and Newton, results in mathematics developed rapidly in a disorganized way. Euler's genius gave coherence to the mathematical landscape. He was the first mathematician to bring the full power of calculus to bear on problems from physics. He made major contributions to virtually every branch of mathematics as well as to the theory of optics, planetary motion, electricity, magnetism, and general mechanics.

FOUR WAYS TO DESCRIBE FUNCTIONS

Functions can be represented in four basic ways:

- Numerically by tables
- Geometrically by graphs
- Algebraically by formulas
- Verbally

The method of representation often depends on how the function arises. For example:

- Table 1.1.1 is a numerical representation of S as a function of t. This is the natural way in which data of this type are recorded.
- Figure 1.1.7 shows a seismic graph of an earthquake's intensity H as a function of the elapsed time t. In this case the function originates as a graph.
- Some of the most familiar examples of functions arise as formulas; for example, the formula $C = 2\pi r$ expresses the circumference C of a circle as a function of its radius r.
- Sometimes functions are described in words. For example, Isaac Newton's Universal Law of Gravitation is often stated as follows: The gravitational force of attraction between two bodies in the Universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This is the verbal description of the formula

$$F = G\frac{m_1 m_2}{r^2} \tag{2}$$

in which F is the force of attraction, m_1 and m_2 are the masses, r is the distance between them, and G is a constant.

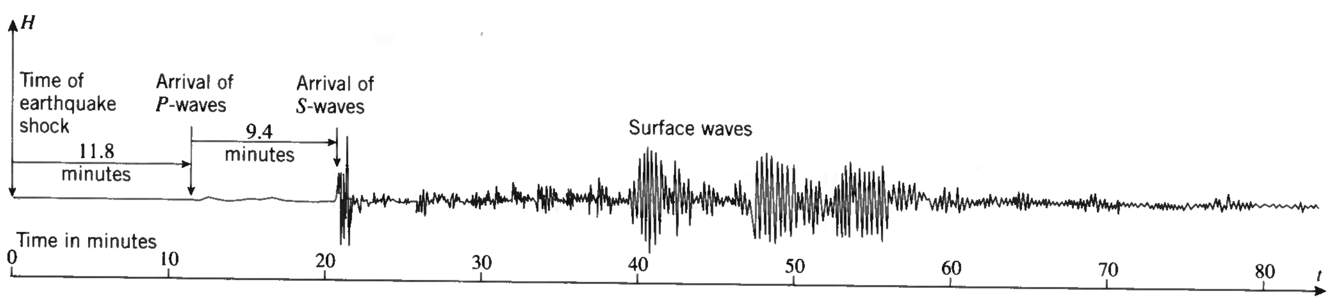


Figure 1.1.7

Table 1.1.3

U.S. POPULATION

YEAR t	POPULATION P (millions)
1790	3.9
1800	5.3
1810	7.2
1820	9.6
1830	12
1840	17
1850	23

Source: The World Almanac.

Sometimes it is desirable to convert one representation of a function into another. For example, in Figure 1.1.1 we converted the numerical relationship between S and t into a graphical relationship, and in writing Formula (2) we converted the verbal representation of the Universal Law of Gravitation into an algebraic relationship.

The problem of converting numerical representations of functions into algebraic formulas often requires special techniques known as ***curve fitting***. For example, Table 1.1.3 gives the U.S. population at 10-year intervals from 1790 to 1850. This table is a numerical representation of the function $P = f(t)$ that relates the U.S. population P to the year t. If we plot P versus t, we obtain the scatter plot in Figure 1.1.8*a*, and if we use curve-fitting methods that will be discussed later, we can obtain the approximation

$$P \approx 3.94(1.03)^{t-1790}$$

Figure 1.1.8*b* shows the graph of this equation imposed on the scatter plot.

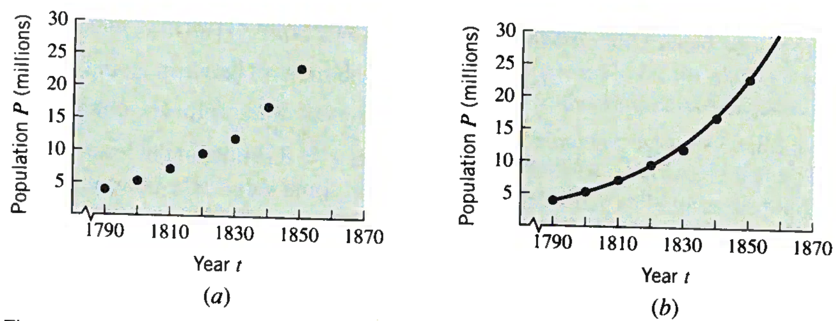


Figure 1.1.8

DISCRETE VERSUS CONTINUOUS DATA

Engineers and physicists distinguish between ***continuous data*** and ***discrete data***. Continuous data have values that vary *continuously* over an interval, whereas discrete data have values that make *discrete* jumps. For example, for the seismic data in Figure 1.1.7 both the time and intensity vary continuously, whereas in Table 1.1.3 and Figure 1.1.8*a* both the year and population make discrete jumps. As a rule, continuous data lead to graphs that are continuous, unbroken curves, whereas discrete data lead to scatter plots consisting of isolated points. Sometimes, as in Figure 1.1.8*b*, it is desirable to approximate a scatter plot by a continuous curve. This is useful for making conjectures about the values of the quantities between the recorded data points.

GRAPHS AS PROBLEM-SOLVING TOOLS

Sometimes a function is buried in the statement of a problem, and it is up to the problem solver to uncover it and use it in an appropriate way to solve the problem. Here is an example that illustrates the power of graphical representations of functions as a problem-solving tool.

Example 2

Figure 1.1.9*a* shows an offshore oil well located at a point W that is 5 km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8 km from A. It costs \$1,000,000/km to lay pipe under water and \$500,000/km over land. In your role as project manager you receive three proposals for piping the oil from W to B. Proposal 1 claims that it is cheapest to pipe directly from W to B, since the shortest distance between two points is a straight line. Proposal 2 claims that it is cheapest to pipe directly to point A and then along the shoreline to B, thereby using the least amount of expensive underwater pipe. Proposal 3 claims that it is cheapest to compromise by piping under water to some well-chosen point between A and B, and then piping along the shoreline to B. Which proposal is correct?

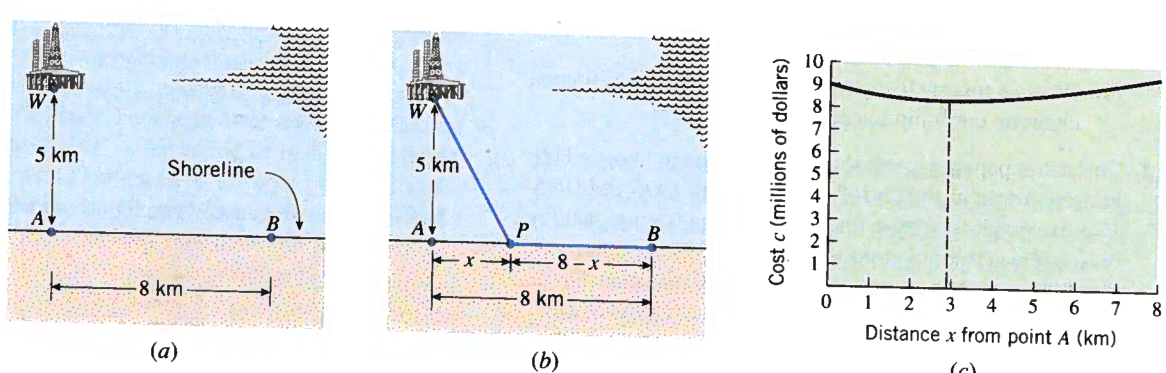


Figure 1.1.9

Solution. Let P be any point between A and B (Figure 1.1.9b), and let

$x =$ distance (in kilometers) between A and P

$c =$ cost (in millions of dollars) for the entire pipeline

Proposal 1 claims that $x = 8$ results in the least cost, Proposal 2 claims that it is $x = 0$, and Proposal 3 claims it is some value of x between 0 and 8. From Figure 1.1.9b the length of pipe along the shore is

$$8 - x \tag{3}$$

and from the Theorem of Pythagoras, the length of pipe under water is

$$\sqrt{x^2 + 25} \tag{4}$$

Thus, from (3) and (4) the total cost c (in millions of dollars) for the pipeline is

$$c = 1\left(\sqrt{x^2 + 25}\right) + 0.5(8 - x) = \sqrt{x^2 + 25} + 0.5(8 - x) \tag{5}$$

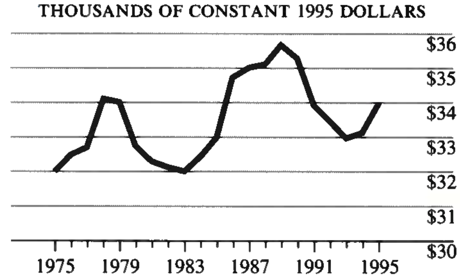
where $0 \le x \le 8$. The graph of Equation (5), shown in Figure 1.1.9c, makes it clear that Proposal 3 is correct—the most cost-effective strategy is to pipe to a point a little less than 3 km from point A. ◀

EXERCISE SET 1.1 Graphing Calculator

1. Use the cigarette consumption graph in Figure 1.1.2b to answer the following questions, making reasonable approximations where needed.
(a) When did the annual cigarette consumption reach 3000 per adult for the first time?
(b) When did the annual cigarette consumption per adult reach its peak, and what was the peak value?
(c) Can you tell from the graph how many cigarettes were consumed in a given year? If not, what additional information would you need to make that determination?
(d) What factors are likely to cause a sharp increase in annual cigarette consumption per adult?
(e) What factors are likely to cause a sharp decline in annual cigarette consumption per adult?

2. The accompanying graph shows the median income in U.S. households (adjusted for inflation) between 1975 and 1995. Use the graph to answer the following questions, making reasonable approximations where needed.
(a) When did the median income reach its maximum value, and what was the median income when that occurred?
(b) When did the median income reach its minimum value, and what was the median income when that occurred?
(c) The median income was declining during the 4-year period between 1989 and 1993. Was it declining more rapidly during the first 2 years or the second 2 years of that period? Explain your reasoning.

MEDIAN U.S. HOUSEHOLD INCOME IN THOUSANDS OF CONSTANT 1995 DOLLARS



Source: Census Bureau, March 1996 [1996 measures 1995 income].

Figure Ex-2

3. Use the accompanying graph to answer the following questions, making reasonable approximations were needed.
(a) For what values of x is $y = 1$?
(b) For what values of x is $y = 3$?
(c) For what values of y is $x = 3$?
(d) For what values of x is $y \le 0$?
(e) What are the maximum and minimum values of y and for what values of x do they occur?

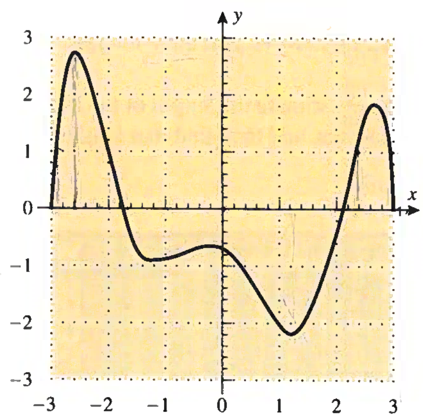


Figure Ex-3

4. Use the table in the accompanying figure to answer the questions posed in Exercise 3.

x	−2	−1	0	2	3	4	5	6
y	5	1	−2	7	−1	1	0	9

Figure Ex-4

5. Use the equation $y = x^2 - 6x + 8$ to answer the following questions.
   (a) For what values of x is $y = 0$?
   (b) For what values of x is $y = -10$?
   (c) For what values of x is $y \geq 0$?
   (d) Does y have a minimum value? A maximum value? If so, find them.

6. Use the equation $y = 1 + \sqrt{x}$ to answer the following questions.
   (a) For what values of x is $y = 4$?
   (b) For what values of x is $y = 0$?
   (c) For what values of x is $y \geq 6$?
   (d) Does y have a minimum value? A maximum value? If so, find them.

7. (a) If you had a device that could record the Earth's population continuously, would you expect the graph of population versus time to be a continuous (unbroken) curve? Explain what might cause breaks in the curve.
   (b) Suppose that a hospital patient receives an injection of an antibiotic every 8 hours and that between injections the concentration C of the antibiotic in the bloodstream decreases as the antibiotic is absorbed by the tissues. What might the graph of C versus the elapsed time t look like?

8. (a) If you had a device that could record the temperature of a room continuously over a 24-hour period, would you expect the graph of temperature versus time to be a continuous (unbroken) curve? Explain your reasoning.
   (b) If you had a computer that could track the number of boxes of cereal on the shelf of a market continuously over a 1-week period, would you expect the graph of the number of boxes on the shelf versus time to be a continuous (unbroken) curve? Explain your reasoning.

9. A construction company wants to build a rectangular enclosure with an area of 1000 square feet by fencing in three sides and using its office building as the fourth side. Your objective as supervising engineer is to design the enclosure so that it uses the least amount of fencing. Proceed as follows.
   (a) Let x and y be the dimensions of the enclosure, and let L be the length of fencing required for those dimensions. Since the area must be 1000 square feet, we must have $xy = 1000$. Find a formula for L in terms of x and y, and then express L in terms of x alone by using the area equation.
   (b) Are there any restrictions on the value of x? Explain.
   (c) Make a graph of L versus x over a reasonable interval, and use the graph to estimate the value of x that results in the smallest value of L.
   (d) Estimate the smallest value of L.

10. A manufacturer constructs open boxes from sheets of cardboard that are 6 inches square by cutting small squares from the corners and folding up the sides (as shown in the accompanying figure). The Research and Development Department asks you to determine the size of the square that produces a box of greatest volume. Proceed as follows.
   (a) Let x be the length of a side of the square to be cut, and let V be the volume of the resulting box. Show that $V = x(6 - 2x)^2$.
   (b) Are there any restrictions on the value of x? Explain.
   (c) Make a graph of V versus x over an appropriate interval, and use the graph to estimate the value of x that results in the largest volume.
   (d) Estimate the largest volume.

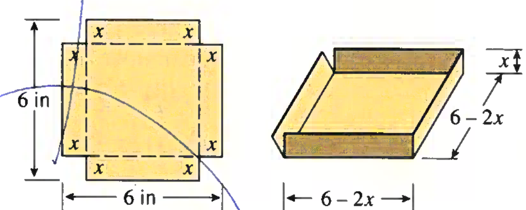


Figure Ex-10

11. A soup company wants to manufacture a can in the shape of a right circular cylinder that will hold 500 cm^3 of liquid. The material for the top and bottom costs 0.02 cent/cm^2, and the material for the sides costs 0.01 cent/cm^2.
   (a) Use the method of Exercises 9 and 10 to estimate the radius r and height h of the can that costs the least to manufacture. [*Suggestion:* Express the cost C in terms of r.]
   (b) Suppose that the tops and bottoms of radius r are punched out from square sheets with sides of length $2r$ and the scraps are waste. If you allow for the cost of

the waste, would you expect the can of least cost to be taller or shorter than the one in part (a)? Explain.

(c) Estimate the radius, height, and cost of the can in part (b), and determine whether your conjecture was correct.

12. The designer of a sports facility wants to put a quarter-mile (1320 ft) running track around a football field, oriented as in the accompanying figure. The football field is 360 ft long (including the end zones) and 160 ft wide. The track consists of two straightaways and two semicircles.

(a) Show that it is possible to construct a quarter-mile track around the football field. [*Suggestion:* Find the shortest track that can be constructed around the field.]

(b) Let L be the length of a straightaway (in feet), and let x be the distance (in feet) between a sideline of the football field and a straightaway. Make a graph of L versus x.

(c) Use the graph to estimate the value of x that produces the shortest straightaways, and then find this value of x exactly.

(d) Use the graph to estimate the length of the longest possible straightaways, and then find that length exactly.

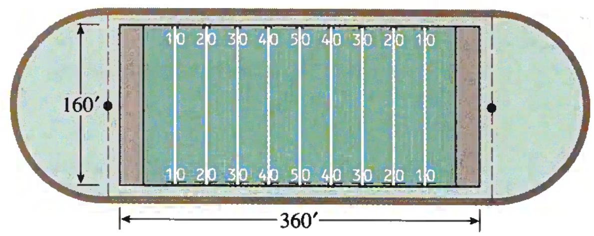


Figure Ex-12

1.2 PROPERTIES OF FUNCTIONS

In this section we will explore properties of functions in more detail. We will assume that you are familiar with the standard notation for intervals and the basic properties of absolute value. Reviews of these topics are provided in Appendices A and B.

INDEPENDENT AND DEPENDENT VARIABLES

Recall from the last section that a function f is a rule that associates a unique output $f(x)$ with each input x. This output is sometimes called the ***value*** of f at x or the ***image*** of x under f. Sometimes we will want to denote the output by a single letter, say y, and write

$$y = f(x)$$

This equation expresses y as a function of x; the variable x is called the ***independent variable*** (or ***argument***) of f, and the variable y is called the ***dependent variable*** of f. This terminology is intended to suggest that x is free to vary, but that once x has a specific value a corresponding value of y is determined. For now we will only consider functions in which the independent and dependent variables are real numbers, in which case we say that f is a ***real-valued function of a real variable***. Later, we will consider other kinds of functions as well.

Table 1.2.1

x	0	1	2	3
y	3	4	−1	6

Table 1.2.1 can be viewed as a numerical representation of a function of f. For this function we have

$f(0) = 3$ — f associates $y = 3$ with $x = 0$.

$f(1) = 4$ — f associates $y = 4$ with $x = 1$.

$f(2) = -1$ — f associates $y = -1$ with $x = 2$.

$f(3) = 6$ — f associates $y = 6$ with $x = 3$.

To illustrate how functions can be defined by equations, consider

$$y = 3x^2 - 4x + 2 \tag{1}$$

This equation has the form $y = f(x)$, where

$$f(x) = 3x^2 - 4x + 2 \tag{2}$$

The outputs of f (the y-values) are obtained by substituting numerical values for x in this formula. For example,

$f(0) = 3(0)^2 - 4(0) + 2 = 2$ — f associates $y = 2$ with $x = 0$.

$f(-1.7) = 3(-1.7)^2 - 4(-1.7) + 2 = 17.47$ — f associates $y = 17.47$ with $x = -1.7$.

$f(\sqrt{2}) = 3(\sqrt{2})^2 - 4\sqrt{2} + 2 = 8 - 4\sqrt{2}$ — f associates $y = 8 - 4\sqrt{2}$ with $x = \sqrt{2}$.

REMARK. Although f, x, and y are the most common notations for functions and variables, any letters can be used. For example, to indicate that the area A of a circle is a function of the radius r, it would be more natural to write $A = f(r)$ [where $f(r) = \pi r^2$]. Similarly, to indicate that the circumference C of a circle is a function of the radius r, we might write $C = g(r)$ [where $g(r) = 2\pi r$]. The area function and the circumference function are different, which is why we denoted them by different letters, f and g.

DOMAIN AND RANGE

If $y = f(x)$, then the set of all possible inputs (x-values) is called the ***domain*** of f, and the set of outputs (y-values) that result when x varies over the domain is called the ***range*** of f. For example, consider the equations

$$y = x^2 \quad \text{and} \quad y = x^2, \quad x \geq 2$$

In the first equation there is no restriction on x, so we may assume that any real value of x is an allowable input. Thus, the equation defines a function $f(x) = x^2$ with domain $-\infty < x < +\infty$. In the second equation, the inequality $x \geq 2$ restricts the allowable inputs to be greater than or equal to 2, so the equation defines a function $g(x) = x^2, x \geq 2$ with domain $2 \leq x < +\infty$.

As x varies over the domain of the function $f(x) = x^2$, the values of $y = x^2$ vary over the interval $0 \leq y < +\infty$, so this is the range of f. By comparison, as x varies over the domain of the function $g(x) = x^2, x \geq 2$, the values of $y = x^2, x \geq 2$ vary over the interval $4 \leq y < +\infty$, so this is the range of g.

It is important to understand here that even though $f(x) = x^2$ and $g(x) = x^2, x \geq 2$ involve the same formula, we regard them to be different functions because they have different domains. In short, *to fully describe a function you must not only specify the rule that relates the inputs and outputs, but you must also specify the domain, that is, the set of allowable inputs.*

GRAPHS OF FUNCTIONS

If f is a real-valued function of a real variable, then the ***graph*** of f in the xy-plane is defined to be the graph of the equation $y = f(x)$. For example, the graph of the function $f(x) = x$ is the graph of the equation $y = x$, shown in Figure 1.2.1. That figure also shows the graphs

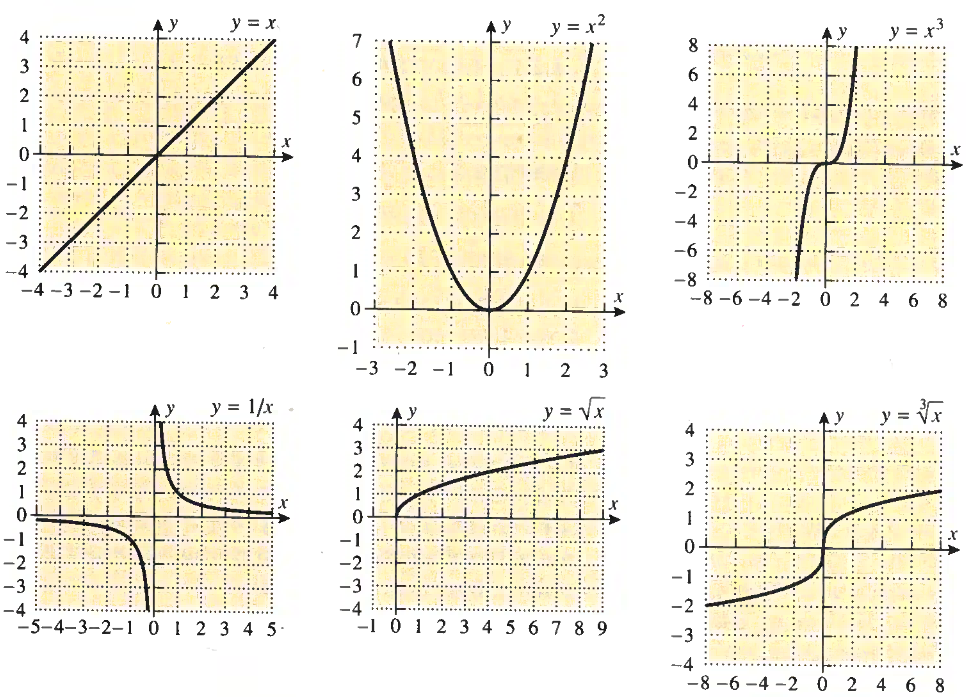


Figure 1.2.1

of some other basic functions that may already be familiar to you. Later in this chapter we will discuss techniques for graphing functions using graphing calculators and computers.

Graphs can provide useful visual information about a function. For example, because the graph of a function f in the xy-plane consists of all points whose coordinates satisfy the equation $y = f(x)$, the points on the graph of f are of the form $(x, f(x))$; hence each y-coordinate is the value of f at the x-coordinate (Figure 1.2.2a). Pictures of the domain and range of f can be obtained by projecting the graph of f onto the coordinate axes (Figure 1.2.2b). The values of x for which $f(x) = 0$ are the x-coordinates of the points where the graph of f intersects the x-axis (Figure 1.2.2c); these values of x are called the ***zeros*** of f, the ***roots*** of $f(x) = 0$, or the ***x-intercepts*** of $y = f(x)$.

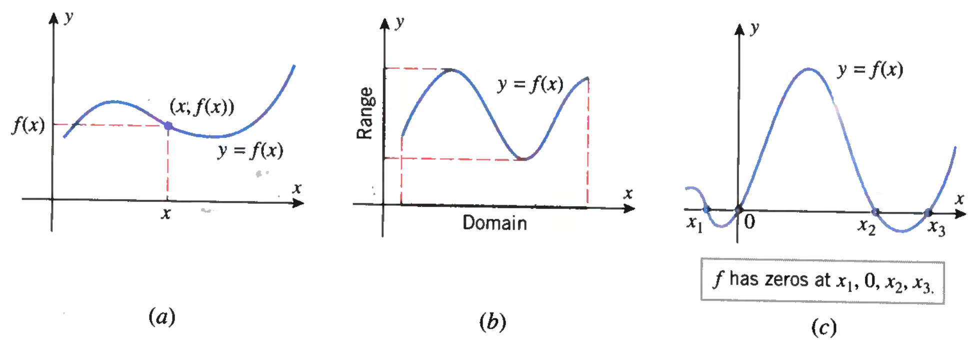


Figure 1.2.2

THE VERTICAL LINE TEST

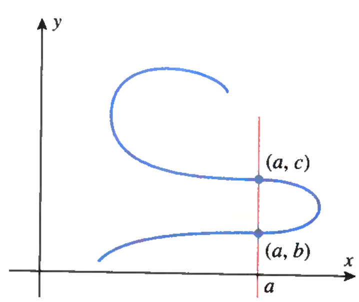


Figure 1.2.3

Not every curve in the xy-plane is the graph of a function. For example, consider the curve in Figure 1.2.3, which is cut at two distinct points, (a, b) and (a, c), by a vertical line. This curve cannot be the graph of $y = f(x)$ for any function f; otherwise, we would have

$$f(a) = b \quad \text{and} \quad f(a) = c$$

which is impossible, since f cannot assign two different values to a. Thus, there is no function f whose graph is the given curve. This illustrates the following general result, which we will call the ***vertical line test***.

> **1.2.1** THE VERTICAL LINE TEST. *A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.*

Example 1

The graph of the equation

$$x^2 + y^2 = 25 \tag{3}$$

is a circle of radius 5, centered at the origin (see Appendix D for a review of circles), and hence there are vertical lines that cut the graph more than once. This can also be seen algebraically by solving (3) for y in terms of x:

$$y = \pm\sqrt{25 - x^2}$$

This equation does not define y as a function of x because the right side is "multiple valued" in the sense that values of x in the interval $(-5, 5)$ produce two corresponding values of y. For example, if $x = 4$, then $y = \pm 3$, and hence $(4, 3)$ and $(4, -3)$ are two points on the circle that lie on the same vertical line (Figure 1.2.4a). However, we can regard the circle as the union of two semicircles:

$$y = \sqrt{25 - x^2} \quad \text{and} \quad y = -\sqrt{25 - x^2}$$

(Figure 1.2.4b), each of which defines y as a function of x. ◀

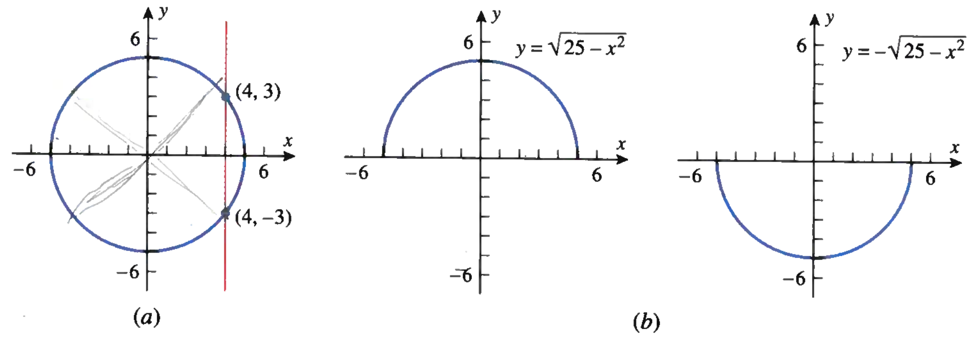


Figure 1.2.4

THE ABSOLUTE VALUE FUNCTION

Recall that the ***absolute value*** or ***magnitude*** of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative. Thus,

$$|5| = 5, \quad \left|-\tfrac{4}{7}\right| = \tfrac{4}{7}, \quad |0| = 0$$

A more detailed discussion of the properties of absolute value is given in Appendix B. However, for convenience we provide the following summary of its algebraic properties.

1.2.2 PROPERTIES OF ABSOLUTE VALUE. *If a and b are real numbers, then*

(a) $|-a| = |a|$	A number and its negative have the same absolute value.
(b) $|ab| = |a||b|$	The absolute value of a product is the product of the absolute values.
(c) $|a/b| = |a|/|b|$	The absolute value of a ratio is the ratio of the absolute values.
(d) $|a+b| \leq |a| + |b|$	The ***triangle inequality***

REMARK. Symbols such as $+x$ and $-x$ are deceptive, since it is tempting to conclude that $+x$ is positive and $-x$ is negative. However, this need not be so, since x itself can be positive or negative. For example, if x is negative, say $x = -3$, then $-x = 3$ is positive and $+x = -3$ is negative.

The graph of the function $f(x) = |x|$ can be obtained by graphing the two parts of the equation

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

separately. For $x \geq 0$, the graph of $y = x$ is a ray of slope 1 with its endpoint at the origin, and for $x < 0$, the graph of $y = -x$ is a ray of slope -1 with its endpoint at the origin. Combining the two parts produces the V-shaped graph in Figure 1.2.5.

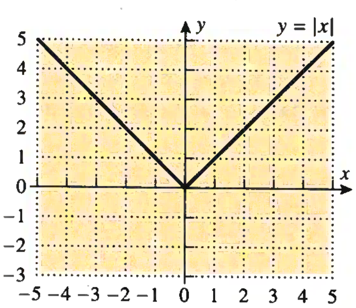


Figure 1.2.5

Absolute values have important relationships to square roots. To see why this is so, recall from algebra that every positive real number x has two square roots, one positive and one negative. By definition, the symbol $\sqrt{x}$ denotes the *positive* square root of x. To denote the negative square root you must write $-\sqrt{x}$. For example, the positive square root of 9 is $\sqrt{9} = 3$, and the negative square root is $-\sqrt{9} = -3$. (Do not make the mistake of writing $\sqrt{9} = \pm 3$.)

Care must be exercised in simplifying expressions of the form $\sqrt{x^2}$, since it is *not* always true that $\sqrt{x^2} = x$. This equation is correct if x is nonnegative, but it is false for negative x. For example, if $x = -4$, then

$$\sqrt{x^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq x$$

A statement that is correct for all real values of x is

$$\sqrt{x^2} = |x|$$

FOR THE READER. Verify this relationship by using a graphing utility to show that the equations $y = \sqrt{x^2}$ and $y = |x|$ have the same graph.

FUNCTIONS DEFINED PIECEWISE

The absolute value function $f(x) = |x|$ is an example of a function that is defined ***piecewise*** in the sense that the formula for f changes, depending on the value of x.

Example 2

Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0, & x \leq -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \geq 1 \end{cases}$$

Solution. The formula for f changes at the points $x = -1$ and $x = 1$. (We call these the ***breakpoints*** for the formula.) A good procedure for graphing functions defined piecewise is to graph the function separately over the open intervals determined by the breakpoints, and then graph f at the breakpoints themselves. For the function f in this example the graph is the horizontal line segment $y = 0$ on the interval $(-\infty, -1)$, it is the semicircle $y = \sqrt{1-x^2}$ on the interval $(-1, 1)$, and it is the line segment $y = x$ on the interval $(1, +\infty)$. The formula for f specifies that the equation $y = 0$ applies at the breakpoint -1 [so $y = f(-1) = 0$], and it specifies that the equation $y = x$ applies at the breakpoint 1 [so $y = f(1) = 1$]. The graph of f is shown in Figure 1.2.6. ◀

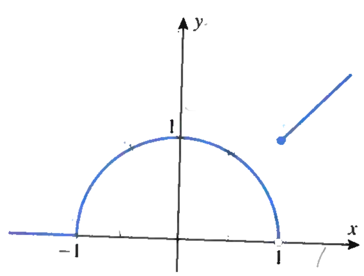


Figure 1.2.6

REMARK. In Figure 1.2.6 the solid dot and open circle at the breakpoint $x = 1$ serve to emphasize that the point on the graph lies on the line segment and not the semicircle. There is no ambiguity at the breakpoint $x = -1$ because the two parts of the graph join together continuously there.

Example 3

Increasing the speed at which air moves over a person's skin increases the rate of moisture evaporation and makes the person feel cooler. (This is why we fan ourselves in hot weather.) The ***windchill index*** is the temperature at a wind speed of 4 mi/h that would produce the same sensation on exposed skin as the current temperature and wind speed combination. An empirical formula (i.e., a formula based on experimental data) for the windchill index W at 32°F for a wind speed of v mi/h is

$$W = \begin{cases} 32, & 0 \leq v \leq 4 \\ 91.4 + 59.4(0.0203v - 0.304\sqrt{v} - 0.474), & 4 < v < 45 \\ -3.6, & v \geq 45 \end{cases}$$

A computer-generated graph of $W(v)$ is shown in Figure 1.2.7. ◀

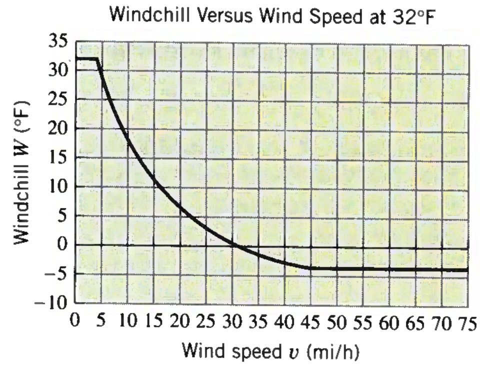


Figure 1.2.7

THE NATURAL DOMAIN

Sometimes, restrictions on the allowable values of an independent variable result from a mathematical formula that defines the function. For example, if $f(x) = 1/x$, then $x = 0$ must be excluded from the domain to avoid division by zero, and if $f(x) = \sqrt{x}$, then negative values of x must be excluded from the domain, since we are only considering real-valued functions of a real variable for now. We make the following definition.

> **1.2.3 DEFINITION.** If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain*** of the function.

Example 4

Find the natural domain of

(a) $f(x) = x^3$ (b) $f(x) = 1/(x-1)(x-3)$

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

Solution (a). The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

Solution (c). Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Solution (d). The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x-3)(x-2) \geq 0$$

This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

$$(-\infty, 2] \cup [3, +\infty)$$

◀

REMARK. In some problems we will want to limit the domain of a function by imposing specific restrictions. For example, by writing

$$f(x) = x^2, \quad x \geq 0$$

we can limit the domain of f to the positive x-axis (Figure 1.2.8).

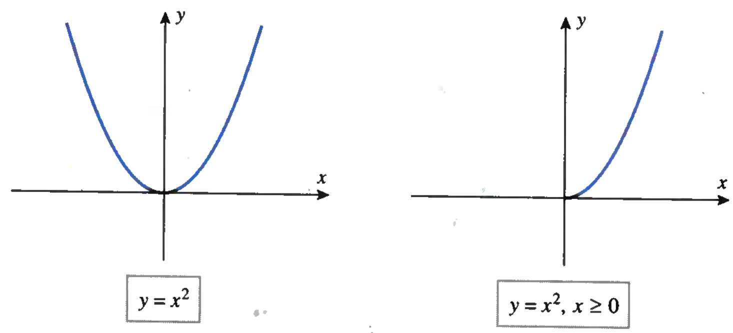


Figure 1.2.8

THE EFFECT OF ALGEBRAIC OPERATIONS ON THE DOMAIN

Algebraic expressions are frequently simplified by canceling common factors in the numerator and denominator. However, care must be exercised when simplifying formulas for functions in this way, since this process can alter the domain.

Example 5

The natural domain of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

consists of all real x except $x = 2$. However, if we factor the numerator and then cancel the common factor in the numerator and denominator, we obtain

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$$

which *is* defined at $x = 2$ [since $f(2) = 4$ for the altered function f]. Thus, the algebraic simplification has altered the domain of the function. Geometrically, the graph of $y = x + 2$ is a line of slope 1 and y-intercept 2, whereas the graph of $y = (x^2 - 4)/(x - 2)$ is the same line, but with a hole in it at $x = 2$, since y is undefined there (Figure 1.2.9). Thus, the geometric effect of the algebraic cancellation is to eliminate the hole in the original graph. In some situations such minor alterations in the domain are irrelevant to the problem under consideration and can be ignored. However, if we wanted to preserve the domain in this example, then we would express the simplified form of the function as

$$f(x) = x + 2, \quad x \neq 2$$

◀

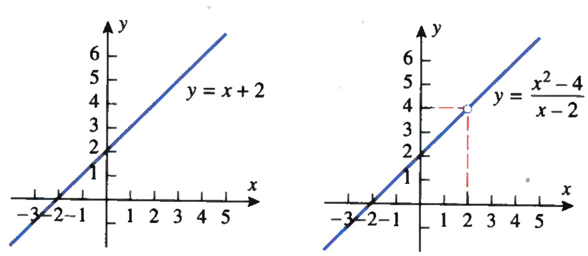


Figure 1.2.9

Example 6

Find the domain and range of

(a) $f(x) = 2 + \sqrt{x-1}$ (b) $f(x) = (x+1)/(x-1)$

Solution (a). Since no domain is stated explicitly, the domain of f is the natural domain $[1, +\infty)$. To determine the range, it will be convenient to introduce a dependent variable $y = 2 + \sqrt{x-1}$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $y = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f. The domain and range are shown graphically in Figure 1.2.10*a*.

Solution (b). The given function f is defined for all real x, except $x = 1$, so the natural domain of f is

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$

As in the preceding part of this example, it will be convenient to introduce a dependent variable

$$y = \frac{x+1}{x-1} \tag{4}$$

Although the set of possible y-values is not immediately evident from this equation, the graph of (4), which is shown in Figure 1.2.10*b*, suggests that the range of f consists of all y, except $y = 1$. To see that this is so, we solve (4) for x in terms of y:

$$\begin{aligned}
(x-1)y &= x+1 \\
xy - y &= x+1 \\
xy - x &= y+1 \\
x(y-1) &= y+1 \\
x &= \frac{y+1}{y-1}
\end{aligned}$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$, which agrees with the result obtained graphically. ◀

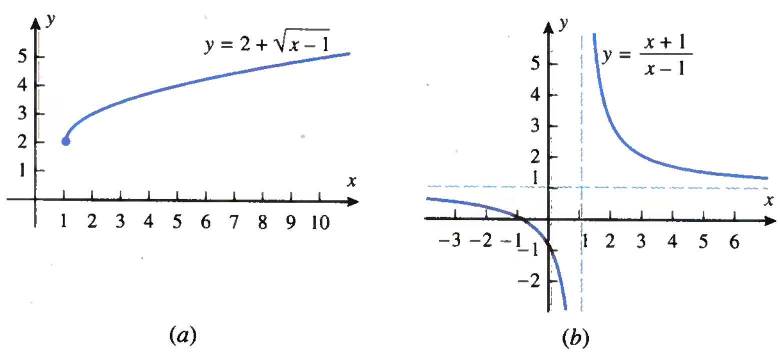


Figure 1.2.10

DOMAIN AND RANGE IN APPLIED PROBLEMS

In applications, physical considerations often impose restrictions on the domain and range of a function.

Example 7

An open box is to be made from a 16 in by 30 in piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 1.2.11*a*).

(a) Let V be the volume of the box that results when the squares have sides of length x. Find a formula for V as a function of x.

(b) Find the domain of V.

(c) Use the graph of V given in Figure 1.2.11*c* to estimate the range of V.

(d) Describe in words what the graph tells you about the volume.

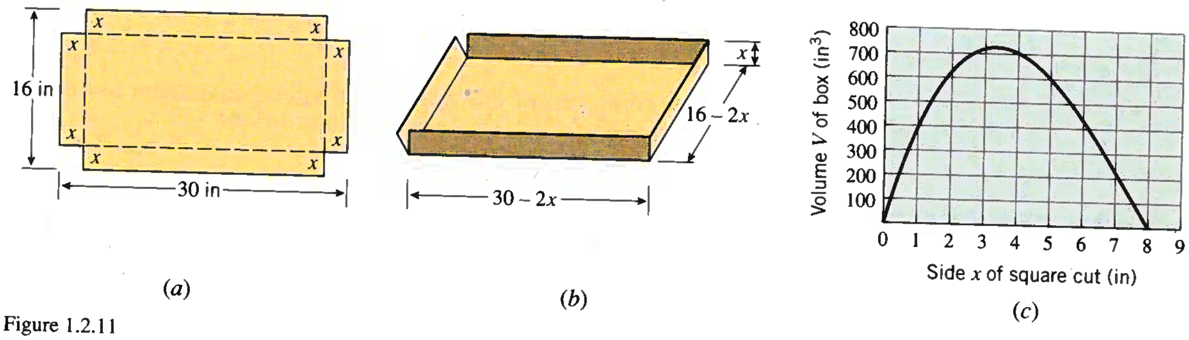


Figure 1.2.11

Solution (a). As shown in Figure 1.2.11*b*, the resulting box has dimensions $16 - 2x$ by $30 - 2x$ by x, so the volume $V(x)$ is given by

$$V(x) = (16 - 2x)(30 - 2x)x = 480x - 92x^2 + 4x^3$$

Solution (b). The domain is the set of x-values and the range is the set of V-values. Because x is a length, it must be nonnegative, and because we cannot cut out squares whose sides are more than 8 in long (why?), the x-values in the domain must satisfy

$$0 \le x \le 8$$

Solution (c). From the graph of V versus x in Figure 1.2.11*c* we estimate that the V-values in the range satisfy

$$0 \le V \le 725$$

Note that this is an approximation. Later we will show how to find the range exactly.

Solution (d). The graph tells us that the box of maximum volume occurs for a value of x that is between 3 and 4 and that the maximum volume is approximately 725 in^3. Moreover, the volume decreases toward zero as x gets closer to 0 or 8. ◀

In applications involving time, formulas for functions are often expressed in terms of a variable t whose starting value is taken to be $t = 0$.

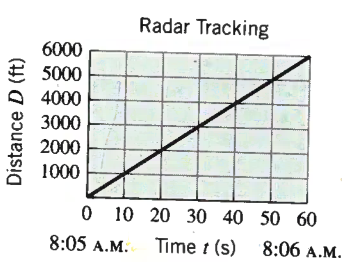


Figure 1.2.12

Example 8

At 8:05 A.M. a car is clocked at 100 ft/s by a radar detector that is positioned at the edge of a straight highway. Assuming that the car maintains a constant speed between 8:05 A.M. and 8:06 A.M., find a function $D(t)$ that expresses the distance traveled by the car during that time interval as a function of the time t.

Solution. It would be clumsy to use clock time for the variable t, so let us agree to measure the elapsed time in seconds, starting with $t = 0$ at 8:05 A.M. and ending with $t = 60$ at

8:06 A.M. At each instant, the distance traveled (in ft) is equal to the speed of the car (in ft/s) multiplied by the elapsed time (in s). Thus,

$$D(t) = 100t, \quad 0 \le t \le 60$$

The graph of D versus t is shown in Figure 1.2.12. ◀

ISSUES OF SCALE AND UNITS

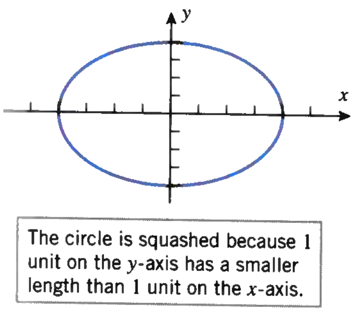


Figure 1.2.13

In geometric problems where you want to preserve the "true" shape of a graph, you must use units of equal length on both axes. For example, if you graph a circle in a coordinate system in which 1 unit in the y-direction is smaller than 1 unit in the x-direction, then the circle will be squashed vertically into an elliptical shape (Figure 1.2.13). You must also use units of equal length when you want to apply the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

to calculate the distance between two points (x_1, y_1) and (x_2, y_2) in the xy-plane.

However, sometimes it is inconvenient or impossible to display a graph using units of equal length. For example, consider the equation

$$y = x^2$$

If we want to show the portion of the graph over the interval $-3 \le x \le 3$, then there is no problem using units of equal length, since y only varies from 0 to 9 over that interval. However, if we want to show the portion of the graph over the interval $-10 \le x \le 10$, then there is a problem keeping the units equal in length, since the value of y varies between 0 and 100. In this case the only reasonable way to show all of the graph that occurs over the interval $-10 \le x \le 10$ is to compress the unit of length along the y-axis, as illustrated in Figure 1.2.14.

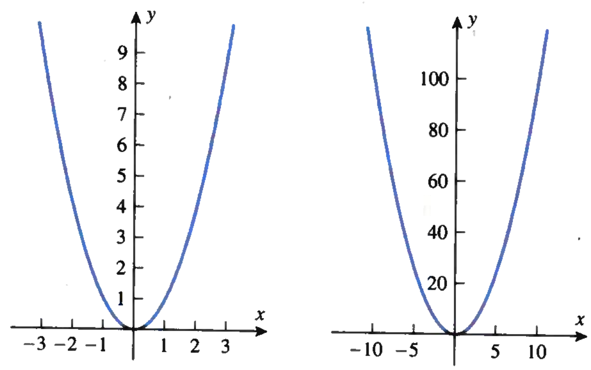


Figure 1.2.14

REMARK. In applications where the variables on the two axes have unrelated units (say, centimeters on the y-axis and seconds on the x-axis), then nothing is gained by requiring the units to have equal lengths; choose the lengths to make the graph as clear as possible.

EXERCISE SET 1.2 Graphing Calculator

1. Find $f(0)$, $f(2)$, $f(-2)$, $f(3)$, $f(\sqrt{2})$, and $f(3t)$.
   (a) $f(x) = 3x^2 - 2$
   (b) $f(x) = \begin{cases} \frac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$

2. Find $g(3)$, $g(-1)$, $g(\pi)$, $g(-1.1)$, and $g(t^2 - 1)$.
   (a) $g(x) = \dfrac{x+1}{x-1}$
   (b) $g(x) = \begin{cases} \sqrt{x+1}, & x \ge 1 \\ 3, & x < 1 \end{cases}$

In Exercises 3–6, find the natural domain of the function algebraically, and confirm that your result is consistent with the graph produced by your graphing utility. [*Note:* Set your graphing utility to the radian mode when graphing trigonometric functions.]

3. (a) $f(x) = \dfrac{1}{x-3}$ (b) $g(x) = \sqrt{x^2 - 3}$

(c) $G(x) = \sqrt{x^2 - 2x + 5}$ (d) $f(x) = \dfrac{x}{|x|}$

(e) $h(x) = \dfrac{1}{1 - \sin x}$

4. (a) $f(x) = \dfrac{1}{5x+7}$ (b) $h(x) = \sqrt{x - 3x^2}$

(c) $G(x) = \sqrt{\dfrac{x^2-4}{x-4}}$ (d) $f(x) = \dfrac{x^2-1}{x+1}$

(e) $h(x) = \dfrac{3}{2 - \cos x}$

5. (a) $f(x) = \sqrt{3-x}$ (b) $g(x) = \sqrt{4 - x^2}$

(c) $h(x) = 3 + \sqrt{x}$ (d) $G(x) = x^3 + 2$

(e) $H(x) = 3 \sin x$

6. (a) $f(x) = \sqrt{3x-2}$ (b) $g(x) = \sqrt{9 - 4x^2}$

(c) $h(x) = \dfrac{1}{3 + \sqrt{x}}$ (d) $G(x) = \dfrac{3}{x}$

(e) $H(x) = \sin^2 \sqrt{x}$

7. In each part of the accompanying figure, determine whether the graph defines y as a function of x.

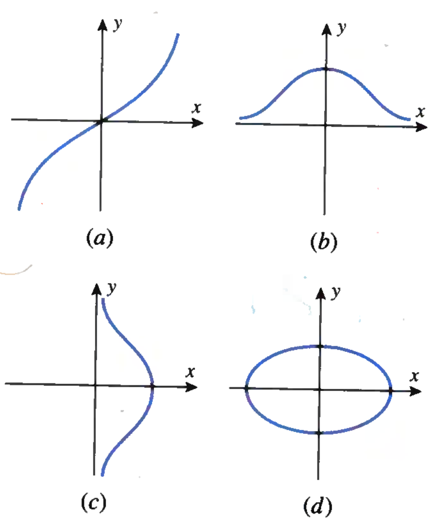


Figure Ex-7

8. Express the length L of a chord of a circle with radius 10 cm as a function of the central angle θ (see the accompanying figure).

9. As shown in the accompanying figure, a pendulum of constant length L makes an angle θ with its vertical position. Express the height h as a function of the angle θ.

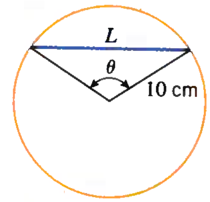


Figure Ex-8

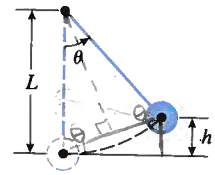


Figure Ex-9

10. A cup of hot coffee sits on a table. You pour in some cool milk and let it sit for an hour. Sketch a rough graph of the temperature of the coffee as a function of time.

11. A boat is bobbing up and down on some gentle waves. Suddenly it gets hit by a large wave and sinks. Sketch a rough graph of the height of the boat above the ocean floor as a function of time.

12. Make a rough sketch of your weight as a function of time from birth to the present.

In Exercises 13 and 14, express the function in piecewise form without using absolute values. [*Suggestion:* It may help to generate the graph of the function.]

13. (a) $f(x) = |x| + 3x + 1$ (b) $g(x) = |x| + |x - 1|$

14. (a) $f(x) = 3 + |2x - 5|$ (b) $g(x) = 3|x - 2| - |x + 1|$

15. As shown in the accompanying figure, an open box is to be constructed from a rectangular sheet of metal, 8 inches by 15 inches, by cutting out squares with sides of length x from each corner and bending up the sides.

(a) Express the volume V as a function of x.

(b) Find the natural domain and the range of the function, ignoring any physical restrictions on the values of the variables.

(c) Modify the domain and range appropriately to account for the physical restrictions on the values of V and x.

(d) In words, describe how the volume V of the box varies with x, and discuss how one might construct boxes of maximum volume and minimum volume.

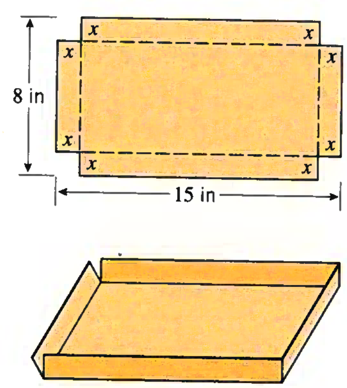


Figure Ex-15

16. As shown in the accompanying figure, a camera is mounted at a point 3000 ft from the base of a rocket launching pad.

The shuttle rises vertically when launched, and the camera's elevation angle is constantly adjusted to follow the bottom of the rocket.

(a) Choose letters to represent the height of the rocket and the elevation angle of the camera, and express the height as a function of the elevation angle.

(b) Find the natural domain and the range of the function, ignoring any physical restrictions on the values of the variables.

(c) Modify the domain and range appropriately to account for the physical restrictions on the values of the variables.

(d) Generate the graph of height versus the elevation on a graphing utility, and use it to estimate the height of the rocket when the elevation angle is $\pi/4 \approx 0.7854$ radian. Compare this estimate to the exact height. [*Suggestion:* If you are using a graphing calculator, the trace and zoom features will be helpful here.]

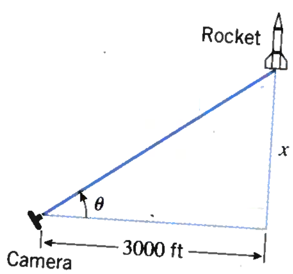


Figure Ex-16

In Exercises 17 and 18: (i) Explain why the function f has one or more holes in its graph, and state the x-values at which those holes occur. (ii) Find a function g whose graph is identical to that of f, but without the holes.

17. $f(x) = \dfrac{(x+2)(x^2-1)}{(x+2)(x-1)}$ **18.** $f(x) = \dfrac{x+\sqrt{x}}{\sqrt{x}}$

19. For a given outside temperature T and wind speed v, the windchill index (WCI) is the equivalent temperature that exposed skin would feel with a wind speed of 4 mi/h. An empirical formula for the WCI (based on experience and observation) is

$$\text{WCI} = \begin{cases} T, & 0 \le v \le 4 \\ 91.4 + (91.4 - T)(0.0203v - 0.304\sqrt{v} - 0.474), & 4 < v < 45 \\ 1.6T - 55, & v \ge 45 \end{cases}$$

where T is the air temperature in °F, v is the wind speed in mi/h, and WCI is the equivalent temperature in °F. Find the WCI to the nearest degree if the air temperature is 25°F and

(a) $v = 3$ mi/h (b) $v = 15$ mi/h
(c) $v = 46$ mi/h.

[Adapted from UMAP Module 658, *Windchill*, W. Bosch and L. Cobb, COMAP, Arlington, MA.]

In Exercises 20–22, use the formula for the windchill index described in Exercise 19.

20. Find the air temperature to the nearest degree if the WCI is reported as −60°F with a wind speed of 48 mi/h.

21. Find the air temperature to the nearest degree if the WCI is reported as −10°F with a wind speed of 8 mi/h.

22. Find the wind speed to the nearest mile per hour if the WCI is reported as −15°F with an air temperature of 20°F.

23. At 9:23 A.M. a lunar lander that is 1000 ft above the Moon's surface begins a vertical descent, touching down at 10:13 A.M. Assuming that the lander maintains a constant speed, find a function $D(t)$ that expresses the altitude of the lander above the Moon's surface as a function of t.

1.3 GRAPHING FUNCTIONS ON CALCULATORS AND COMPUTERS; COMPUTER ALGEBRA SYSTEMS

In this section we will discuss issues that relate to generating graphs of equations and functions with graphing utilities (graphing calculators and computers). Because graphing utilities vary widely, it is difficult to make general statements about them. Therefore, at various places in this section we will ask you to refer to the documentation for your own graphing utility for specific details about the way it operates.

GRAPHING CALCULATORS AND COMPUTER ALGEBRA SYSTEMS

The development of new technology has significantly changed how and where mathematicians, engineers, and scientists perform their work, as well as their approach to problem solving. Not only have portable computers and handheld calculators with graphing capabilities become standard tools in the scientific community, but there have been major new innovations in computer software. Among the most significant of these innovations are programs called ***Computer Algebra Systems*** (abbreviated CAS), the most common being *Mathematica*, *Maple*, and *Derive*.* Computer algebra systems not only have powerful graphing capabilities, but, as their name suggests, they can perform many of the symbolic computations that occur in algebra, calculus, and branches of higher mathematics. For example, it is a trivial task for a CAS to perform the factorization

$$x^6 + 23x^5 + 147x^4 - 139x^3 - 3464x^2 - 2112x + 23040 = (x+5)(x-3)^2(x+8)^3$$

or the exact numerical computation

$$\left(\frac{63456}{3177295} - \frac{43907}{22854377}\right)^3 = \frac{2251912457164208291259320230122866923}{382895955819369204449565945369203764688375}$$

Technology has also made it possible to generate graphs of equations and functions in seconds that in the past might have taken hours to produce. Graphing technology includes handheld graphing calculators, computer algebra systems, and software designed for that purpose. Figure 1.3.1 shows the graphs of the function $f(x) = x^4 - x^3 - 2x^2$ produced with various graphing utilities; the first two were generated with the CAS programs, *Mathematica* and *Maple*, and the third with a graphing calculator. Graphing calculators produce coarser graphs than most computer programs but have the advantage of being compact and portable.

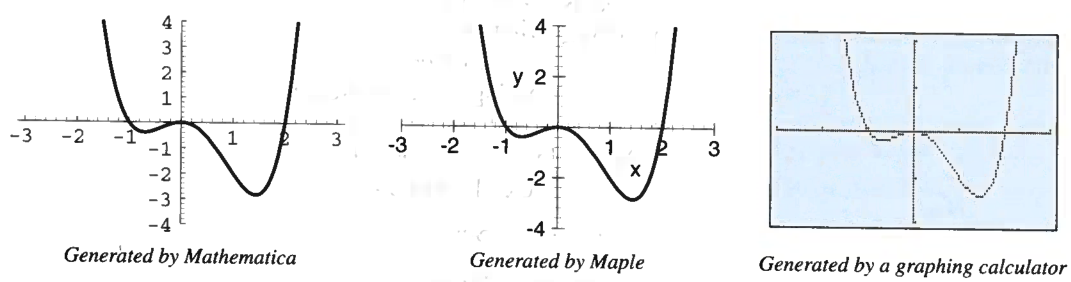


Figure 1.3.1

VIEWING WINDOWS

Graphing utilities can only show a portion of the xy-plane in the viewing screen, so the first step in graphing an equation is to determine which rectangular portion of the xy-plane you want to display. This region is called the ***viewing window*** (or ***viewing rectangle***). For example, in Figure 1.3.1 the viewing window extends over the interval $[-3, 3]$ in the x-direction and over the interval $[-4, 4]$ in the y-direction, so we say that the viewing window is $[-3, 3] \times [-4, 4]$ (read "$[-3, 3]$ by $[-4, 4]$"). In general, if the viewing window is $[a, b] \times [c, d]$, then the window extends between $x = a$ and $x = b$ in the x-direction and between $y = c$ and $y = d$ in the y-direction. We will call $[a, b]$ the ***x-interval*** for the window and $[c, d]$ the ***y-interval*** for the window (Figure 1.3.2).

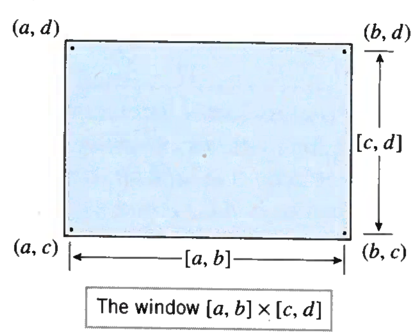


Figure 1.3.2

** Mathematica* is a product of Wolfram Research, Inc.; *Maple* is a product of Waterloo Maple Software, Inc.; and *Derive* is a product of Soft Warehouse, Inc.

Different graphing utilities designate viewing windows in different ways. For example, the first two graphs in Figure 1.3.1 were produced by the commands

```
Plot[x^4 - x^3 -2*x^2, {x, -3, 3}, PlotRange->{-4, 4}]
```
(*Mathematica*)

```
plot( x^4 - x^3 -2*x^2, x = -3..3, y = -4..4);
```
(*Maple*)

and the last graph was produced on a graphing calculator by pressing the GRAPH button after setting the following values for the variables that determine the x-interval and y-intervals:

$x\text{Min} = -3, \quad x\text{Max} = 3, \quad y\text{Min} = -4, \quad y\text{Max} = 4$

FOR THE READER. Use your own graphing utility to generate the graph of the function $f(x) = x^4 - x^3 - 2x^2$ in the window $[-3, 3] \times [-4, 4]$.

TICK MARKS AND GRID LINES

To help locate points in a viewing window visually, graphing utilities provide methods for drawing ***tick marks*** (also called ***scale marks***) on the coordinate axes or at other locations in the viewing window. With computer programs such as *Mathematica* and *Maple*, there are specific commands for designating the spacing between tick marks, but if the user does not specify the spacing, then the programs make certain *default* choices. For example, in the first two parts of Figure 1.3.1, the tick marks shown were the default choices.

On graphing calculators the spacing between tick marks is determined by two ***scale variables*** (also called ***scale factors***), which we will denote by

$x\text{Scl}$ and $y\text{Scl}$

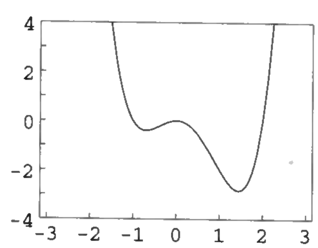
(The notation varies among calculators.) These variables specify the spacing between the tick marks in the x- and y-directions, respectively. For example, in the third part of Figure 1.3.1 the window and tick marks were designated by the settings

$x\text{Min} = -3 \quad x\text{Max} = 3$

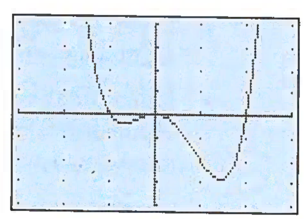
$y\text{Min} = -4 \quad y\text{Max} = 4$

$x\text{Scl} = 1 \quad y\text{Scl} = 1$

Most graphing utilities allow for variations in the design and positioning of tick marks. For example, Figure 1.3.3 shows two variations of the graphs in Figure 1.3.1; the first was generated on a computer using an option for placing the ticks and numbers on the edges of a box, and the second was generated on a graphing calculator using an option for drawing grid lines to simulate graph paper.



Generated by Mathematica



Generated by a graphing calculator

Figure 1.3.3

Example 1

Figure 1.3.4*a* shows the window $[-5, 5] \times [-5, 5]$ with the tick marks spaced .5 unit apart in the x-direction and 10 units apart in the y-direction. Note that no tick marks are actually visible in the y-direction because the tick mark at the origin is covered by the x-axis, and all other tick marks in the y-direction fall outside of the viewing window. ◀

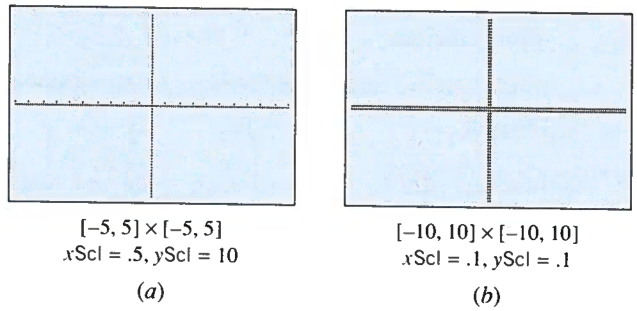


Figure 1.3.4

Example 2

Figure 1.3.4*b* shows the window $[-10, 10] \times [-10, 10]$ with the tick marks spaced .1 unit apart in the x- and y-directions. In this case the tick marks are so close together that they create the effect of thick lines on the coordinate axes. When this occurs you will usually want to increase the scale factors to reduce the number of tick marks and make them legible. ◀

FOR THE READER. Graphing calculators provide a way of clearing all settings and returning them to *default values*. For example, on the author's calculator the default window is $[-10, 10] \times [-10, 10]$ and the default scale factors are $x\mathrm{Scl} = 1$ and $y\mathrm{Scl} = 1$. Check your documentation to determine the default values for your calculator and how to reset the calculator to its default configuration. If you are using a computer program, check your documentation to determine the commands for specifying the spacing between tick marks.

CHOOSING A VIEWING WINDOW

When the graph of a function extends indefinitely in some direction, no single viewing window can show the entire graph. In such cases the choice of the viewing window can drastically affect one's perception of how the graph looks. For example, Figure 1.3.5 shows a computer-generated graph of $y = 9 - x^2$, and Figure 1.3.6 shows four views of this graph generated on the author's calculator:

- In part (a) the graph falls completely outside of the window, so the window is blank (except for the ticks and axes).
- In part (b) the graph is broken into two pieces because it passes in and out of the window.
- In part (c) the graph appears to be a straight line because we have zoomed in on such a small segment of the curve.
- In part (d) we have a more complete picture of the graph shape because the window encompasses all of the important points, namely the high point on the graph and the intersections with the x-axis.

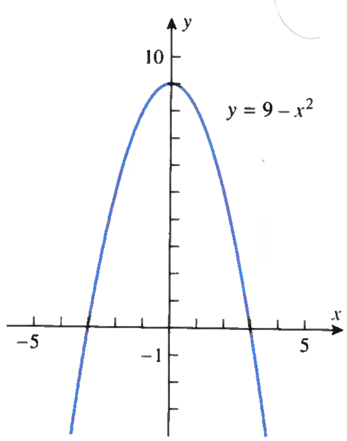


Figure 1.3.5

For a function whose graph does not extend indefinitely in either the x- or y-directions, the domain and range of the function can be used to obtain a viewing window that contains the entire graph.

Example 3

Use the domain and range of the function $f(x) = \sqrt{12 - 3x^2}$ to determine a viewing window that contains the entire graph.

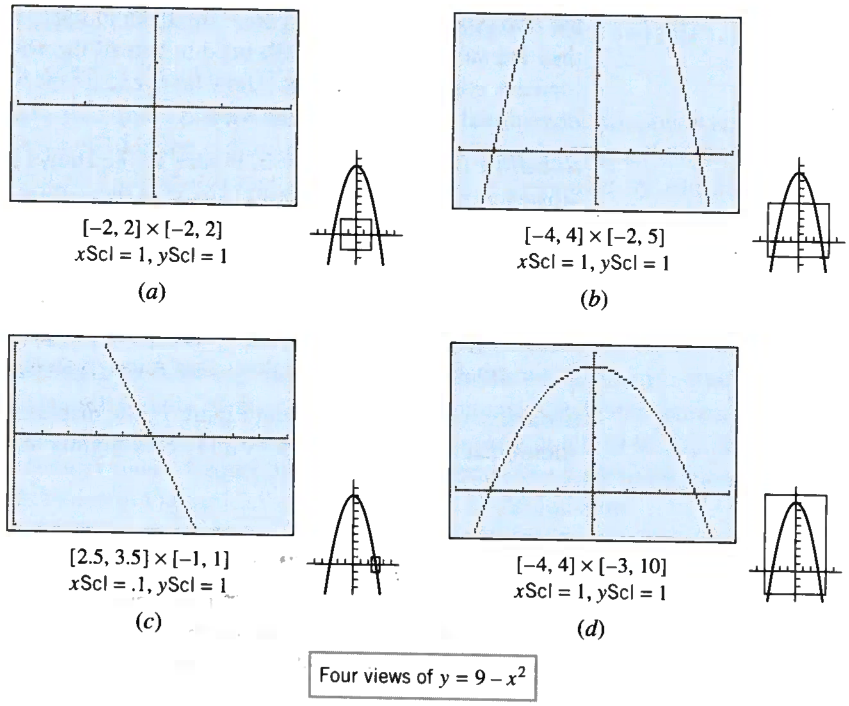
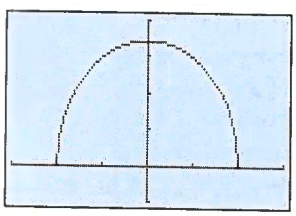


Figure 1.3.6



$[-3, 3] \times [-1, 4]$
$x\text{Scl} = 1, y\text{Scl} = 1$

Figure 1.3.7

Solution. The natural domain of f is $[-2, 2]$ and the range is $[0, \sqrt{12}]$ (verify), so the entire graph will be contained in the viewing window $[-2, 2] \times [0, \sqrt{12}]$. For clarity, it is desirable to use a slightly larger window to avoid having the graph too close to the ends of the screen. For example, taking the viewing window to be $[-3, 3] \times [-1, 4]$ yields the graph in Figure 1.3.7. ◀

If the graph of f extends indefinitely in either the x- or y-direction, then it will not be possible to show the entire graph in any one viewing window. In such cases one tries to choose the window to show all of the important features for the problem at hand. (Of course, what is important in one problem may not be important in another, so the choice of the viewing window will often depend on the objectives in the problem.)

Example 4

Graph the equation $y = x^3 - 12x^2 + 18$ in the following windows and discuss the advantages and disadvantages of each window.

(a) $[-10, 10] \times [-10, 10]$ with $x\text{Scl} = 1$, $y\text{Scl} = 1$
(b) $[-20, 20] \times [-20, 20]$ with $x\text{Scl} = 1$, $y\text{Scl} = 1$
(c) $[-20, 20] \times [-300, 20]$ with $x\text{Scl} = 1$, $y\text{Scl} = 20$
(d) $[-5, 15] \times [-300, 20]$ with $x\text{Scl} = 1$, $y\text{Scl} = 20$
(e) $[1, 2] \times [-1, 1]$ with $x\text{Scl} = .1$, $y\text{Scl} = .1$

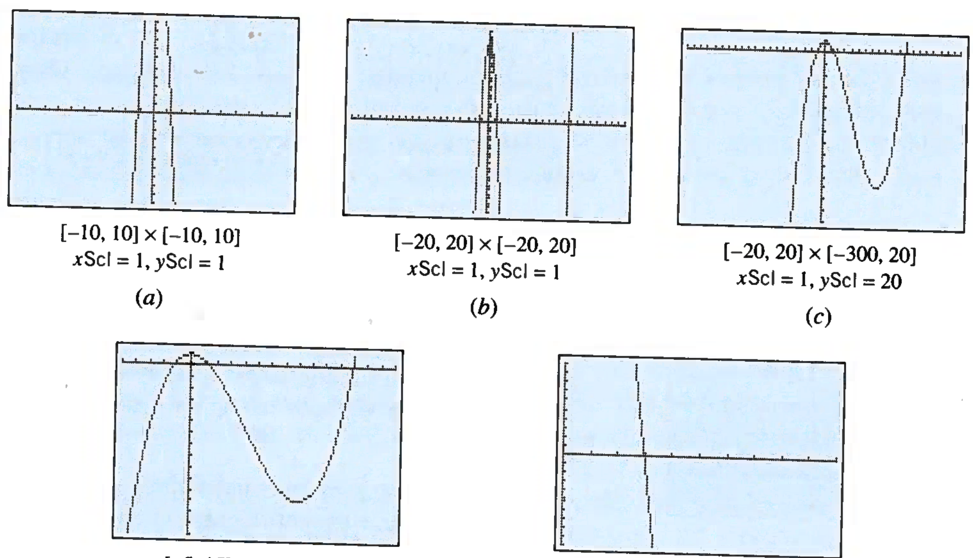
Solution (a). The window in Figure 1.3.8*a* has chopped off the portion of the graph that intersects the y-axis, and it shows only two of three possible real roots for the given cubic polynomial. To remedy these problems we need to widen the window in both the x- and y-directions.

Solution (b). The window in Figure 1.3.8*b* shows the intersection of the graph with the y-axis and the three real roots, but it has chopped off the portion of the graph between the two positive roots. Moreover, the ticks in the y-direction are nearly illegible because they are so close together. We need to extend the window in the negative y-direction and increase yScl. We do not know how far to extend the window, so some experimentation will be required to obtain what we want.

Solution (c). The window in Figure 1.3.8c shows all of the main features of the graph. However, we have some wasted space in the x-direction. We can improve the picture by shortening the window in the x-direction appropriately.

Solution (d). The window in Figure 1.3.8d shows all of the main features of the graph without a lot of wasted space. However, the window does not provide a clear view of the roots. To get a closer view of the roots we must forget about showing all of the main features of the graph and choose windows that zoom in on the roots themselves.

Solution (e). The window in Figure 1.3.8e displays very little of the graph, but it clearly shows that the root in the interval $[1, 2]$ is slightly less than 1.3, say $x \approx 1.29$. ◀



$[-10, 10] \times [-10, 10]$
xScl = 1, yScl = 1
(a)

$[-20, 20] \times [-20, 20]$
xScl = 1, yScl = 1
(b)

$[-20, 20] \times [-300, 20]$
xScl = 1, yScl = 20
(c)

$[-5, 15] \times [-300, 20]$
xScl = 1, yScl = 20
(d)

$[1, 2] \times [-1, 1]$
xScl = .1, yScl = .1
(e)

Figure 1.3.8

FOR THE READER. Sometimes you will want to determine the viewing window by choosing the x-interval for the window and allowing the graphing utility to determine a y-interval that encompasses the maximum and minimum values of the function over the x-interval. Most graphing utilities provide some method for doing this, so check your documentation to determine how to use this feature. Allowing the graphing utility to determine the y-interval of the window takes some of the guesswork out of problems like that in part (b) of the preceding example.

ZOOMING

The process of enlarging or reducing the size of a viewing window is called ***zooming***. If you reduce the size of the window, you see less of the graph as a whole, but more detail of the part shown; this is called ***zooming in***. In contrast, if you enlarge the size of the window, you see more of the graph as a whole, but less detail of the part shown; this is called ***zooming out***. Most graphing calculators provide menu items for zooming in or zooming out by fixed factors. For example, on the author's calculator the amount of enlargement or reduction is controlled by setting values for two ***zoom factors***, xFact and yFact. If

$$x\text{Fact} = 10 \quad \text{and} \quad y\text{Fact} = 5$$

then each time a zoom command is executed the viewing window is enlarged or reduced by a factor of 10 in the x-direction and a factor of 5 in the y-direction. With computer programs such as *Mathematica* and *Maple*, zooming is controlled by adjusting the x-interval and y-interval directly; however, there are ways to automate this by programming.

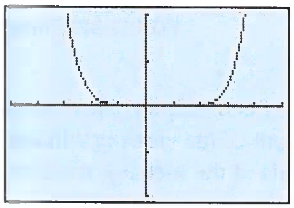
FOR THE READER. If you are using a graphing calculator, read your documentation to determine how to use the zooming feature.

COMPRESSION

Enlarging the viewing window for a graph has the geometric effect of compressing the graph, since more of the graph is packed into the calculator screen. If the compression is sufficiently great, then some of the detail in the graph may be lost. Thus, the choice of the viewing window frequently depends on whether you want to see more of the graph or more of the detail. Figure 1.3.9 shows two views of the equation

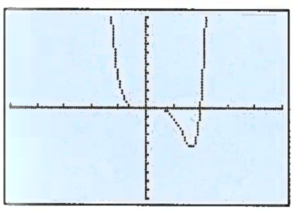
$$y = x^5(x-2)$$

In part (*a*) of the figure the y-interval is very large, resulting in a vertical compression that obscures the detail in the vicinity of the x-axis. In part (*b*) the y-interval is smaller, and consequently we see more of the detail in the vicinity of the x-axis but less of the graph in the y-direction.



$[-5, 5] \times [-1000, 1000]$
xScl = 1, yScl = 500

(*a*)



$[-5, 5] \times [-10, 10]$
xScl = 1, yScl = 1

(*b*)

Figure 1.3.9

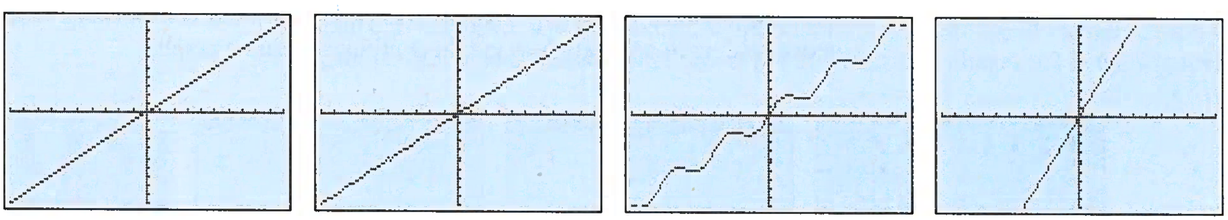
Example 5

Describe the graph of the function $f(x) = x + 0.01 \sin 50\pi x$; then graph the function in the following windows and explain why the graphs do or do not differ from your description.

(a) $[-10, 10] \times [-10, 10]$ (b) $[-1, 1] \times [-1, 1]$
(c) $[-.1, .1] \times [-.1, .1]$ (d) $[-.01, .01] \times [-.01, .01]$

Solution. The formula for f is the sum of the function x (whose graph is a straight line) and the function $0.01 \sin 50\pi x$ (whose graph is a sinusoidal curve with an amplitude of 0.01 and a period of $2\pi/50\pi = 0.04$). Intuitively, this suggests that the graph of f will follow the general path of the line $y = x$ but will have small bumps resulting from the contributions of the sinusoidal oscillations.

To generate the four graphs, we first set the calculator to the radian mode.* Because the windows in successive parts of this example are decreasing in size by a factor of 10, it will be convenient to use the zoom in feature of the calculator with the zoom factors set to 10 in the x- and y-directions. In Figure 1.3.10*a* the graph appears to be a straight line



$[-10, 10] \times [-10, 10]$
xScl = 1, yScl = 1
(*a*)

$[-1, 1] \times [-1, 1]$
xScl = .1, yScl = .1
(*b*)

$[-.1, .1] \times [-.1, .1]$
xScl = .01, yScl = .01
(*c*)

$[-.01, .01] \times [-.01, .01]$
xScl = .001, yScl = .001
(*d*)

Figure 1.3.10

*In this text we follow the convention that angles are measured in radians unless degree measure is specified.

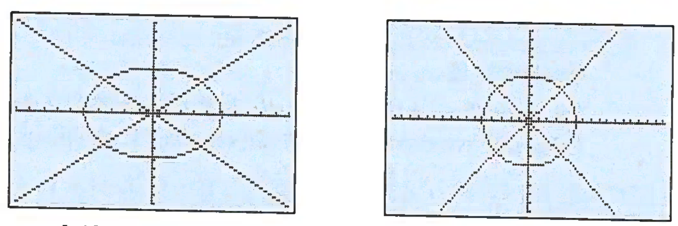
because compression has hidden the small sinusoidal oscillations. (Keep in mind that the amplitude of the sinusoidal portion of the function is only 0.01.) In part (*b*) the oscillations have begun to appear since the y-interval has been reduced, and in part (*c*) the oscillations have become very clear because the vertical scale is more in keeping with the amplitude of the oscillations. In part (*d*) the graph appears to be a line segment because we have zoomed in on such a small portion of the curve. ◀

ASPECT RATIO DISTORTION

Figure 1.3.11*a* shows a circle of radius 5 and two perpendicular lines graphed in the window $[-10, 10] \times [-10, 10]$ with $x\text{Scl} = 1$ and $y\text{Scl} = 1$. However, the circle is distorted and the lines do not appear perpendicular because the calculator has not used the same length for 1 unit on the x-axis and 1 unit on the y-axis. (Compare the spacing between the ticks on the axes.) This is called ***aspect ratio distortion***. Many calculators provide a menu item for automatically correcting the distortion by adjusting the viewing window appropriately. For example, the author's calculator makes this correction to the viewing window $[-10, 10] \times [-10, 10]$ by changing it to

$$[-16.9970674487, 16.9970674487] \times [-10, 10]$$

(Figure 1.3.11*b*). With computer programs such as *Mathematica* and *Maple*, aspect ratio distortion is controlled with adjustments to the physical dimensions of the viewing window on the computer screen, rather than altering the x- and y-intervals of the viewing window.



$[-10, 10] \times [-10, 10]$
$x\text{Scl} = 1, y\text{Scl} = 1$
(*a*)

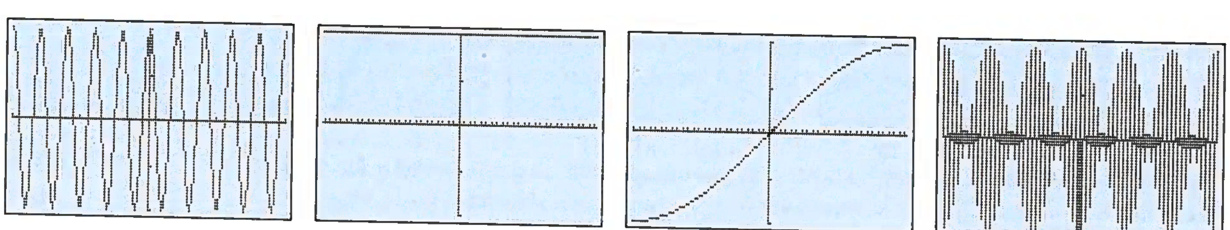
$[-16.9970674487, 16.9970674487] \times [-10, 10]$
$x\text{Scl} = 1, y\text{Scl} = 1$
(*b*)

Figure 1.3.11

FOR THE READER. Read the documentation for your graphing utility to determine how to control aspect ratio distortion.

PIXELS AND RESOLUTION

Sometimes graphing utilities produce unexpected results. For example, Figure 1.3.12 shows the graph of $y = \cos(10\pi x)$ generated on the author's graphing calculator in four different windows. (Your own calculator may produce different results.) The first graph has the correct shape, but the remaining three do not. To explain what is happening here we need to understand more precisely how graphing utilities generate graphs.



$[-1, 1] \times [-1, 1]$
$x\text{Scl} = .5, y\text{Scl} = .5$
(*a*)

$[-12.6, 12.6] \times [-1, 1]$
$x\text{Scl} = 1, y\text{Scl} = .5$
(*b*)

$[-12.5, 12.6] \times [-1, 1]$
$x\text{Scl} = 1, y\text{Scl} = .5$
(*c*)

$[-6, 6] \times [-1, 1]$
$x\text{Scl} = 1, y\text{Scl} = .5$
(*d*)

Figure 1.3.12

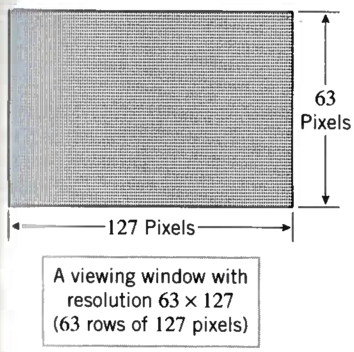


Figure 1.3.13

Screen displays for graphing utilities are divided into rows and columns of rectangular blocks, called ***pixels***. For black-and-white displays each pixel has two possible states—an activated (or dark) state and a deactivated (or light) state. Since graphical elements are produced by activating pixels, the more pixels that a screen has to work with, the greater the amount of detail it can show. For example, the author's calculator has a ***resolution*** of 63×127, meaning that there are 63 rows with 127 pixels per row (Figure 1.3.13). In contrast, the author's computer screen has a resolution of 1024×1280 (1024 rows with 1280 pixels per row), so the computer screen is capable of displaying much smoother graphs than the calculator.

FOR THE READER. If you are using a graphing calculator, check the documentation to determine its resolution.

SAMPLING ERROR

The procedure that a graphing utility follows to generate a graph is similar to the procedure for plotting points by hand. When a viewing window is selected and an equation is entered, the graphing utility determines the x-coordinates of certain pixels on the x-axis and computes the corresponding points (x, y) on the graph. It then activates the pixels whose coordinates most closely match those of the calculated points and uses some built-in algorithm to activate additional intermediate pixels to create the curve shape. The point to keep in mind here is that *changing the window changes the points plotted by the graphing utility*. Thus, it is possible that a particular window will produce a false impression about the graph shape because significant characteristics of the graph occur *between* the plotted pixels. This is called ***sampling error***. This is exactly what occurred in Figure 1.3.12 when we graphed $y = \cos(10\pi x)$. In part (*b*) of the figure the plotted pixels happened to fall at the peaks of the cosine curve, giving the false impression that the graph is a horizontal line at $y = 1$. In part (*c*) the plotted pixels fell at successively higher points along the graph, and in part (*d*) the plotted pixels fell in a strange way that created yet another misleading impression of the graph shape.

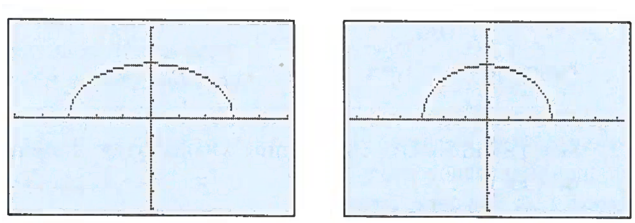
REMARK. Figure 1.3.12 suggests that for trigonometric graphs with rapid oscillations, restricting the x-interval to a few periods is likely to produce a more accurate representation about the graph shape.

FALSE GAPS

Sometimes graphs that are continuous appear to have gaps when they are generated on a calculator. These ***false gaps*** typically occur where the graph rises so rapidly that vertical space is opened up between successive pixels.

Example 6

Figure 1.3.14 shows the graph of the semicircle $y = \sqrt{9 - x^2}$ in two viewing windows. Although this semicircle has x-intercepts at the points $x = \pm 3$, part (*a*) of the figure shows false gaps at those points because there are no pixels with x-coordinates ± 3 in the window



$[-5, 5] \times [-5, 5]$
$x\text{Scl} = 1, y\text{Scl} = 1$
(*a*)

$[-6.3, 6.3] \times [-5, 5]$
$x\text{Scl} = 1, y\text{Scl} = 1$
(*b*)

Figure 1.3.14

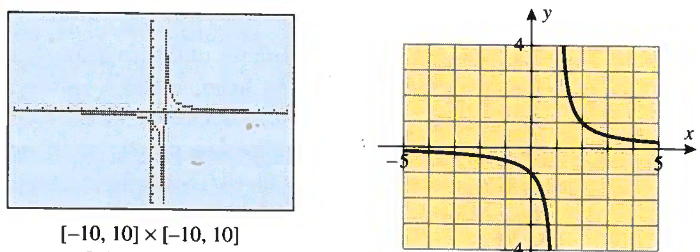
selected. In part (*b*) no gaps occur because there are pixels with x-coordinates $x = \pm 3$ in the window being used. ◀

FALSE LINE SEGMENTS

In addition to creating false gaps in continuous graphs, calculators can err in the opposite direction by placing ***false line segments*** in the gaps of discontinuous curves.

Example 7

Figure 1.3.15*a* shows the graph of $y = 1/(x-1)$ in the default window on the author's calculator. Although the graph appears to contain vertical line segments near $x = 1$, they should not be there. There is actually a gap in the curve at $x = 1$, since a division by zero occurs at that point (Figure 1.3.15*b*). ◀



$[-10, 10] \times [-10, 10]$
xScl = 1, yScl = 1

$y = 1/(x-1)$ with false line segments

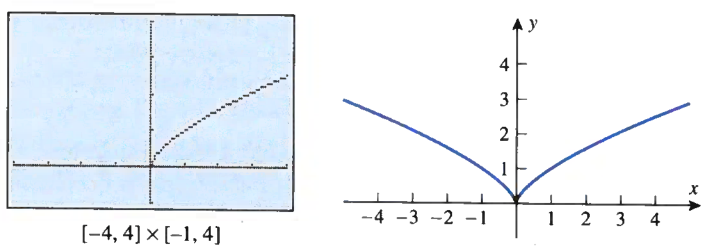
Actual curve shape of $y = 1/(x-1)$

(*a*) (*b*)

Figure 1.3.15

ERRORS OF OMISSION

Most graphing utilities use logarithms to evaluate functions with fractional exponents such as $f(x) = x^{2/3} = \sqrt[3]{x^2}$. However, because logarithms are only defined for positive numbers, many (but not all) graphing utilities will omit portions of the graphs of functions with fractional exponents. For example, the author's calculator graphs $y = x^{2/3}$ as in Figure 1.3.16*a*, whereas the actual graph is as in Figure 1.3.16*b*. (See the discussion preceding Exercise 29 for a way of circumventing this problem.)



$[-4, 4] \times [-1, 4]$
xScl = 1, yScl = 1

(*a*) (*b*)

Figure 1.3.16

FOR THE READER. Determine whether your graphing utility produces the complete graph of $y = x^{2/3}$.

WHAT IS THE TRUE SHAPE OF A GRAPH?

Although graphing utilities are powerful tools for generating graphs quickly, they can produce misleading graphs as a result of compression, sampling error, false gaps, and false line segments. In short, *graphing utilities can suggest graph shapes, but they cannot establish them with certainty*. Thus, the more you know about the functions you are graphing, the easier it will be to choose good viewing windows, and the better you will be able to judge the reasonableness of the results produced by your graphing utility.

MORE INFORMATION ON GRAPHING AND CALCULATING UTILITIES

The main source of information about your graphing utility is its own documentation, and from time to time we will suggest that you refer to that documentation to learn some particular technique.

EXERCISE SET 1.3

1. Use a graphing utility to generate the graph of the function $f(x) = x^4 - x^2$ in the given viewing windows, and specify the window that you think gives the best view of the graph.
   (a) $-50 \le x \le 50,\ -50 \le y \le 50$
   (b) $-5 \le x \le 5,\ -5 \le y \le 5$
   (c) $-2 \le x \le 2,\ -2 \le y \le 2$
   (d) $-2 \le x \le 2,\ -1 \le y \le 1$
   (e) $-1.5 \le x \le 1.5,\ -0.5 \le y \le 0.5$
2. Use a graphing utility to generate the graph of the function $f(x) = x^5 - x^3$ in the given viewing windows, and specify the window that you think gives the best view of the graph.
   (a) $-50 \le x \le 50,\ -50 \le y \le 50$
   (b) $-5 \le x \le 5,\ -5 \le y \le 5$
   (c) $-2 \le x \le 2,\ -2 \le y \le 2$
   (d) $-2 \le x \le 2,\ -1 \le y \le 1$
   (e) $-1.5 \le x \le 1.5,\ -0.5 \le y \le 0.5$
3. Use a graphing utility to generate the graph of the function $f(x) = x^2 + 12$ in the given viewing windows, and specify the window that you think gives the best view of the graph.
   (a) $-1 \le x \le 1,\ 13 \le y \le 15$
   (b) $-2 \le x \le 2,\ 11 \le y \le 15$
   (c) $-4 \le x \le 4,\ 10 \le y \le 28$
   (d) A window of your choice
4. Use a graphing utility to generate the graph of the function $f(x) = -12 - x^2$ in the given viewing windows, and specify the window that you think gives the best view of the graph.
   (a) $-1 \le x \le 1,\ -15 \le y \le -13$
   (b) $-2 \le x \le 2,\ -15 \le y \le -11$
   (c) $-4 \le x \le 4,\ -28 \le y \le -10$
   (d) A window of your choice

In Exercises 5 and 6, use the domain and range of f to determine a viewing window that contains the entire graph, and generate the graph in that window.

5. $f(x) = \sqrt{16 - 2x^2}$
6. $f(x) = \sqrt{3 - 2x - x^2}$
7. Graph the function $f(x) = x^3 - 15x^2 - 3x + 45$ using the stated windows and tick spacing, and discuss the advantages and disadvantages of each window.
   (a) $-10 \le x \le 10,\ -10 \le y \le 10$ with $x\text{Scl} = 1$ and $y\text{Scl} = 1$
   (b) $-20 \le x \le 20,\ -20 \le y \le 20$ with $x\text{Scl} = 1$ and $y\text{Scl} = 1$
   (c) $-5 \le x \le 20,\ -500 \le y \le 50$ with $x\text{Scl} = 5$ and $y\text{Scl} = 50$
   (d) $-2 \le x \le -1,\ -1 \le y \le 1$ with $x\text{Scl} = 0.1$ and $y\text{Scl} = 0.1$
   (e) $9 \le x \le 11,\ -486 \le y \le -484$ with $x\text{Scl} = 0.1$ and $y\text{Scl} = 0.1$
8. Graph the function $f(x) = -x^3 - 12x^2 + 4x + 48$ using the stated windows and tick spacing, and discuss the advantages and disadvantages of each window.
   (a) $-10 \le x \le 10,\ -10 \le y \le 10$ with $x\text{Scl} = 1$ and $y\text{Scl} = 1$
   (b) $-20 \le x \le 20,\ -20 \le y \le 20$ with $x\text{Scl} = 1$ and $y\text{Scl} = 1$
   (c) $-16 \le x \le 4,\ -250 \le y \le 50$ with $x\text{Scl} = 2$ and $y\text{Scl} = 25$
   (d) $-3 \le x \le -1,\ -1 \le y \le 1$ with $x\text{Scl} = 0.1$ and $y\text{Scl} = 0.1$
   (e) $-9 \le x \le -7,\ -241 \le y \le -239$ with $x\text{Scl} = 0.1$ and $y\text{Scl} = 0.1$

In Exercises 9–16, generate the graph of f in a viewing window that you think is appropriate.

9. $f(x) = x^2 - 9x - 36$
10. $f(x) = \dfrac{x+7}{x-9}$
11. $f(x) = 2\cos 80x$
12. $f(x) = 12\sin(x/80)$
13. $f(x) = 300 - 10x^2 + 0.01x^3$
14. $f(x) = x(30 - 2x)(25 - 2x)$
15. $f(x) = x^2 + \dfrac{1}{x}$
16. $f(x) = \sqrt{11x - 18}$

In Exercises 17 and 18, generate the graph of f and determine whether your graphs contain false line segments. Sketch the actual graph and see if you can make the false line segments disappear by changing the viewing window.

17. $f(x) = \dfrac{x}{x^2 - 1}$
18. $f(x) = \dfrac{x^2}{4 - x^2}$

19. The graph of the equation $x^2 + y^2 = 16$ is a circle of radius 4 centered at the origin.
(a) Find a function whose graph is the upper semicircle and graph it.
(b) Find a function whose graph is the lower semicircle and graph it.
(c) Graph the upper and lower semicircles together. If the combined graphs do not appear circular, see if you can adjust the viewing window to eliminate the aspect ratio distortion.
(d) Graph the portion of the circle in the first quadrant.
(e) Is there a function whose graph is the right half of the circle? Explain.

20. In each part, graph the equation by solving for y in terms of x and graphing the resulting functions together.
(a) $x^2/4 + y^2/9 = 1$ (b) $y^2 - x^2 = 1$

21. Read the documentation for your graphing utility to determine how to graph functions involving absolute values, and graph the given equation.
(a) $y = |x|$ (b) $y = |x - 1|$
(c) $y = |x| - 1$ (d) $y = |\sin x|$
(e) $y = \sin |x|$ (f) $y = |x| - |x + 1|$

22. Based on your knowledge of the absolute value function, sketch the graph of $f(x) = |x|/x$. Check your result using a graphing utility.

23. Make a conjecture about the relationship between the graph of $y = f(x)$ and the graph of $y = |f(x)|$; check your conjecture with some specific functions.

24. Make a conjecture about the relationship between the graph of $y = f(x)$ and the graph of $y = f(|x|)$; check your conjecture with some specific functions.

25. (a) Based on your knowledge of the absolute value function, sketch the graph of $y = |x - a|$, where a is a constant. Check your result using a graphing utility and some specific values of a.
(b) Sketch the graph of $y = |x - 1| + |x - 2|$; check your result with a graphing utility.

26. How are the graphs of $y = |x|$ and $y = \sqrt{x^2}$ related? Check your answer with a graphing utility.

> Most graphing utilities provide some way of graphing functions that are defined piecewise; read the documentation for your graphing utility to find out how to do this. However, if your goal is just to find the general shape of the graph, you can graph each portion of the function separately and combine the pieces with a hand-drawn sketch. Use this method in Exercises 27 and 28.

27. Draw the graph of

$$f(x) = \begin{cases} \sqrt[3]{x - 2}, & x \le 2 \\ x^3 - 2x - 4, & x > 2 \end{cases}$$

28. Draw the graph of

$$f(x) = \begin{cases} x^3 - x^2, & x \le 1 \\ \dfrac{1}{1 - x}, & 1 < x < 4 \\ x^2 \cos \sqrt{x}, & 4 \le x \end{cases}$$

> We noted in the text that for functions involving fractional exponents (or radicals), graphing utilities sometimes omit portions of the graph. If $f(x) = x^{p/q}$, where p/q is a positive fraction in *lowest terms*, then you can circumvent this problem as follows:
> - If p is even and q is odd, then graph $g(x) = |x|^{p/q}$ instead of $f(x)$.
> - If p is odd and q is odd, then graph $g(x) = (|x|/x)|x|^{p/q}$ instead of $f(x)$.
>
> We will explain why this works in the exercises of the next section.

29. (a) Generate the graphs of $f(x) = x^{2/5}$ and $g(x) = |x|^{2/5}$, and determine whether your graphing utility missed part of the graph of f.
(b) Generate the graphs of the functions $f(x) = x^{1/5}$ and $g(x) = (|x|/x)|x|^{1/5}$, and then determine whether your graphing utility missed part of the graph of f.
(c) Generate a complete graph of the equation

$$y = (x - 1)^{4/5}$$

(d) Generate a complete graph of the equation

$$y = (x + 1)^{3/4}$$

30. The graphs of $y = (x^2 - 4)^{2/3}$ and $y = [(x^2 - 4)^2]^{1/3}$ should be the same. Does your graphing utility produce the same graph for both equations? If not, what do you think is happening?

31. In each part, graph the function for various values of c, and write a paragraph or two that describes how changes in c affect the graph in each case.
(a) $y = cx^2$ (b) $y = x^2 + cx$
(c) $y = x^2 + x + c$

32. The graph of an equation of the form $y^2 = x(x - a)(x - b)$ (where $0 < a < b$) is called a ***bipartite cubic***. The accompanying figure shows a typical graph of this type.

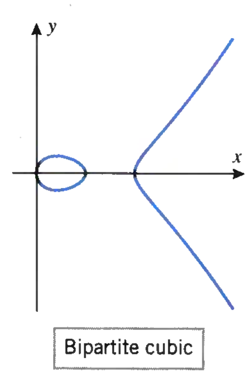


Figure Ex-32

(a) Graph the bipartite cubic $y^2 = x(x-1)(x-2)$ by solving for y in terms of x and graphing the two resulting functions.

(b) Find the x-intercepts of the bipartite cubic

$$y^2 = x(x-a)(x-b)$$

and make a conjecture about how changes in the values of a and b would affect the graph. Test your conjecture by graphing the bipartite cubic for various values of a and b.

33. Based on your knowledge of the graphs of $y = x$ and $y = \sin x$, make a sketch of the graph of $y = x \sin x$. Check your conclusion using a graphing utility.

34. What do you think the graph of $y = \sin(1/x)$ looks like? Test your conclusion using a graphing utility. [*Suggestion:* Examine the graph on a succession of smaller and smaller intervals centered at $x = 0$.]

1.4 NEW FUNCTIONS FROM OLD

Just as numbers can be added, subtracted, multiplied, and divided to produce other numbers, so functions can be added, subtracted, multiplied, and divided to produce other functions. In this section we will discuss these operations and some others that have no analogs in ordinary arithmetic.

ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions $f+g$, $f-g$, fg, and f/g. For example, $f+g$ is defined by the formula

$$(f+g)(x) = f(x) + g(x) \tag{1}$$

which states that for each input the value of $f+g$ is obtained by adding the values of f and g. For example, if

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

then

$$(f+g)(x) = f(x) + g(x) = x + x^2$$

Equation (1) provides a formula for $f+g$ but does not say anything about the domain of $f+g$. However, for the right side of this equation to be defined, x must lie in the domain of f *and* in the domain of g, so we define the domain of $f+g$ to be the intersection of those two domains. More generally, we make the following definition:

1.4.1 DEFINITION. Given functions f and g, we define

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(f/g)(x) = f(x)/g(x)$$

For the functions $f+g$, $f-g$, and fg we define the domain to be the intersection of the domains of f and g, and for the function f/g we define the domain to be the intersection of the domains of f and g but with the points where $g(x) = 0$ excluded (to avoid division by zero).

REMARK. If f is a constant function, say $f(x) = c$ for all x, then the product of f and g is cg, so multiplying a function by a constant is a special case of multiplying two functions.

Example 1

Let

$$f(x) = 1 + \sqrt{x-2} \quad \text{and} \quad g(x) = x - 3$$

Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $(f/g)(x)$, and $(7f)(x)$; state the domains of $f+g$, $f-g$, fg, f/g, and $7f$.

Solution. First, we will find formulas for the functions and then the domains. The formulas are

$$(f+g)(x) = f(x) + g(x) = (1 + \sqrt{x-2}) + (x-3) = x - 2 + \sqrt{x-2} \tag{2}$$
$$(f-g)(x) = f(x) - g(x) = (1 + \sqrt{x-2}) - (x-3) = 4 - x + \sqrt{x-2} \tag{3}$$
$$(fg)(x) = f(x)g(x) = (1 + \sqrt{x-2})(x-3) \tag{4}$$
$$(f/g)(x) = f(x)/g(x) = \frac{1 + \sqrt{x-2}}{x-3} \tag{5}$$
$$(7f)(x) = 7f(x) = 7 + 7\sqrt{x-2} \tag{6}$$

In all five cases the natural domain determined by the formula is the same as the domain specified in Definition 1.4.1, so there is no need to state the domain explicitly in any of these cases. For example, the domain of f is $[2, +\infty)$, the domain of g is $(-\infty, +\infty)$, and the natural domain for $f(x) + g(x)$ determined by Formula (2) is $[2, +\infty)$, which is precisely the intersection of the domains of f and g. ◀

REMARK. There are situations in which the natural domain associated with the formula resulting from an operation on two functions is not the correct domain for the new function. For example, if $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x}$, then according to Definition 1.4.1 the domain of fg should be $[0, +\infty) \cap [0, +\infty) = [0, +\infty)$. However, $(fg)(x) = \sqrt{x}\sqrt{x} = x$, which has a natural domain of $(-\infty, +\infty)$. Thus, to be precise in describing the formula for fg, we must write $(fg)(x) = x$, $x \geq 0$.

STRETCHES AND COMPRESSIONS

Multiplying a function f by a *nonnegative* constant c has the geometric effect of stretching or compressing the graph of f vertically. For example, examine the graphs of $y = f(x)$, $y = 2f(x)$, and $y = \frac{1}{2}f(x)$ shown in Figure 1.4.1*a*. Multiplying by 2 doubles each y-coordinate, thereby stretching the graph, and multiplying by $\frac{1}{2}$ cuts each y-coordinate in

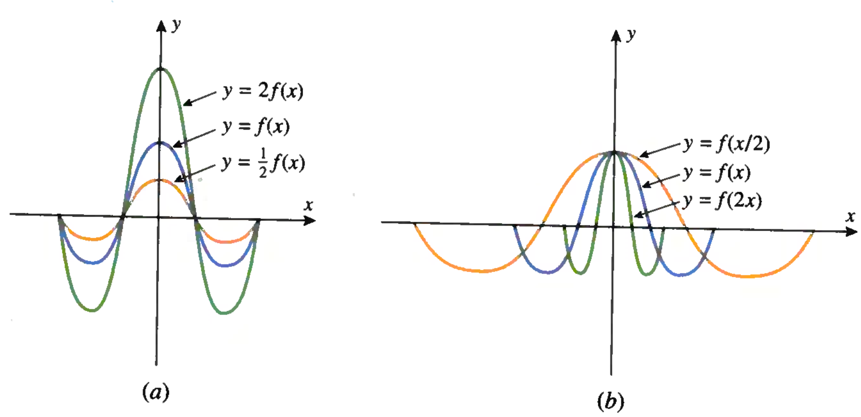


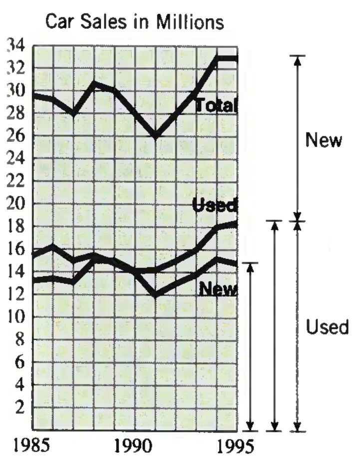
Figure 1.4.1

half, thereby compressing the graph. In general, if $c > 0$, then the graph of $y = cf(x)$ can be obtained from the graph of $y = f(x)$ by compressing the graph of $y = f(x)$ vertically by a factor of $1/c$ if $0 < c < 1$, or stretching it by a factor of c if $c > 1$.

Analogously, multiplying the independent variable of a function f by a ***nonnegative*** constant c has the geometric effect of stretching or compressing the graph of f horizontally. For example, examine the graphs of $y = f(x)$, $y = f(2x)$, and $y = f(x/2)$ shown in Figure 1.4.1*b*. Multiplying x by 2 compresses the graph by a factor of 2 and multiplying x by $\frac{1}{2}$ stretches the graph by a factor of 2. [This is a little confusing, but think of it this way: The value of $2x$ changes twice as fast as the value of x, so a point moving along the x-axis will only have to move half as far from the origin for $y = f(2x)$ to have the same value as $y = f(x)$.] In general, if $c > 0$, then the graph of $y = f(cx)$ can be obtained from the graph of $y = f(x)$ by stretching the graph of $y = f(x)$ horizontally by a factor of c if $0 < c < 1$, or compressing it by a factor of c if $c > 1$.

SUMS OF FUNCTIONS

Adding two functions can be accomplished geometrically by adding the corresponding y-coordinates of their graphs. For example, Figure 1.4.2 shows line graphs of yearly new car sales $N(t)$ and used car sales $U(t)$ in the United States between 1985 and 1995. The sum of these functions, $T(t) = N(t) + U(t)$, represents the yearly total car sales for that period. As illustrated in the figure, the graph of $T(t)$ can be obtained by adding the values of $N(t)$ and $U(t)$ together at each time t and plotting the resulting value.



Source: NADA.

Figure 1.4.2

Example 2

Referring to Figure 1.2.2 for the graphs of $y = \sqrt{x}$ and $y = 1/x$, make a sketch that shows the general shape of the graph of $y = \sqrt{x} + 1/x$ for $x \geq 0$.

Solution. To add the corresponding y-values of $y = \sqrt{x}$ and $y = 1/x$ graphically, just imagine them to be "stacked" on top of one another. This yields the sketch in Figure 1.4.3. ◀

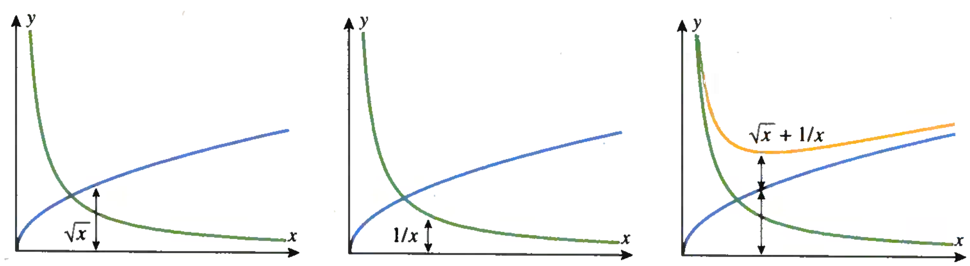


Figure 1.4.3

COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called *composition*, which has no direct analog in ordinary arithmetic. Informally stated, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1$$

If we substitute $g(x)$ for x in the formula for f, we obtain a new function

$$f(g(x)) = (g(x))^2 = (x + 1)^2$$

which we denote by $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x + 1)^2$$

In general, we make the following definition.

1.4.2 DEFINITION. Given functions f and g, the ***composition*** of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f.

REMARK. Although the domain of $f \circ g$ may seem complicated at first glance, it makes sense intuitively: To compute $f(g(x))$ one needs x in the domain of g to compute $g(x)$, then one needs $g(x)$ in the domain of f to compute $f(g(x))$.

COMPOSITIONS VIEWED AS COMPUTER PROGRAMS

In Section 1.1 we noted that a function f can be viewed as a computer program that takes an input x, operates on it, and produces an output $f(x)$. From this viewpoint composition can be viewed as two programs, g and f, operating in succession: An input x is fed first to a program g, which produces the output $g(x)$; then this output is fed as input to a program f, which produces the output $f(g(x))$ (Figure 1.4.4). However, rather than have two separate programs operating in succession, we could create a ***single*** program that takes the input x and directly produces the output $f(g(x))$. This program is the composition $f \circ g$ since $(f \circ g)(x) = f(g(x))$.

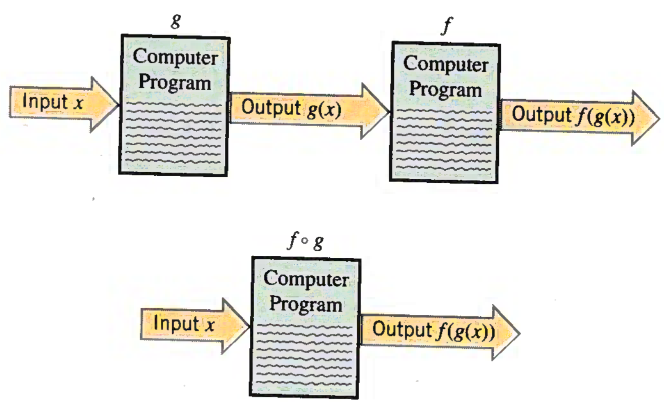


Figure 1.4.4

Example 3

Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). The formula for $f(g(x))$ is

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x}\,)^2 + 3 = x + 3$$

Since the domain of g is $[0, +\infty)$ and the domain of f is $(-\infty, +\infty)$, the domain of $f \circ g$ consists of all x in $[0, +\infty)$ such that $g(x) = \sqrt{x}$ lies in $(-\infty, +\infty)$; thus, the domain of $f \circ g$ is $[0, +\infty)$. Therefore,

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

Solution (b). The formula for $g(f(x))$ is

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$$

Since the domain of f is $(-\infty, +\infty)$ and the domain of g is $[0, +\infty)$, the domain of $g \circ f$ consists of all x in $(-\infty, +\infty)$ such that $f(x) = x^2 + 3$ lies in $[0, +\infty)$. Thus, the domain of

$g \circ f$ is $(-\infty, +\infty)$. Therefore,

$$(g \circ f)(x) = \sqrt{x^2 + 3}$$

There is no need to indicate that the domain is $(-\infty, +\infty)$, since this is the natural domain of $\sqrt{x^2 + 3}$. ◀

REMARK. Note that the functions $f \circ g$ and $g \circ f$ in the preceding example are not the same. Thus, the order in which functions are composed can (and usually will) make a difference in the end result.

Compositions can also be defined for three or more functions; for example, $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

In other words, first find $h(x)$, then find $g(h(x))$, and then find $f(g(h(x)))$.

Example 4

Find $(f \circ g \circ h)(x)$ if

$$f(x) = \sqrt{x}, \quad g(x) = 1/x, \quad h(x) = x^3$$

Solution.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3)) = f(1/x^3) = \sqrt{1/x^3} = 1/x^{3/2}$$ ◀

EXPRESSING A FUNCTION AS A COMPOSITION

Many problems in mathematics are attacked by "decomposing" functions into compositions of simpler functions. For example, consider the function h given by

$$h(x) = (x + 1)^2$$

To evaluate $h(x)$ for a given value of x, we would first compute $x + 1$ and then square the result. These two operations are performed by the functions

$$g(x) = x + 1 \quad \text{and} \quad f(x) = x^2$$

We can express h in terms of f and g by writing

$$h(x) = (x + 1)^2 = [g(x)]^2 = f(g(x))$$

so we have succeeded in expressing h as the composition $h = f \circ g$.

The thought process in this example suggests a general procedure for decomposing a function h into a composition $h = f \circ g$:

- Think about how you would evaluate $h(x)$ for a specific value of x, trying to break the evaluation into two steps performed in succession.
- The first operation in the evaluation will determine a function g and the second a function f.
- The formula for h can then be written as $h(x) = f(g(x))$.

For descriptive purposes, we will refer to g as the "inside function" and f as the "outside function" in the expression $f(g(x))$. The inside function performs the first operation and the outside function performs the second.

Example 5

Express $h(x) = (x - 4)^5$ as a composition of two functions.

Solution. To evaluate $h(x)$ for a given value of x we would first compute $x - 4$ and then raise the result to the fifth power. Therefore, the inside function (first operation) is

$$g(x) = x - 4$$

and the outside function (second operation) is

$$f(x) = x^5$$

so $h(x) = f(g(x))$. As a check,

$$f(g(x)) = [g(x)]^5 = (x-4)^5 = h(x)$$ ◀

Example 6

Express $\sin(x^3)$ as a composition of two functions.

Solution. To evaluate $\sin(x^3)$, we would first compute x^3 and then take the sine, so $g(x) = x^3$ is the inside function and $f(x) = \sin x$ the outside function. Therefore,

$$\sin(x^3) = f(g(x)) \qquad \boxed{g(x) = x^3 \text{ and } f(x) = \sin x}$$ ◀

Example 7

Table 1.4.1 gives some more examples of decomposing functions into compositions.

Table 1.4.1

FUNCTION	$g(x)$ INSIDE	$f(x)$ OUTSIDE	COMPOSITION
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10} = f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5) = f(g(x))$
$\sqrt{4-3x}$	$4-3x$	$\sqrt{x}$	$\sqrt{4-3x} = f(g(x))$
$8+\sqrt{x}$	$\sqrt{x}$	$8+x$	$8+\sqrt{x} = f(g(x))$
$\frac{1}{x+1}$	$x+1$	$\frac{1}{x}$	$\frac{1}{x+1} = f(g(x))$

◀

REMARK. It should be noted that there is always more than one way to express a function as a composition. For example, here are two ways to express $(x^2+1)^{10}$ as a composition that differ from that in Table 1.4.1:

$$(x^2+1)^{10} = \left[(x^2+1)^2\right]^5 = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^2 \text{ and } f(x) = x^5}$$

$$(x^2+1)^{10} = \left[(x^2+1)^3\right]^{10/3} = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^3 \text{ and } f(x) = x^{10/3}}$$

SYMMETRY

Figure 1.4.5 shows the graphs of three curves that have certain obvious symmetries. The graph in part (*a*) is ***symmetric about the x-axis*** in the sense that for each point (x, y) on the graph the point $(x, -y)$ is also on the graph; the graph in part (*b*) is ***symmetric about the y-axis*** in the sense that for each point (x, y) on the graph the point $(-x, y)$ is also on the graph; and the graph in part (*c*) is ***symmetric about the origin*** in the sense that for each point (x, y) on the graph the point $(-x, -y)$ is also on the graph. Geometrically, symmetry about the origin occurs if rotating the graph 180° about the origin leaves the graph unchanged.

Symmetries can often be detected from the equation of a curve. For example, the graph of

$$y = x^3 \qquad (7)$$

must be symmetric about the origin because for any point (x, y) whose coordinates satisfy (7), the coordinates of the point $(-x, -y)$ also satisfy (7), since substituting these

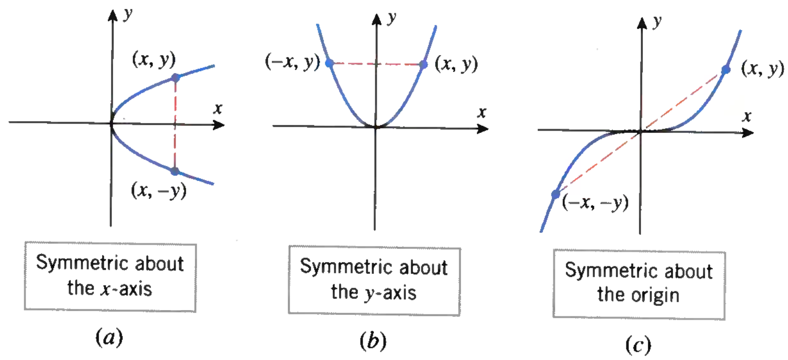


Figure 1.4.5

coordinates in (7) yields

$$-y = (-x)^3$$

which simplifies to (7). This suggests the following symmetry tests (Figure 1.4.6).

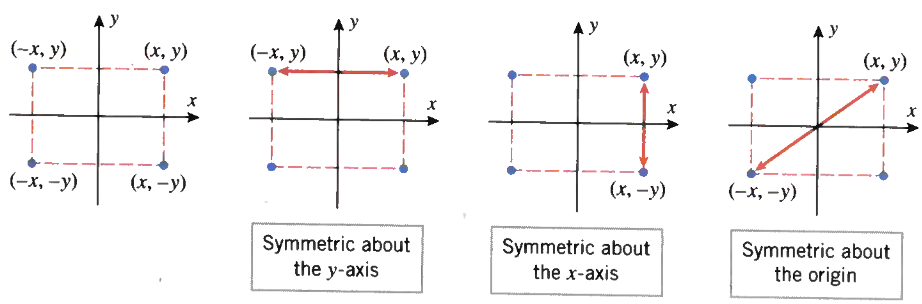


Figure 1.4.6

1.4.3 THEOREM (*Symmetry Tests*).

(*a*) *A plane curve is symmetric about the y-axis if and only if replacing x by $-x$ in its equation produces an equivalent equation.*

(*b*) *A plane curve is symmetric about the x-axis if and only if replacing y by $-y$ in its equation produces an equivalent equation.*

(*c*) *A plane curve is symmetric about the origin if and only if replacing both x by $-x$ and y by $-y$ in its equation produces an equivalent equation.*

EVEN AND ODD FUNCTIONS

For the graph of a function f to be symmetric about the y-axis, the equations $y = f(x)$ and $y = f(-x)$ must be equivalent; for this to happen we must have

$$f(x) = f(-x)$$

A function with this property is called an ***even function***. Some examples are x^2, x^4, x^6, and $\cos x$. Similarly, for the graph of a function f to be symmetric about the origin, the equations $y = f(x)$ and $-y = f(-x)$ must be equivalent; for this to happen we must have

$$f(x) = -f(-x)$$

A function with this property is called an ***odd function***. Some examples are x, x^3, x^5, and $\sin x$.

FOR THE READER. Explain why the graph of a nonzero function cannot by symmetric about the x-axis.

TRANSLATIONS

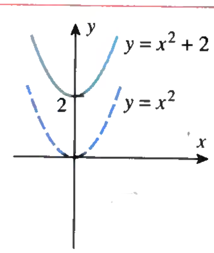
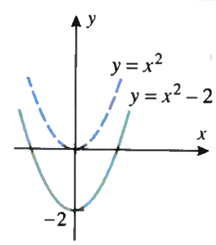
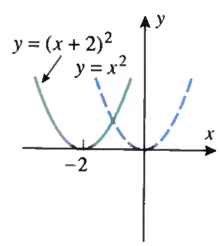
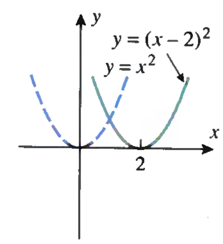
Once you know the graph of an equation $y = f(x)$, there are some techniques that can be used to help visualize the graphs of the equations

$$y = f(x) + c, \quad y = f(x) - c, \quad y = f(x + c), \quad y = f(x - c)$$

where c is any positive constant.

If a positive constant is added to or subtracted from $f(x)$, the geometric effect is to translate the graph of $y = f(x)$ parallel to the y-axis; addition translates the graph in the positive direction and subtraction translates it in the negative direction. This is illustrated in Table 1.4.2. Similarly, if a positive constant is added to or subtracted from the independent variable x, the geometric effect is to translate the graph of the function parallel to the x-axis; subtraction translates the graph in the positive direction, and addition translates it in the negative direction. This is also illustrated in Table 1.4.2.

Table 1.4.2

OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				

Before proceeding to the following examples, it will be helpful to review the graphs in Figure 1.2.1.

Example 8

Sketch the graph of

(a) $y = \sqrt{x - 3}$ (b) $y = \sqrt{x + 3}$

Solution. The graph of the equation $y = \sqrt{x - 3}$ can be obtained by translating the graph of $y = \sqrt{x}$ right 3 units, and the graph of $y = \sqrt{x + 3}$ by translating the graph of $y = \sqrt{x}$ left 3 units (Figure 1.4.7). ◀

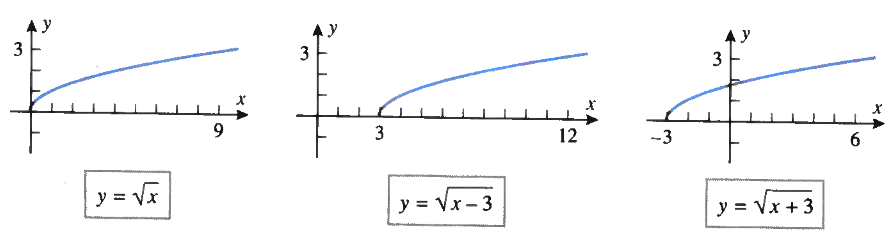


Figure 1.4.7

Example 9

Sketch the graph of $y = |x - 3| + 2$.

Solution. The graph can be obtained by two translations: first translate the graph of $y = |x|$ right 3 units to obtain the graph of $y = |x - 3|$, then translate this graph up 2 units to obtain the graph of $y = |x - 3| + 2$ (Figure 1.4.8). ◀

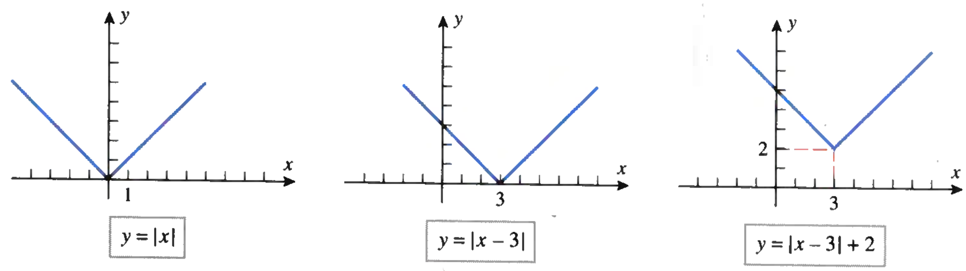


Figure 1.4.8

REMARK. The graph in the preceding example could also have been obtained by performing the translations in the opposite order: first translating the graph of $y = |x|$ up 2 units to obtain the graph of $y = |x| + 2$, then translating this graph right 3 units to obtain the graph of $y = |x - 3| + 2$.

Example 10

Sketch the graph of $y = x^2 - 4x + 5$.

Solution. Completing the square on the first two terms yields

$$y = (x^2 - 4x + 4) - 4 + 5 = (x - 2)^2 + 1$$

(see Appendix D for a review of this technique). In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $x - 2$, and up 1 unit because of the $+1$ (Figure 1.4.9). ◀

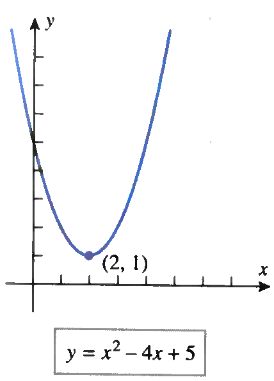


Figure 1.4.9

Example 11

By completing the square, an equation of the form $y = ax^2 + bx + c$ with $a \neq 0$ can be expressed as

$$y = a(x - h)^2 + k \tag{8}$$

Sketch the graph of this equation.

Solution. We can build up Equation (8) in three steps from the equation $y = x^2$. First, we can multiply by a to obtain $y = ax^2$. If $a > 0$, this operation has the geometric effect of stretching or compressing the graph of $y = x^2$; and if $a < 0$, it has the geometric effect of reflecting the graph about the x-axis, in addition to stretching or compressing it. Since stretching or compressing does not alter the general parabolic shape of the original curve, the graph of $y = ax^2$ looks roughly like one of those in Figure 1.4.10*a*. Next, we can subtract h from x to obtain the equation $y = a(x - h)^2$, and then we can add k to obtain $y = a(x - h)^2 + k$. Subtracting h causes a horizontal translation (right or left, depending on the sign of h), and adding k causes a vertical translation (up or down, depending on the sign of k). Thus, the graph of (8) looks roughly like one of those in Figure 1.4.10*b*, which are shown with $h > 0$ and $k > 0$ for simplicity. ◀

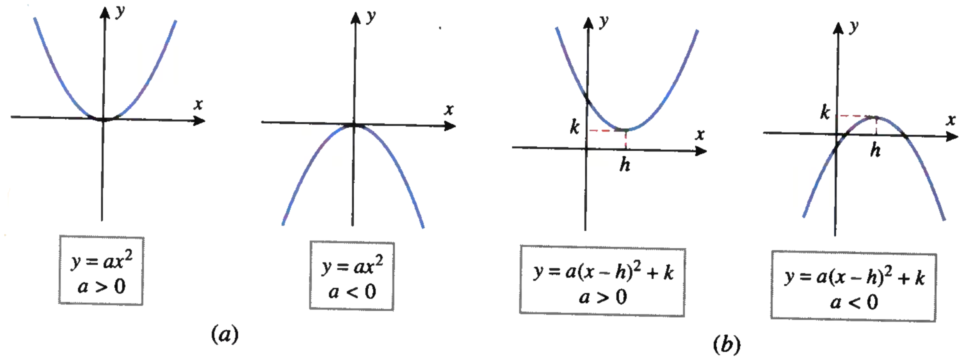
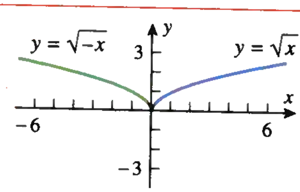
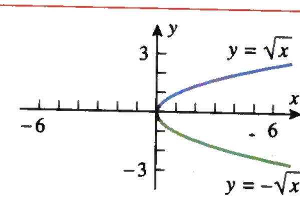


Figure 1.4.10

REFLECTIONS

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis, and the graph of $y = -f(x)$ [or equivalently, $-y = f(x)$] is the reflection of the graph of $y = f(x)$ about the x-axis. Thus, if you know what the graph of $y = f(x)$ looks like, you can obtain the graphs of $y = f(-x)$ and $y = -f(x)$ by making appropriate reflections. This is illustrated in Table 1.4.3.

Table 1.4.3

OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
EXAMPLE		

Example 12

Sketch the graph of $y = \sqrt[3]{2 - x}$.

Solution. The graph can be obtained by a reflection and a translation: first reflect the graph of $y = \sqrt[3]{x}$ about the y-axis to obtain the graph of $y = \sqrt[3]{-x}$, then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x-2)} = \sqrt[3]{2 - x}$ (Figure 1.4.11). ◀

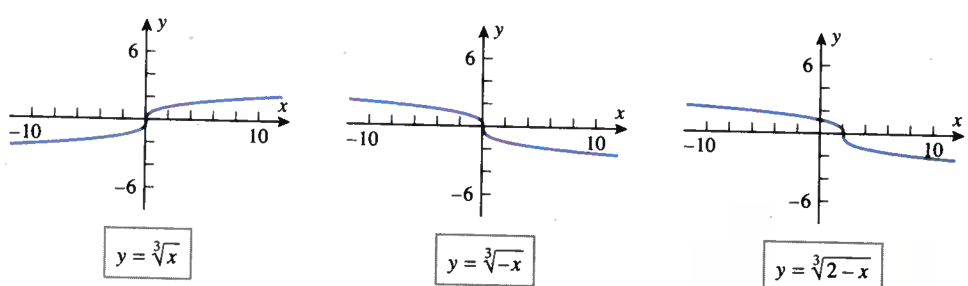


Figure 1.4.11

Example 13

Sketch the graph of $y = 4 - |x - 2|$.

Solution. The graph can be obtained by a reflection and two translations: first translate the graph of $y = |x|$ right 2 units to obtain the graph of $y = |x - 2|$; then reflect this graph about the x-axis to obtain the graph of $y = -|x - 2|$; and then translate this graph up 4 units to obtain the graph of the equation $y = -|x - 2| + 4 = 4 - |x - 2|$ (Figure 1.4.12). ◀

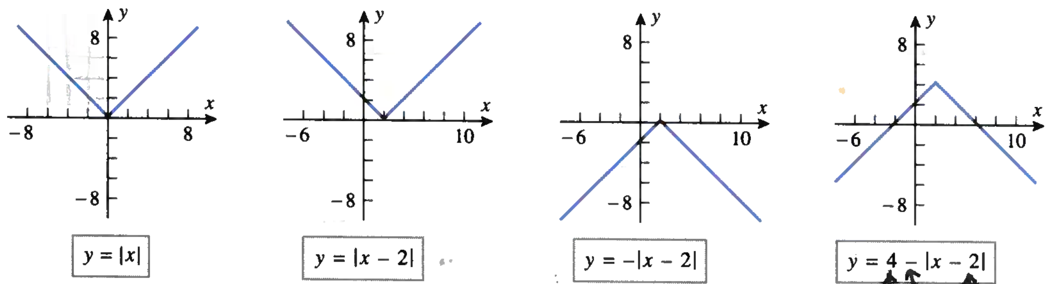


Figure 1.4.12

Exercise Set 1.4 Graphing Calculator

1. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
(a) $y = f(x) - 1$ (b) $y = f(x - 1)$
(c) $y = \frac{1}{2}f(x)$ (d) $y = f\left(-\frac{1}{2}x\right)$

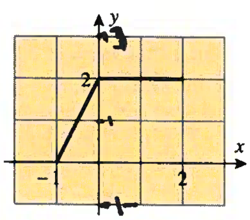


Figure Ex-1

2. Use the graph in Exercise 1.4.1 to sketch the graphs of the following equations.
(a) $y = -f(-x)$ (b) $y = f(2 - x)$
(c) $y = 1 - f(2 - x)$ (d) $y = \frac{1}{2}f(2x)$

3. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
(a) $y = f(x + 1)$ (b) $y = f(2x)$
(c) $y = |f(x)|$ (d) $y = 1 - |f(x)|$

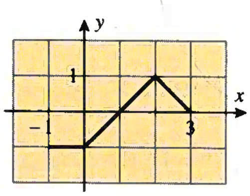


Figure Ex-3

4. Use the graph in Exercise 1.4.3 to sketch the graph of the equation $y = f(|x|)$.

In Exercises 5–12, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = x^2$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

5. $y = 1 + (x - 2)^2$
6. $y = 2 - (x + 1)^2$
7. $y = -2(x + 1)^2 - 3$
8. $y = \frac{1}{2}(x - 3)^2 + 2$
9. $y = x^2 + 6x$
10. $y = x^2 + 6x - 10$
11. $y = 1 + 2x - x^2$
12. $y = \frac{1}{2}(x^2 - 2x + 3)$

In Exercises 13–16, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = \sqrt{x}$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

13. $y = 3 - \sqrt{x + 1}$
14. $y = 1 + \sqrt{x - 4}$
15. $y = \frac{1}{2}\sqrt{x} + 1$
16. $y = -\sqrt{3x}$

In Exercises 17–20, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = 1/x$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

17. $y = \dfrac{1}{x - 3}$
18. $y = \dfrac{1}{1 - x}$

19. $y = 2 - \dfrac{1}{x+1}$

20. $y = \dfrac{x-1}{x}$

In Exercises 21–24, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = |x|$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

21. $y = |x+2| - 2$

22. $y = 1 - |x-3|$

23. $y = |2x-1| + 1$

24. $y = \sqrt{x^2 - 4x + 4}$

In Exercises 25–28, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = \sqrt[3]{x}$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

25. $y = 1 - 2\sqrt[3]{x}$

26. $y = \sqrt[3]{x-2} - 3$

27. $y = 2 + \sqrt[3]{x+1}$

28. $y + \sqrt[3]{x-2} = 0$

29. (a) Sketch the graph of $y = x + |x|$ by adding the corresponding y-coordinates on the graphs of $y = x$ and $y = |x|$.
(b) Express the equation $y = x + |x|$ in piecewise form with no absolute values, and confirm that the graph you obtained in part (a) is consistent with this equation.

30. Sketch the graph of $y = x + (1/x)$ by adding corresponding y-coordinates on the graphs of $y = x$ and $y = 1/x$. Use a graphing utility to confirm that your sketch is correct.

In Exercises 31–34, find formulas for $f+g$, $f-g$, fg, and f/g, and state the domains of the functions.

31. $f(x) = 2x,\ g(x) = x^2 + 1$

32. $f(x) = 3x - 2,\ g(x) = |x|$

33. $f(x) = 2\sqrt{x-1},\ g(x) = \sqrt{x-1}$

34. $f(x) = \dfrac{x}{1+x^2},\ g(x) = \dfrac{1}{x}$

35. Let $f(x) = \sqrt{x}$ and $g(x) = x^3 + 1$. Find
(a) $f(g(2))$ (b) $g(f(4))$
(c) $f(f(16))$ (d) $g(g(0))$.

36. Let $g(x) = \pi - x^2$ and $h(x) = \cos x$. Find
(a) $g(h(0))$ (b) $h(g(\sqrt{\pi/2}))$
(c) $g(g(1))$ (d) $h(h(\pi/2))$.

37. Let $f(x) = x^2 + 1$. Find
(a) $f(t^2)$ (b) $f(t+2)$ (c) $f(x+2)$
(d) $f\left(\dfrac{1}{x}\right)$ (e) $f(x+h)$ (f) $f(-x)$
(g) $f(\sqrt{x})$ (h) $f(3x)$.

38. Let $g(x) = \sqrt{x}$. Find
(a) $g(5s+2)$ (b) $g(\sqrt{x}+2)$ (c) $3g(5x)$
(d) $\dfrac{1}{g(x)}$ (e) $g(g(x))$ (f) $(g(x))^2 - g(x^2)$
(g) $g(1/\sqrt{x})$ (h) $g((x-1)^2)$.

In Exercises 39–44, find formulas for $f \circ g$ and $g \circ f$, and state the domains of the functions.

39. $f(x) = 2x + 1,\ g(x) = x^2 - x$

40. $f(x) = 2 - x^2,\ g(x) = x^3$

41. $f(x) = x^2,\ g(x) = \sqrt{1-x}$

42. $f(x) = \sqrt{x-3},\ g(x) = \sqrt{x^2+3}$

43. $f(x) = \dfrac{1+x}{1-x},\ g(x) = \dfrac{x}{1-x}$

44. $f(x) = \dfrac{x}{1+x^2},\ g(x) = \dfrac{1}{x}$

In Exercises 45 and 46, find a formula for $f \circ g \circ h$.

45. $f(x) = x^2 + 1,\ g(x) = \dfrac{1}{x},\ h(x) = x^3$

46. $f(x) = \dfrac{1}{1+x},\ g(x) = \sqrt[3]{x},\ h(x) = \dfrac{1}{x^3}$

In Exercises 47–50, express f as a composition of two functions; that is, find g and h such that $f = g \circ h$. [*Note:* Each exercise has more than one solution.]

47. (a) $f(x) = \sqrt{x+2}$ (b) $f(x) = |x^2 - 3x + 5|$

48. (a) $f(x) = x^2 + 1$ (b) $f(x) = \dfrac{1}{x-3}$

49. (a) $f(x) = \sin^2 x$ (b) $f(x) = \dfrac{3}{5 + \cos x}$

50. (a) $f(x) = 3\sin(x^2)$ (b) $f(x) = 3\sin^2 x + 4\sin x$

In Exercises 51 and 52, express F as a composition of three functions; that is, find f, g, and h such that $F = f \circ g \circ h$. [*Note:* Each exercise has more than one solution.]

51. (a) $F(x) = \left(1 + \sin(x^2)\right)^3$ (b) $F(x) = \sqrt{1 - \sqrt[3]{x}}$

52. (a) $F(x) = \dfrac{1}{1-x^2}$ (b) $F(x) = |5 + 2x|$

53. Use the table in the accompanying figure to make a scatter plot of $y = f(g(x))$.

x	−3	−2	−1	0	1	2	3
$f(x)$	−4	−3	−2	−1	0	1	2
$g(x)$	−1	0	1	2	3	−2	−3

Figure Ex-53

54. Find the domain of $g \circ f$ for the functions f and g in Exercise 53.

55. Sketch the graph of $y = f(g(x))$ for the functions graphed in the accompanying figure.

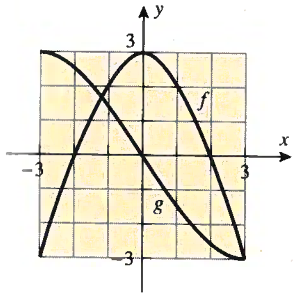


Figure Ex-55

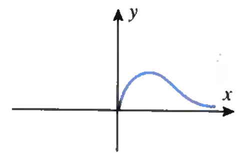


Figure Ex-64

56. Sketch the graph of $y = g(f(x))$ for the functions graphed in Exercise 55.

57. Use the graphs of f and g in Exercise 55 to estimate the solutions of the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

58. Use the table in Exercise 53 to solve the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

In Exercises 59–62, find

$$\frac{f(x+h) - f(x)}{h}$$

and simplify as much as possible.

59. $f(x) = 3x^2 - 5$

60. $f(x) = x^2 + 6x$

61. $f(x) = 1/x$

62. $f(x) = 1/x^2$

63. In each part of the accompanying figure determine whether the graph is symmetric about the x-axis, the y-axis, the origin, or none of the preceding.

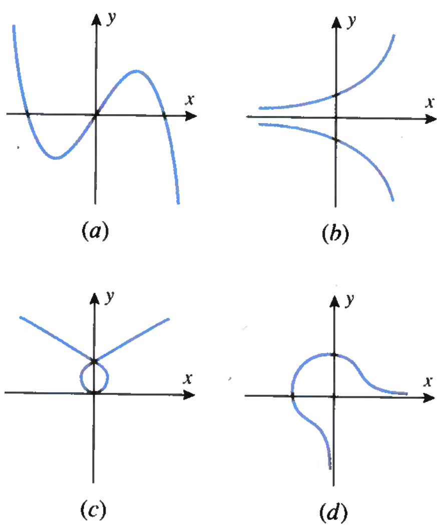


Figure Ex-63

64. The accompanying figure shows a portion of a graph. Complete the graph so that the entire graph is symmetric about
(a) the x-axis (b) the y-axis (c) the origin.

65. Complete the table in the accompanying figure so that the graph of $y = f(x)$ (which is a scatter plot) is symmetric about
(a) the y-axis (b) the origin.

x	−3	−2	−1	0	1	2	3
$f(x)$	1		−1	0		−5	

Figure Ex-65

66. The accompanying figure shows a portion of the graph of a function f. Complete the graph assuming that
(a) f is an even function (b) f is an odd function.

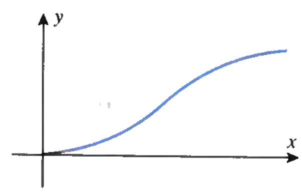


Figure Ex-66

67. Classify the functions graphed in the accompanying figure as even, odd, or neither.

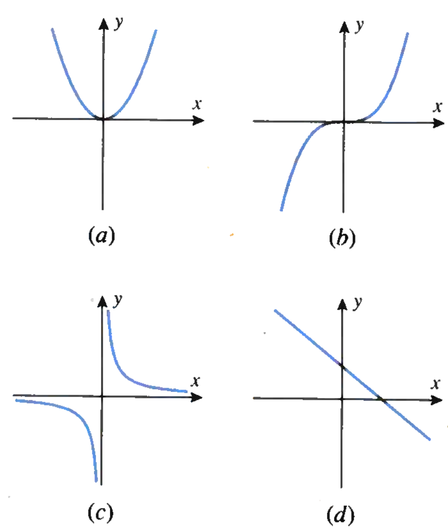


Figure Ex-67

68. Classify the functions whose values are given in the following table as even, odd, or neither.

x	−3	−2	−1	0	1	2	3
$f(x)$	5	3	2	3	1	−3	5
$g(x)$	4	1	−2	0	2	−1	−4
$h(x)$	2	−5	8	−2	8	−5	2

69. In each part, classify the function as even, odd, or neither.
(a) $f(x) = x^2$ (b) $f(x) = x^3$
(c) $f(x) = |x|$ (d) $f(x) = x + 1$
(e) $f(x) = \dfrac{x^5 - x}{1 + x^2}$ (f) $f(x) = 2$

In Exercises 70 and 71, use Theorem 1.4.3 to determine whether the graph has symmetries about the x-axis, the y-axis, or the origin.

70. (a) $x = 5y^2 + 9$ (b) $x^2 - 2y^2 = 3$
(c) $xy = 5$

71. (a) $x^4 = 2y^3 + y$ (b) $y = \dfrac{x}{3 + x^2}$
(c) $y^2 = |x| - 5$

In Exercises 72 and 73: (i) Use a graphing utility to graph the equation in the first quadrant. [*Note:* To do this you will have to solve the equation for y in terms of x.] (ii) Use symmetry to make a hand-drawn sketch of the entire graph. (iii) Confirm your work by generating the graph of the equation in the remaining three quadrants.

72. $9x^2 + 4y^2 = 36$ **73.** $4x^2 + 16y^2 = 16$

74. The graph of the equation $x^{2/3} + y^{2/3} = 1$, which is shown in the accompanying figure, is called a ***four-cusped hypocycloid***.
(a) Use Theorem 1.4.3 to confirm that this graph is symmetric about the x-axis, the y-axis, and the origin.
(b) Find a function f whose graph in the first quadrant coincides with the four-cusped hypocycloid, and use a graphing utility to confirm your work.
(c) Repeat part (b) for the remaining three quadrants.

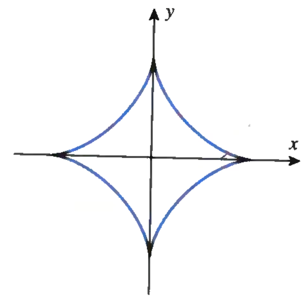


Figure Ex-74

75. The equation $y = |f(x)|$ can be written as
$$y = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$
which shows that the graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by retaining the portion that lies on or above the x-axis and reflecting about the x-axis the portion that lies below the x-axis. Use this method to obtain the graph of $y = |2x - 3|$ from the graph of $y = 2x - 3$.

In Exercises 76 and 77, use the method described in Exercise 75.

76. Sketch the graph of $y = |1 - x^2|$.

77. Sketch the graph of
(a) $f(x) = |\cos x|$ (b) $f(x) = \cos x + |\cos x|$.

78. The ***greatest integer function***, $[x]$, is defined to be the greatest integer that is less than or equal to x. For example, $[2.7] = 2$, $[-2.3] = -3$, and $[4] = 4$. Sketch the graph of
(a) $f(x) = [x]$ (b) $f(x) = [x^2]$
(c) $f(x) = [x]^2$ (d) $f(x) = [\sin x]$.

79. Is it ever true that $f \circ g = g \circ f$ if f and g are nonconstant functions? If not, prove it; if so, give some examples for which it is true.

80. In the discussion preceding Exercise 29 of Section 1.3, we gave a procedure for generating a complete graph of $f(x) = x^{p/q}$ in which we suggested graphing the function $g(x) = |x|^{p/q}$ instead of $f(x)$ when p is even and q is odd and graphing $g(x) = (|x|/x)|x|^{p/q}$ if p is odd and q is odd. Show that in both cases $f(x) = g(x)$ if $x > 0$ or $x < 0$. [*Hint:* Show that $f(x)$ is an even function if p is even and q is odd and is an odd function if p is odd and q is odd.]

1.5 MATHEMATICAL MODELS; LINEAR MODELS

In this section we will discuss mathematical modeling, which is the process that is used to express scientific laws in mathematical form. We will also review some results about lines and apply those results to mathematical modeling.

This section includes a quick review of precalculus material on lines. Readers who want to review this material in more depth are referred to Appendix C.

MATHEMATICAL MODELS

A ***mathematical model*** of a physical law is a description of that law in the language of mathematics. The process of constructing a mathematical model is called ***mathematical modeling***. For example, suppose that two variables, x and y, are related by some physical law that we would like to describe by a mathematical model. Models can be expressed in terms of graphs, tables, or equations, ranging from simple to extremely complicated. However, many important mathematical models are simply equations of the form

$$y = f(x)$$

that relate x and y. For such models the fundamental problem is to find a function f that accurately describes the physical relationship between the variables. Sometimes an appropriate function f might be suggested by experimental data, in which case we say that the model is obtained *inductively*, and sometimes it might be derived from some general theory proposed by a researcher, in which case we say that the model is obtained *deductively*.

The more factors one takes into account when creating a mathematical model, the more complicated the model tends to become, so there is always a balance to be struck between keeping a model mathematically simple and accounting for all of the physical factors that might affect the relationship between the variables. For example, if a meteorologist were trying to model the relationship between the speed of a raindrop when it hits the ground and the height of the cloud in which it was formed, then he or she would certainly have to take air resistance into account. However, the meteorologist would likely ignore the gravitational pull of the planet Pluto since its effect is so small.

Once a mathematical model of a physical law is obtained, it may be possible to use mathematical methods to deduce results about the physical world that are not self-evident or have never been observed. For example, the possibility of placing a satellite in orbit around the Earth was deduced mathematically from Isaac Newton's model of mechanics nearly 200 years before the launching of *Sputnik*, and Albert Einstein's relativistic model of mechanics in 1915 explained a precession (position shift) in the perihelion of the planet Mercury that was not confirmed by physical measurement until 1967.

A good mathematical model is one that produces results that are consistent with observations in the physical world. If a time comes when the mathematical results produced by the model do not agree with real-world observations, then the model must be abandoned in favor of a new model that does. This is the nature of the scientific method—old models constantly being replaced by new models that more accurately describe the real world.

A QUICK REVIEW OF LINES

An equation that is expressible in the form

$$Ax + By + C = 0 \tag{1}$$

where A and B are not both zero, is called a ***first-degree equation*** or a ***linear equation*** in x and y. It is shown in precalculus that every first-degree equation in x and y has a straight line as its graph and, conversely, every straight line can be represented by a first-degree equation in x and y. For this reason (1) is sometimes called the ***general equation*** of a line.

Recall that equations of lines can be written in several different forms:

$$y = mx + b \qquad \text{Slope-intercept form} \tag{2}$$

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form} \tag{3}$$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{Double-intercept form} \tag{4}$$

In these equations m is the slope of the line, a is the x-intercept, b is the y-intercept, and (x_1, y_1) is any point on the line (Figure 1.5.1). Keep in mind that these equations do not apply to vertical lines. For vertical lines the slope is *undefined*, or stated informally, a vertical line has ***infinite slope***. Vertical and horizontal lines have particularly simple equations:

$$x = a \qquad \text{The vertical line with } x\text{-intercept } a \tag{5}$$

$$y = b \qquad \text{The horizontal line with } y\text{-intercept } b \tag{6}$$

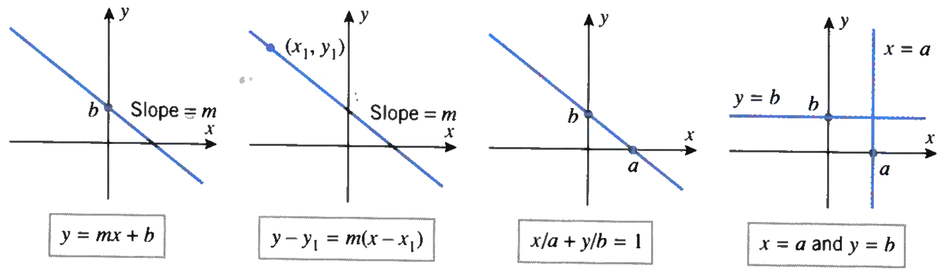


Figure 1.5.1

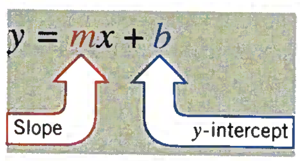


Figure 1.5.2

Equation (2) is especially useful because the slope and the y-intercept of the line can be determined by inspection: the slope is the coefficient of x, and the y-intercept is the constant term (Figure 1.5.2). This equation expresses y as a function of x, the function being $f(x) = mx + b$. A function of this form is called a ***linear function*** of x.

INTERPRETATIONS OF SLOPE

The slope m of a nonvertical line $y = mx + b$ has two important interpretations (which are related but different in viewpoint):

- m is a measure of the *steepness* of the line.
- m is the rate of change of y with respect to x.

The steepness interpretation has an analog in surveying. Surveyors measure the grade or slope of a hill as the ratio of its rise over its run (Figure 1.5.3a). The same idea applies to lines. Consider a particle that moves left to right along a nonvertical line from a point

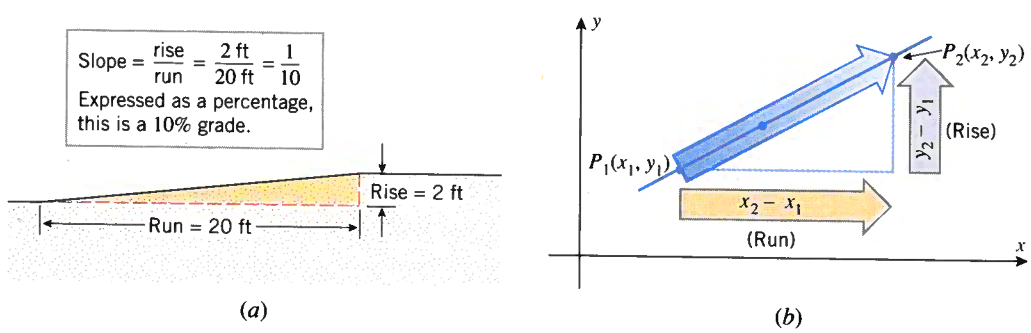


Figure 1.5.3

$P_1(x_1, y_1)$ to a point $P_2(x_2, y_2)$. In the course of its travel the point moves $y_2 - y_1$ units vertically as it travels $x_2 - x_1$ units horizontally (Figure 1.5.3*b*). The vertical change, which is denoted by $\Delta y = y_2 - y_1$, is called the ***rise***, and the horizontal change, which is denoted by $\Delta x = x_2 - x_1$, is called the ***run***. The ratio of the rise over the run is always equal to the slope, regardless of where the points P_1 and P_2 are located on the line; that is,

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{7}$$

REMARK. The symbols Δx and Δy should not be interpreted as products; rather, Δx should be viewed as a single entity representing the *change* in the value of x, and Δy as a single entity representing the *change* in the value of y. In general, if v is any variable whose value changes from an initial value of v_1 to a final value of v_2, then we call $\Delta v = v_2 - v_1$ (final value minus initial value) an ***increment*** in v. Increments can be positive or negative, depending on whether the final value is larger or smaller than the initial value.

ANGLE OF INCLINATION

The slope of a nonvertical line L is related to the angle that L makes with the positive x-axis. If ϕ is the smallest positive angle measured counterclockwise from the x-axis to L, then the slope of the line can be expressed as

$$m = \tan\phi \tag{8}$$

(Figure 1.5.4*a*). The angle ϕ, which is called the ***angle of inclination*** of the line, satisfies $0° \le \phi < 180°$ in degree measure (or, equivalently, $0 \le \phi < \pi$ in radian measure). If ϕ is an acute angle, then $m = \tan\phi$ is positive and the line slopes up to the right, and if ϕ is an obtuse angle, then $m = \tan\phi$ is negative and the line slopes down to the right. For example, a line whose angle of inclination is 45° has slope $m = \tan 45° = 1$, and a line whose angle of inclination is 135° has a slope of $m = \tan 135° = -1$ (Figure 1.5.4*b*). Figure 1.5.5 shows a convenient way of using the line $x = 1$ as a "ruler" for visualizing the relationship between lines of various slopes.

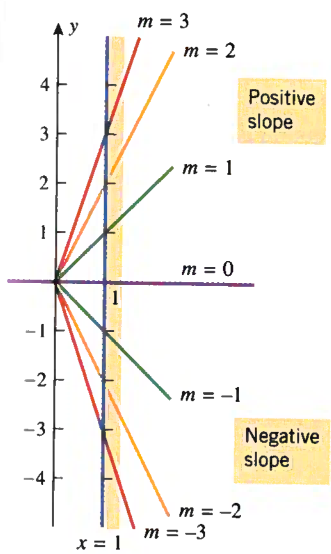


Figure 1.5.5

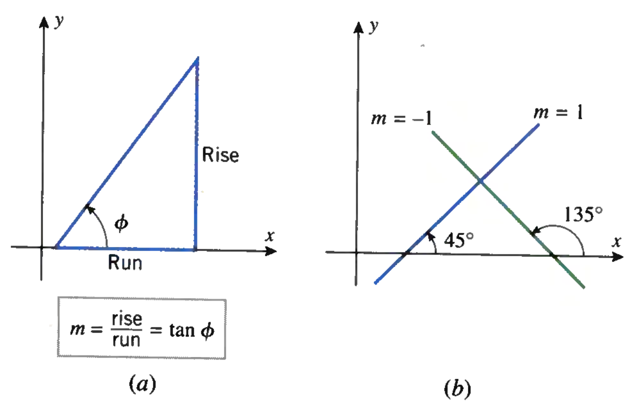


Figure 1.5.4

SLOPES OF LINES IN APPLIED PROBLEMS

In applied problems, changing the units of measurement can change the slope of a line, so it is essential to include the units when calculating the slope. The following example illustrates this.

Example 1

Suppose that a uniform rod of length 40 cm (= 0.4 m) is thermally insulated around the lateral surface and that the exposed ends of the rod are held at constant temperatures of 25°C and 5°C, respectively (Figure 1.5.6*a*). It is shown in physics that under appropriate

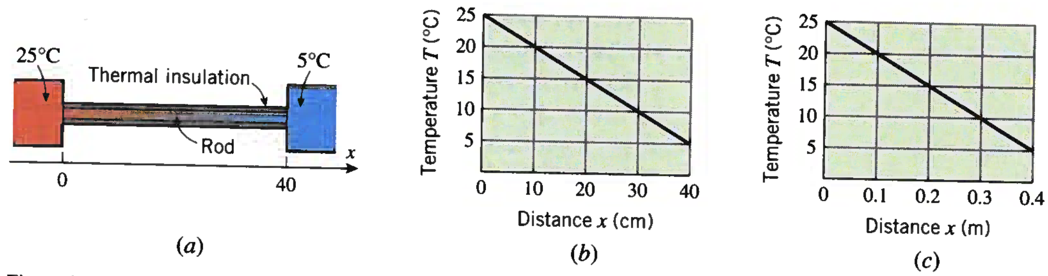


Figure 1.5.6

conditions the graph of the temperature T versus the distance x from the left-hand end of the rod will be a straight line. Parts (*b*) and (*c*) of Figure 1.5.6 show two such graphs: one in which x is measured in centimeters and one in which it is measured in meters. The slopes in the two cases are

$$m = \frac{5-25}{40-0} = \frac{-20}{40} = -0.5°\text{C/cm} \qquad (9)$$

$$m = \frac{5-25}{0.4-0} = \frac{-20}{0.4} = -50°\text{C/m} \qquad (10)$$

The slope in (9) implies that the temperature *decreases* at a rate of 0.5°C per centimeter of distance from the left end of the rod, and the slope in (10) implies that the temperature decreases at a rate of 50°C per meter of distance from the left end of the rod. The two statements are equivalent physically, even though the slopes differ. ◀

Example 2

Find the slope-intercept form of the equation of the temperature distribution in the preceding example if the temperature T is measured in degrees Celsius (°C) and the distance x is measured in (a) centimeters and (b) meters.

Solution (a). The slope is $m = -0.5°\text{C/cm}$ and the intercept on the T-axis is 25°, so

$$T = -0.5x + 25, \quad 0 \le x \le 40$$

where the restriction on x is required because the rod is 40 cm in length. The graph of this equation with the restriction is a line segment rather than a line.

Solution (b). The slope is $m = -50°\text{C/m}$, the intercept on the T-axis is 25°, and the restriction on x is $0 \le x \le 0.4$. Thus, the equation is

$$T = -50x + 25, \quad 0 \le x \le 0.4$$

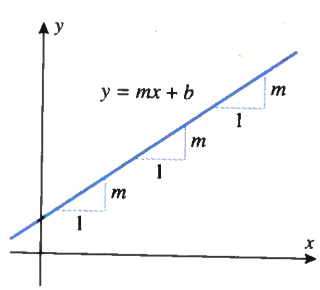
◀

LINEAR MATHEMATICAL MODELS

If y is a linear function of x, say $y = mx + b$, then it follows from (7) that

$$\Delta y = m\Delta x$$

Thus, a 1-unit increase in x ($\Delta x = 1$) produces an m-unit change in y ($\Delta y = m$). Moreover, this is true at every point on the line (Figure 1.5.7), so we say that y changes at a *constant rate* with respect to x, and we call m the ***rate of change of y with respect to x***. This idea can be summarized as follows.



A 1-unit increase in x always produces an m-unit change in y.

Figure 1.5.7

1.5.1 LINEAR MATHEMATICAL MODELS. *If a variable y is related to a variable x in such a way that the rate of change of y with respect to x is constant, say m, then y is a linear function of x of the form*

$$y = mx + b$$

and we say that y is related to x by a ***linear mathematical model****. Conversely, if y is a linear function of x whose graph has slope m, then the rate of change of y with respect to x is constant and equal to m.*

It follows from this that linear models are appropriate whenever experimentation or theory suggests that the rate of change of y with respect to x is constant.

UNIFORM RECTILINEAR MOTION

One of the important themes in calculus is the study of motion. To describe the motion of an object completely, one must specify its ***speed*** (how fast it is going) and the direction in which it is moving. The speed and the direction of motion together comprise what is called the ***velocity*** of the object. For example, knowing that the speed of an aircraft is 500 mi/h tells us how fast it is going, but not which way it is moving. In contrast, knowing that the velocity of the aircraft is 500 mi/h *due south* pins down the speed and the direction of motion.

Later, we will study the motion of particles that move along curves in two- or three-dimensional space, but for now we will focus on motion along a line; this is called ***rectilinear motion***. In general rectilinear motion, a particle can move back and forth along the line (as with a piston moving up and down in a cylinder); however, for now we will only consider the simple case in which the particle moves in just *one direction* along a line (as with a car traveling on a straight road).

For simplicity, we will assume that the motion is along a coordinate line, such as an x-axis or y-axis, and that the particle is moving in the positive direction. In general discussions we will usually name the coordinate line the s-axis to avoid being specific. A graphical description of rectilinear motion along an s-axis can be obtained by making a plot of the s-coordinate of the particle versus the elapsed time t. This is called the ***position versus time curve*** for the particle. Figure 1.5.8a shows a typical position versus time curve for a car moving in the positive direction along an s-axis.

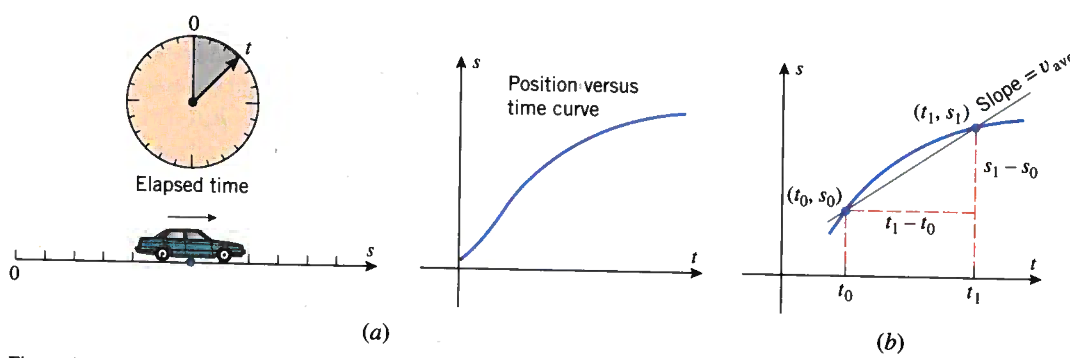


Figure 1.5.8

FOR THE READER. How can you tell from the position versus time curve in Figure 1.5.8a that the car does not reverse direction?

Because we are assuming that the particle is moving in the positive direction of the s-axis, there is no ambiguity about the direction of motion, and hence the terms "speed" and "velocity" can be used interchangeably. However, later, when we consider general rectilinear motion or motion along a curved path, it will be necessary to distinguish between these terms, since the direction of motion may vary.

For a particle in rectilinear motion along a coordinate axis, we define the ***average velocity*** v_{ave} of the particle during the time interval from t_0 to t_1 to be

$$v_{\text{ave}} = \frac{s_1 - s_0}{t_1 - t_0} = \frac{\Delta s}{\Delta t} \tag{11}$$

where s_0 and s_1 are the s-coordinates of the particle at times t_0 and t_1, respectively. Geometrically, this is the slope of the secant line connecting the points (t_0, s_0) and (t_1, s_1) on the position versus time curve (Figure 1.5.8b). The quantity $\Delta s = s_1 - s_0$ is called the ***displacement*** or ***change in position*** of the particle during the time interval from t_0 to t_1. With this terminology, Formula (11) states that for a particle in rectilinear motion *the average velocity over a time interval is the displacement during the time interval divided by the length of the time interval.* For example, if a car moving in one direction along a straight road travels 75 miles in 3 hours, then its average velocity is $75/3 = 25$ mi/h.

In the special case where the average velocity of a particle in rectilinear motion is the same over every time interval, the particle is said to have ***constant velocity*** and ***uniform rectilinear motion***. If the average velocity over every time interval is v, then we will refer to v as the ***velocity*** of the particle (dropping the adjective "average").

For a particle with uniform rectilinear motion the displacement over *any* time interval is given by the formula

$$\text{displacement} = \text{velocity} \times \text{elapsed time} \tag{12}$$

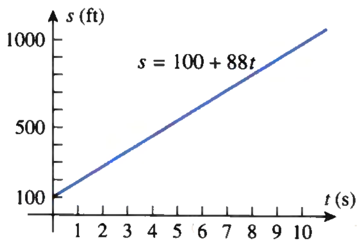


Figure 1.5.9

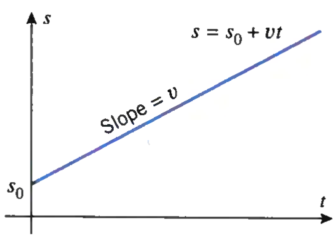
Example 3

Suppose that a car moves with a constant velocity of 88 ft/s in the positive direction of an s-axis. Given that the s-coordinate of the car at time $t = 0$ is $s = 100$, find an equation for s as a function of t, and graph the position versus time curve.

Solution. It follows from (12) that in a period of t seconds, the car will move $88t$ feet from its starting point, so its coordinate s at time t will be

$$s = 100 + 88t$$

The graph of this equation is the line in Figure 1.5.9. ◀



Position versus time curve for a particle with coordinate s_0 at time $t = 0$ and moving with constant velocity v

Figure 1.5.10

It is not accidental that the position versus time curve turned out to be a line in the last example; this will always be the case for uniform rectilinear motion. To see why this is so, suppose that a particle moves with constant velocity v in the positive direction along an s-axis, starting at the point s_0 at time $t = 0$. It follows from (12) that in t units of time the particle will move vt units from its starting point s_0, so its coordinate s at time t will be

$$s = s_0 + vt$$

which is a line with s-intercept s_0 and slope v (Figure 1.5.10). It follows from this equation and 1.5.1 that we can view the velocity v as the rate of change of s with respect to t, that is, the rate of change of position with respect to time.

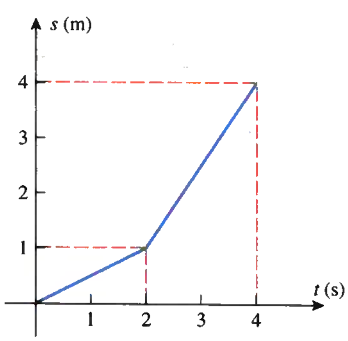


Figure 1.5.11

Example 4

Figure 1.5.11 shows the position versus time curve for a particle moving along an s-axis. Describe the motion of the particle in words.

Solution. At time $t = 0$ the particle is at the origin. From time $t = 0$ to $t = 2$ the slope of the line segment is $\frac{1}{2}$, so the particle is moving with a constant velocity of $\frac{1}{2} = 0.5$ m/s. At time $t = 2$ the particle is at the point $s = 1$ (i.e., 1 meter from the origin). From time $t = 2$ to $t = 4$ the slope of the line segment is $\frac{3}{2}$, so the particle is moving with a constant velocity of $\frac{3}{2} = 1.5$ m/s. At time $t = 4$ it is at the point $s = 4$. ◀

CONSTANT ACCELERATION

In everyday language we say that an object is "accelerating" if it is speeding up and "decelerating" if it is slowing down. Mathematically, the ***acceleration*** of a particle in rectilinear motion is defined to be the *rate of change of velocity with respect to time*, where the acceleration is positive if the velocity is increasing and negative if it is decreasing. Thus, for a particle that moves in the positive direction of an s-axis, negative acceleration means the particle is "decelerating" in everyday language. Acceleration, like velocity, can be variable or constant. For example, by pressing the gas pedal of a car toward the floor smoothly, the driver can make the car's velocity increase at a constant rate (a constant acceleration); however, if the driver suddenly slams the pedal to the floor, the car will lurch forward, reflecting a nonconstant acceleration. Later in the text we will study acceleration in more depth, but for now we will only consider the case in which acceleration is constant.

REMARK. The units of acceleration are units of velocity divided by units of time. For example, if the velocity of a particle is increasing at a rate of 3 feet per second each second, then its acceleration is 3 ft/s/s (velocity in ft/s divided by time in s); this is usually written as 3 ft/s^2 (read "3 feet per second per second" or "3 feet per second squared"). Similarly, if the velocity of a particle is decreasing at a rate of 3 feet per second each second, then it has an acceleration of −3 ft/s^2.

Graphical information about the acceleration of a particle can be obtained from the graph of velocity versus time; this is called the ***velocity versus time curve***. In the case where the particle has constant acceleration, the velocity versus time curve will be linear, and its slope, which is the rate of change of velocity with time, will be the acceleration (Figure 1.5.12).

Example 5

Suppose that a car moves in the positive direction of an s-axis in such a way that its velocity v increases at a constant rate of 2 ft/s^2.

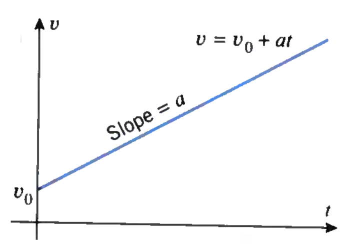
(a) Assuming that the velocity of the car is 88 ft/s at time $t = 0$, find an equation for v as a function of t.

(b) Make a graph of velocity versus time, and mark the point on the graph at which the car attains a velocity of 100 ft/s.

Solution (a). Since the rate of change of v with respect to t is 2 ft/s^2, and since $v = 88$ ft/s if $t = 0$, the equation for velocity as a function of time is

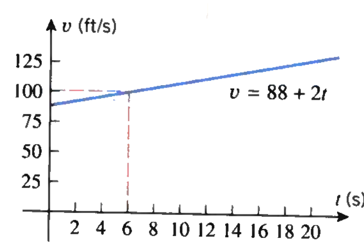
$$v = 88 + 2t \tag{13}$$

Solution (b). To find the time it takes for the car to reach a velocity of 100 ft/s, we substitute $v = 100$ in (13) and solve for t. This yields $t = 6$. The graph of (13) and the point at which the velocity reaches 100 ft/s is shown in Figure 1.5.13. ◀



Velocity versus time curve for a particle with velocity v_0 at time $t = 0$ and moving with constant acceleration a

Figure 1.5.12



Velocity versus time curve for a particle with a velocity of 88 ft/s at time $t = 0$ and moving with a constant acceleration of 2 ft/s^2

Figure 1.5.13

LINEAR MODELS FROM DIRECT PROPORTION

Recall that a variable y is said to be ***directly proportional*** to a variable x if there is a positive constant k, called the ***constant of proportionality***, such that

$$y = kx \tag{14}$$

The graph of this equation is a line through the origin whose slope k is the constant of proportionality. Thus, linear models are appropriate in physical problems where one variable is directly proportional to another.

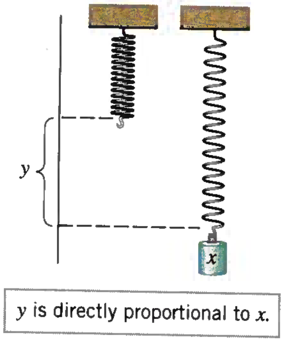


Figure 1.5.14

Hooke's law* in physics provides a nice example of direct proportion. It follows from this law that if a weight of x units is suspended from a spring, then the spring will be stretched by an amount y that is directly proportional to x, that is, $y = kx$ (Figure 1.5.14). The constant k depends on the stiffness of the spring: the stiffer the spring, the smaller the value of k (why?).

Example 6

Figure 1.5.15 shows an old-fashioned spring scale that is calibrated in pounds.

(a) Given that the pound scale marks are 0.5 in apart, find an equation that expresses the length y that the spring is stretched (in inches) in terms of the suspended weight x in pounds).

(b) Graph the equation obtained in part (a).

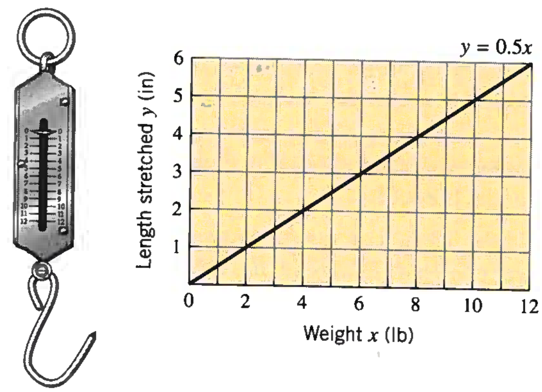


Figure 1.5.15

Solution (a). It follows from Hooke's law that y is related to x by an equation of the form $y = kx$. To find k we rewrite this equation as $k = y/x$, and use the fact that a weight of $x = 1$ lb stretches the spring $y = 0.5$ in. Thus,

$$k = \frac{y}{x} = \frac{0.5}{1} = 0.5 \qquad \text{and hence} \qquad y = 0.5x$$

Solution (b). The graph of the equation $y = 0.5x$ is shown in Figure 1.5.15. ◀

LINEAR MODELS FROM GRAPHICAL DATA

Sometimes linear models are suggested by graphical data. For example, Figure 1.5.16*a* shows a graph of temperature versus altitude that was transmitted by the *Magellan* spacecraft when it entered the atmosphere of Venus in October 1991. The graph strongly suggests that there is a linear relationship between temperature and altitude for altitudes between 35 km and 60 km.

Example 7

(a) Use the graph transmitted by the *Magellan* spacecraft to find a linear model of temperature versus altitude in the Venusian atmosphere that is valid for altitudes between 35 km and 60 km.

(b) Use the model to estimate the temperature at the surface of Venus, and discuss the assumptions you are making in obtaining the estimate.

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*Hooke's law, named for the English physicist Robert Hooke (1635–1703), applies only for small displacements that do not stretch the spring to the point of permanently distorting it.

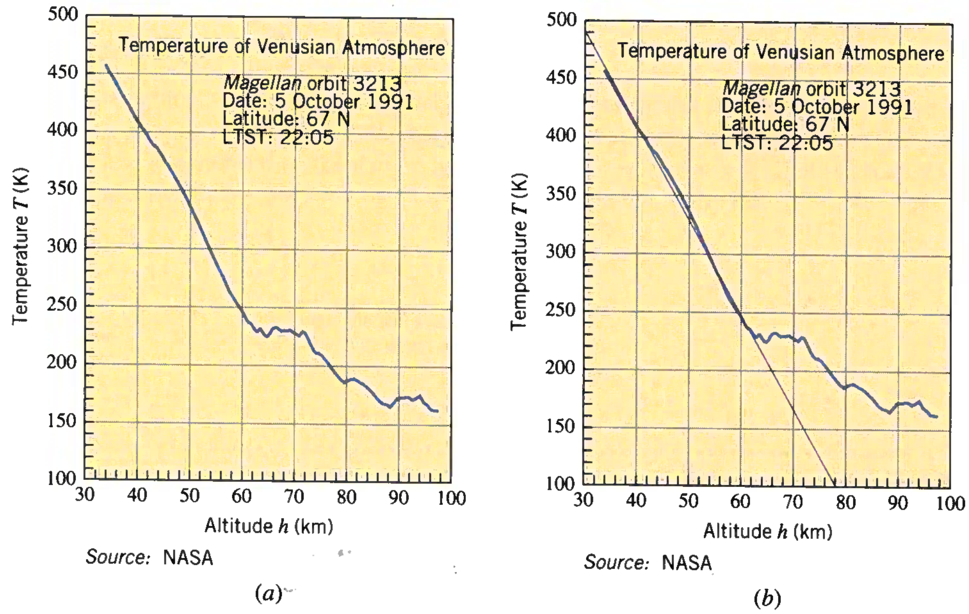


Figure 1.5.16

Solution (a). Let T be the temperature in kelvins and h the altitude in kilometers. We will first estimate the slope m of the linear portion of the graph, then estimate the coordinates of a data point (h_1, T_1) on that portion of the graph, and then use the point-slope form of a line

$$T - T_1 = m(h - h_1) \tag{15}$$

The graph nearly passes through the point (60, 250), so we will take $h_1 \approx 60$ and $T_1 \approx 250$. In Figure 1.5.16*b* we have sketched a line that closely approximates the linear portion of the data. Using the intersections of that line with the edges of the grid box, we estimate the slope to be

$$m \approx \frac{100 - 490}{78 - 30} = -\frac{390}{48} \approx -8.125 \text{ K/km}$$

Substituting our estimates of h_1, T_1, and m into (15) yields the equation

$$T - 250 = -8.125(h - 60)$$

or equivalently,

$$T = -8.125h + 737.5 \tag{16}$$

Solution (b). The *Magellan* spacecraft stopped transmitting data at an altitude of approximately 35 km, so we cannot be certain that the linear model applies at lower altitudes. However, since we have no other data to work with, let us *assume* that the model is valid at all lower altitudes, in which case we can approximate the temperature at the surface of Venus by setting $h = 0$ in (16). We obtain $T \approx 737.5$ K. ◀

REMARK. The method of the preceding example is crude, at best, since it relies on extracting rough estimates of numerical data from a graph. Nevertheless, the final result is quite good, since the most recent information from NASA places the surface temperature of Venus at about 740 K (hot enough to melt lead).

LINEAR MODELS FROM NUMERICAL DATA

One method for determining whether n points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

lie on a line is to compare the slopes of the line segments joining successive points. The points lie on a line if and only if those slopes are equal (Figure 1.5.17).

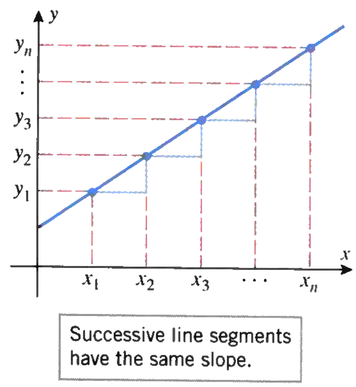


Figure 1.5.17

Table 1.5.1

x	y
1.5	0.3
2.5	1.1
3.5	1.9
5.5	3.5
9.5	6.7

Example 8

Consider the data in Table 1.5.1.

(a) Explain why a linear model is appropriate for the data in the table.

(b) Find a linear equation that relates x and y, and graph the equation and the data together.

Solution (a). The five data points lie on a line, since each 1-unit increase in x produces a corresponding 0.8-unit increase in y. Thus, the slope of the line segment joining any two successive data points is

$$m = \frac{\Delta y}{\Delta x} = \frac{0.8}{1} = 0.8$$

Solution (b). A linear equation relating x and y can be obtained from the point-slope form of the line using the slope $m = 0.8$ calculated in part (a) and any one of the five data points. If we use the first data point, $(1.5, 0.3)$, we obtain

$$y - 0.3 = 0.8(x - 1.5)$$

or in slope-intercept form,

$$y = 0.8x - 0.9$$

The graph of this equation together with the given data are shown in Figure 1.5.18. ◀

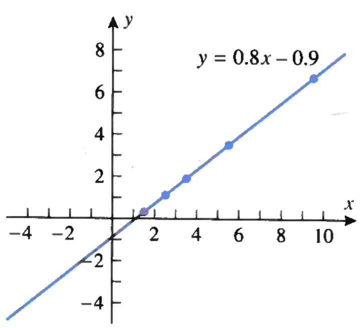


Figure 1.5.18

REMARK. Sometimes, data points that should theoretically lie on a line do not because of experimental error and other factors. In such cases curve-fitting techniques are used to find a line that most closely fits the data. Such techniques will be discussed later in the text.

OTHER APPLICATIONS OF LINEAR FUNCTIONS

Linear functions arise in a variety of practical problems. Here is a typical example.

Example 9

A university parking lot charges \$3.00 per day but offers a \$40.00 monthly sticker with which the student pays only \$0.25 per day.

(a) Find equations for the cost C of parking for x days per month under both payment methods, and graph the equations for $0 \le x \le 30$. (Treat C as a continuous function of x, even though x only assumes integer values.)

(b) Find the value of x for which the graphs intersect, and discuss the significance of this value.

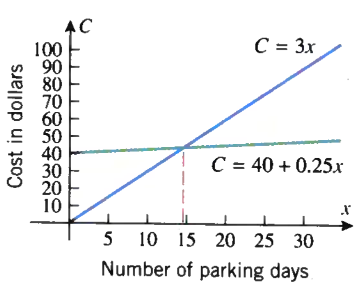


Figure 1.5.19

Solution (a). The cost in dollars of parking for x days at \$3.00 per day is $C = 3x$, and the cost for the \$40.00 sticker plus x days at \$0.25 per day is $C = 40 + 0.25x$ (Figure 1.5.19).

Solution (b). The graphs intersect at the point where

$$3x = 40 + 0.25x$$

which is $x = 40/2.75 \approx 14.5$. This value of x is not an option for the student, since x must be an integer. However, it is the dividing point at which the monthly sticker method becomes less expensive than the daily payment method; that is, for $x \geq 15$ it is cheaper to buy the monthly sticker and for $x \leq 14$ it is cheaper to pay the daily rate. ◀

EXERCISE SET 1.5 Graphing Calculator

Exercises 1–26 involve the basic properties of lines and slope. In some of these exercises you will need to use slopes to determine whether two lines are parallel or perpendicular. If you have forgotten how to do this, review Appendix C.

1. (a) Find the slopes of the sides of the triangle with vertices $(0, 3)$, $(2, 0)$, and $(6, \frac{8}{3})$.
   (b) Is this a right triangle? Explain.

2. (a) Find the slopes of the sides of the quadrilateral with vertices $(-3, -1)$, $(5, -1)$, $(7, 3)$, and $(-1, 3)$.
   (b) Is this a parallelogram? Explain.

3. List the lines in the accompanying figure in the order of increasing slope.

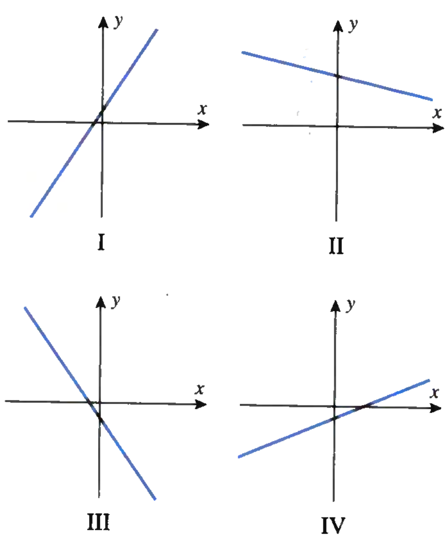


Figure Ex-3

4. List the lines in the accompanying figure in the order of increasing slope.

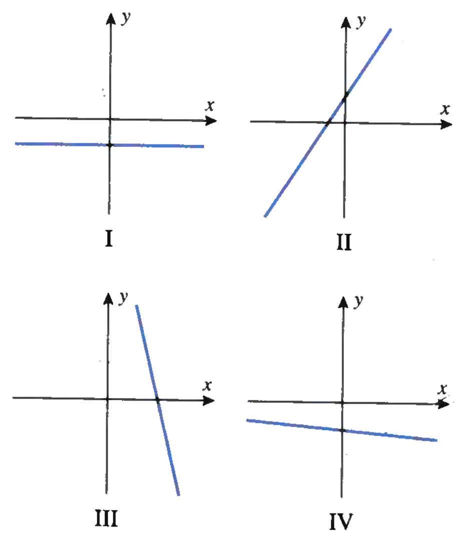


Figure Ex-4

5. Use slopes to determine whether the given points lie on the same line.
   (a) $(1, 1)$, $(-2, -5)$, and $(0, -1)$
   (b) $(-2, 4)$, $(0, 2)$, and $(1, 5)$

6. A particle, initially at $(7, 5)$, moves along a line of slope $m = -2$ to a new position (x, y).
   (a) Find y if $x = 9$.
   (b) Find x if $y = 12$.

7. A particle, initially at $(1, 2)$, moves along a line of slope $m = 3$ to a new position (x, y).
   (a) Find y if $x = 5$.
   (b) Find x if $y = -2$.

8. Find x and y if the line through $(0, 0)$ and (x, y) has slope $\frac{1}{2}$, and the line through (x, y) and $(7, 5)$ has slope 2.

9. Find x if the slope of the line through $(1, 2)$ and $(x, 0)$ is the negative of the slope of the line through $(4, 5)$ and $(x, 0)$.

In Exercises 10 and 11, find the angle of inclination of the line with slope m to the nearest degree. Use a calculating utility, where needed.

10. (a) $m = \frac{1}{2}$ (b) $m = -1$
(c) $m = 2$ (d) $m = -57$

11. (a) $m = -\frac{1}{2}$ (b) $m = 1$
(c) $m = -2$ (d) $m = 57$

In Exercises 12 and 13, find the angle of inclination of the line to the nearest degree. Use a calculating utility, where needed.

12. (a) $3y = 2 - \sqrt{3}x$ (b) $y - 4x + 7 = 0$

13. (a) $y = \sqrt{3}x + 2$ (b) $y + 2x + 5 = 0$

14. Find equations for the x- and y-axes.

In Exercises 15–22, find the slope-intercept form of the equation of the line satisfying the stated conditions, and check your answer using a graphing utility.

15. Slope $= -2$, y-intercept $= 4$

16. $m = 5, b = -3$

17. The line is parallel to $y = 4x - 2$ and its y-intercept is 7.

18. The line is parallel to $3x + 2y = 5$ and passes through $(-1, 2)$.

19. The line is perpendicular to the equation $y = 5x + 9$ and has y-intercept 6.

20. The line is perpendicular to $x - 4y = 7$ and passes through $(3, -4)$.

21. The line passes through $(2, 4)$ and $(1, -7)$.

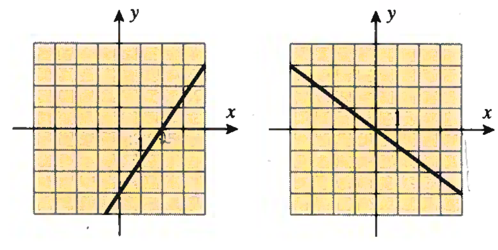
22. The line passes through $(-3, 6)$ and $(-2, 1)$.

23. In each part, classify the lines as parallel, perpendicular, or neither.
(a) $y = 4x - 7$ and $y = 4x + 9$
(b) $y = 2x - 3$ and $y = 7 - \frac{1}{2}x$
(c) $5x - 3y + 6 = 0$ and $10x - 6y + 7 = 0$
(d) $Ax + By + C = 0$ and $Bx - Ay + D = 0$
(e) $y - 2 = 4(x - 3)$ and $y - 7 = \frac{1}{4}(x - 3)$

24. In each part, classify the lines as parallel, perpendicular, or neither.
(a) $y = -5x + 1$ and $y = 3 - 5x$
(b) $y - 1 = 2(x - 3)$ and $y - 4 = -\frac{1}{2}(x + 7)$
(c) $4x + 5y + 7 = 0$ and $5x - 4y + 9 = 0$
(d) $Ax + By + C = 0$ and $Ax + By + D = 0$
(e) $y = \frac{1}{2}x$ and $x = \frac{1}{2}y$

In Exercises 25 and 26, use the graph to find the equation of the line in slope-intercept form, and then check your result by using a graphing utility to graph the equation.

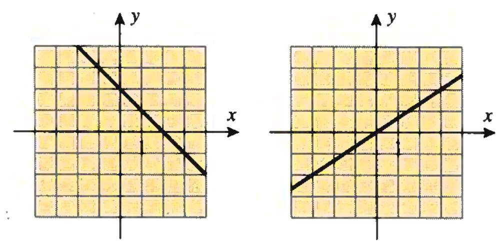
25.



(a) (b)

Figure Ex-25

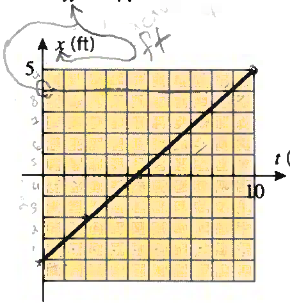
26.



(a) (b)

Figure Ex-26

27. The accompanying figure shows the position versus time curve for a particle moving along an x-axis.
(a) What is the velocity of the particle?
(b) What is the x-coordinate of the particle at time $t = 0$?
(c) What is the x-coordinate of the particle at time $t = 2$?
(d) At what time does the particle have an x-coordinate of $x = 4$?



(d) 9 or 8.888 (s) which is equivalent to 80/9 (s)

Figure Ex-27

28. A particle moving along an x-axis with constant velocity is at the point $x = 1$ when $t = 2$ and is at the point $x = 5$ when $t = 4$.
(a) Find the velocity of the particle if x is in meters and t is in seconds.
(b) Find an equation that expresses x as a function of t.
(c) What is the coordinate of the particle at time $t = 0$?

29. A particle moving along an x-axis with constant acceleration has velocity $v = 3$ ft/s at time $t = 1$ and velocity $v = -1$ ft/s at time $t = 4$.
(a) Find the acceleration of the particle.
(b) Find an equation that expresses v as a function of t.
(c) What is the velocity of the particle at time $t = 0$?

30. The accompanying figure shows the velocity versus time curve for a particle moving along the x-axis.
(a) What is the acceleration of the particle?
(b) What is the velocity of the particle at time $t = 0$?
(c) What is the velocity of the particle at time $t = 2$?
(d) At what time does the particle have a velocity of $v = 3$ ft/s?

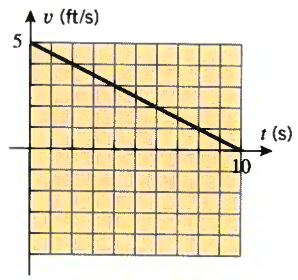


Figure Ex-30

31. The accompanying figure shows the position versus time curve for a particle moving along an x-axis.
(a) Describe the motion of the particle in words.
(b) Find the average velocity of the particle from $t = 0$ to $t = 10$.
(c) Find the average speed of the particle from $t = 0$ to $t = 10$.

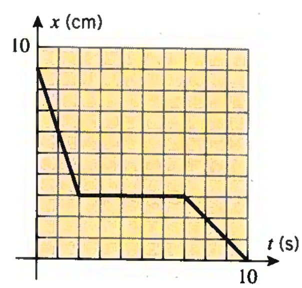


Figure Ex-31

32. The accompanying figure shows the velocity versus time curve for a particle moving along an x-axis. Describe the motion of the particle in words.

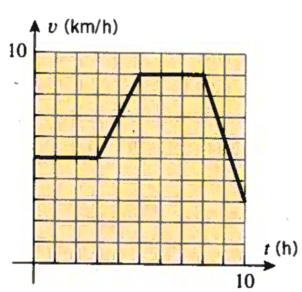


Figure Ex-32

33. A locomotive travels on a straight track at a constant speed of 40 mi/h, then reverses direction and returns to its starting point, traveling at a constant speed of 60 mi/h.
(a) What is the average velocity for the round-trip?
(b) What is the average speed for the round-trip?
(c) What is the total distance traveled by the train if the total trip took 5 h?

34. A ball is tossed straight up at time $t = 0$ with an initial velocity of 64 ft/s. We will show later using basic principles of physics that the velocity of the ball as a function of time is $v = 64 - 32t$.
(a) What direction is the ball traveling 3 s after it is released? Explain your reasoning.
(b) At what time does the ball reach its maximum height above the ground? Explain your reasoning.
(c) What can you say about the acceleration of the ball?

35. A car is stopped at a toll booth on a straight highway. Starting at time $t = 0$ it accelerates at a constant rate of 10 ft/s^2 for 10 s. It then travels at a constant speed of 100 ft/s for 90 s. At that time it begins to decelerate at a constant rate of 5 ft/s^2 for 20 s, at which point in time it reaches a full stop at a traffic light.
(a) Sketch the velocity versus time curve.
(b) Express v as a piecewise function of t.

36. Make a reasonable sketch of a position versus time curve for a particle that moves in the positive x-direction with positive constant acceleration.

37. A spring with a natural length of 15 in stretches to a length of 20 in when a 45-lb object is suspended from it.
(a) Use Hooke's law to find an equation that expresses the length y that the spring is stretched (in inches) in terms of the suspended weight x (in pounds).
(b) Graph the equation obtained in part (b).
(c) Find the length of the spring when a 100-lb object is suspended from it.
(d) What is the largest weight that can be suspended from the spring if the spring cannot be stretched to more than twice its natural length?

38. The spring in a heavy-duty shock absorber has a natural length of 3 ft and is compressed 0.2 ft by a load of 1 ton. An additional load of 5 tons compresses the spring an additional 1 ft.
(a) Assuming that Hooke's law applies to compression as well as extension, find an equation that expresses the length y that the spring is compressed from its natural length (in feet) in terms of the load x (in tons).
(b) Graph the equation obtained in part (a).
(c) Find the amount that the spring is compressed from its natural length by a load of 3 tons.
(d) Find the maximum load that can be applied if safety regulations prohibit compressing the spring to less than half its natural length.

In Exercises 39 and 40, confirm that a linear model is appropriate for the relationship between x and y. Find a linear equation relating x and y, and verify that the data points lie on the graph of your equation.

39.

x	0	1	2	4	6
y	2	3.2	4.4	6.8	9.2

40.

x	−1	0	2	5	8
y	12.6	10.5	6.3	0	−6.3

41. There are two common systems for measuring temperature, Celsius and Fahrenheit. Water freezes at 0°Celsius (0°C) and 32°Fahrenheit (32°F); it boils at 100°C and 212°F.
(a) Assuming that the Celsius temperature T_C and the Fahrenheit temperature T_F are related by a linear equation, find the equation.
(b) What is the slope of the line relating T_F and T_C if T_F is plotted on the horizontal axis?
(c) At what temperature is the Fahrenheit reading equal to the Celsius reading?
(d) Normal body temperature is 98.6°F. What is it in °C?

42. Thermometers are calibrated using the so-called "triple point" of water, which is 273.16 K on the Kelvin scale and 0.01°C on the Celsius scale. A one-degree difference on the Celsius scale is the same as a one-degree difference on the Kelvin scale, so there is a linear relationship between the temperature T_C in degrees Celsius and the temperature T_K in kelvins.
(a) Find an equation that relates T_C and T_K.
(b) Absolute zero (0 K on the Kelvin scale) is the temperature below which a body's temperature cannot be lowered. Express absolute zero in °C.

43. To the extent that water can be assumed to be incompressible, the pressure p in a body of water varies linearly with the distance h below the surface.
(a) Given that the pressure is 1 atmosphere (1 atm) at the surface and 5.9 atm at a depth of 50 m, find an equation that relates pressure to depth.
(b) At what depth is the pressure twice that at the surface?

44. A resistance thermometer is a device that determines temperature by measuring the resistance of a fine wire whose resistance varies with temperature. Suppose that the resistance R in ohms (Ω) varies linearly with the temperature T in °C and that $R = 123.4\ \Omega$ when $T = 20$°C and that $R = 133.9\ \Omega$ when $T = 45$°C.
(a) Find an equation for R in terms of T.
(b) If R is measured experimentally as 128.6 Ω, what is the temperature?

45. Suppose that the mass of a spherical mothball decreases with time, due to evaporation, at a rate that is proportional to its surface area. Assuming that it always retains the shape of a sphere, it can be shown that the radius r of the sphere decreases linearly with the time t.
(a) If, at a certain instant, the radius is 0.80 mm and 4 days later it is 0.75 mm, find an equation for r (in millimeters) in terms of the elapsed time t (in days).
(b) How long will it take for the mothball to completely evaporate?

46. The accompanying figure shows three masses suspended from a spring: a mass of 11 g, a mass of 24 g, and an unknown mass of W g.
(a) What will the pointer indicate on the scale if no mass is suspended?
(b) Find W.

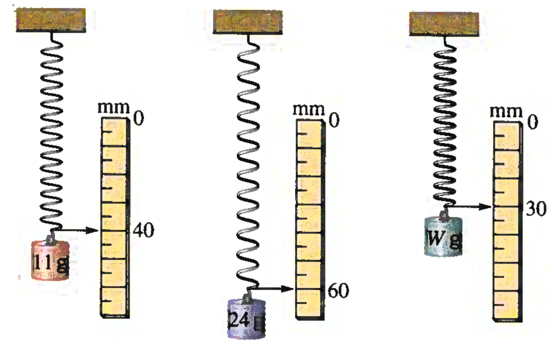


Figure Ex-46

47. The price for a round-trip bus ride from a university to center city is \$2.00, but it is possible to purchase a monthly commuter pass for \$25.00 with which each round-trip ride costs an additional \$0.25.
(a) Find equations for the cost C of making x round-trips per month under both payment plans, and graph the equations for $0 \le x \le 30$ (treating C as a continuous function of x, even though x assumes only integer values).
(b) How many round-trips per month would a student have to make for the commuter pass to be worthwhile?

48. A student must decide between buying one of two used cars: car A for \$4000 or car B for \$5500. Car A gets 20 miles per gallon of gas, and car B gets 30 miles per gallon. The student estimates that gas will run \$1.25 per gallon. Both cars are in excellent condition, so the student feels that repair costs should be negligible for the foreseeable future. How many miles would the student have to drive before car B becomes the better buy?

49. **(The Age of the Universe)** In the early 1900s the astronomer Edwin P. Hubble (1889–1953) noted an unexpected relationship between the radial velocity of a galaxy and its distance d from any reference point (Earth, for example). That relationship, now known as ***Hubble's law***, states that the galaxies are receding with a velocity v that is directly proportional to the distance d. This is usually expressed as $v = Hd$, where H (the constant of proportionality) is called ***Hubble's constant.*** When applying this formula it is usual to express v in kilometers per second (km/s) and d in millions of light-years (Mly), in which case H has units of km/s/Mly. The accompanying figure shows an original plot and trend line of the velocity-distance relationship obtained by Hubble and a collaborator Milton L. Humason (1891–1972).

(a) Use the trend line in the figure to estimate Hubble's constant.

(b) An estimate of the age of the universe can be obtained by assuming that the galaxies move with constant velocity v, in which case v and d are related by $d = vt$. Assuming that the Universe began with a "big bang" that initiated its expansion, show that the Universe is roughly 1.5×10^{10} years old. [Take $H = 20$ km/s/Mly, which is in keeping with current estimates that place H between 15 and 27 km/s/Mly. (Note that the current estimates are significantly less than that resulting from Hubble's data.)]

(c) In a more realistic model of the Universe, the velocity v would decrease with time. What effect would that have on your estimate in part (b)?

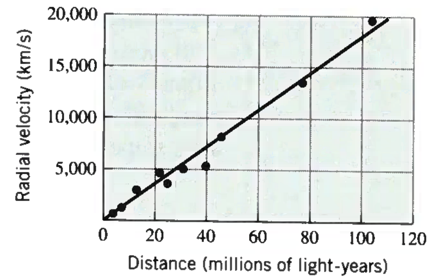


Figure Ex-49

1.6 FAMILIES OF FUNCTIONS

Functions are often grouped into families according to the form of their defining formulas or other common characteristics. In this section we will discuss some of the most basic families of functions.

This section includes quick reviews of precalculus material on polynomials and trigonometry. Readers who want to review this material in more depth are referred to Appendices E and F. Instructors who want to spend some additional time on precalculus review can divide this section into two parts, covering the trigonometry material in a second lecture.

FAMILIES OF LINES

A function f whose values are all the same is called a ***constant function***. For example, the formula $f(x) = c$ defines the constant function whose value is c for all x. The graph of the constant function $f(x) = c$ is the horizontal line $y = c$ (Figure 1.6.1*a*). If we vary c, then we obtain a set or ***family*** of horizontal lines (Figure 1.6.1*b*).

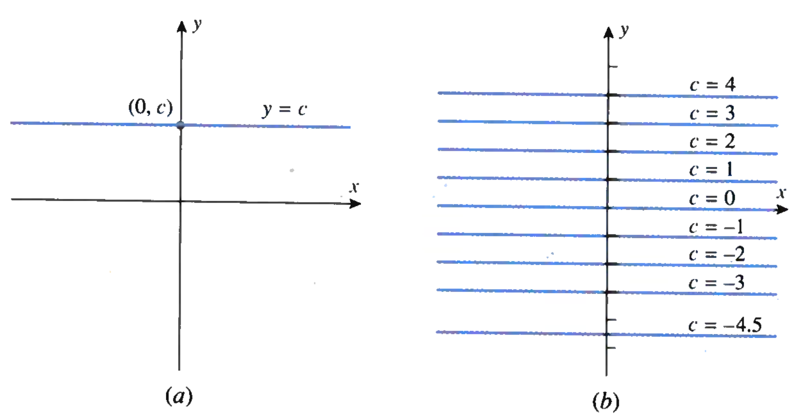


Figure 1.6.1

REMARK. The expression $f(x) = c$ can be confusing because it can be interpreted either as an equation that is satisfied for certain x (as in $x^2 = c$) or as an identity that is satisfied for all x; it is the latter interpretation that defines a constant function. Thus, when you see an expression of the form $f(x) = c$, you will have to determine from its context whether it is intended as an equation or a constant function.

The quantities m and b in the equation $y = mx+b$ can be viewed as unspecified constants whose values may change from one application to another; such changeable constants are called ***parameters***.

If we keep b fixed and vary the parameter m in the equation $y = mx + b$, then we obtain a family of lines whose members all have y-intercept b (Figure 1.6.2*a*); and if we keep m fixed and vary the parameter b, then we obtain a family of parallel lines whose members all have slope m (Figure 1.6.2*b*).

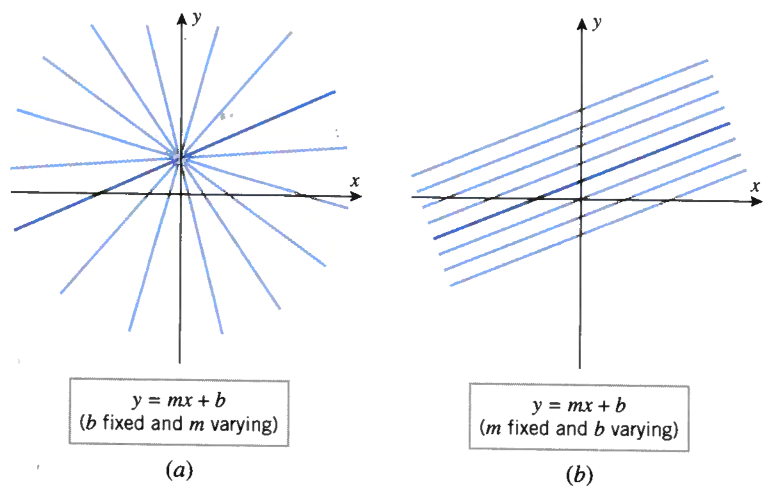


Figure 1.6.2

Example 1

(a) Find an equation for the family of lines with slope $\frac{1}{2}$.

(b) Find the member of the family in part (a) that passes through the point (4, 1).

(c) Find an equation for the family of lines whose members are perpendicular to the lines in part (a).

Solution (a). The lines of slope $\frac{1}{2}$ are of the form

$$y = \tfrac{1}{2}x + b \qquad (1)$$

where the parameter b can have any real value.

Solution (b). To find the line in the family that passes through the point (4, 1), we must find the value of b for which the coordinates $x = 4$ and $y = 1$ satisfy (1). Substituting these coordinates into (1) and solving for b yields $b = -1$, and hence the equation of the line is

$$y = \tfrac{1}{2}x - 1 \qquad (2)$$

(Figure 1.6.3*a*).

Solution (c). Since the slopes of perpendicular lines are negative reciprocals, it follows that the lines perpendicular to those in part (a) have slope -2 and hence are of the form

$$y = -2x + b$$

Some typical lines in families (1) and (2) are graphed in Figure 1.6.3*b*. ◀

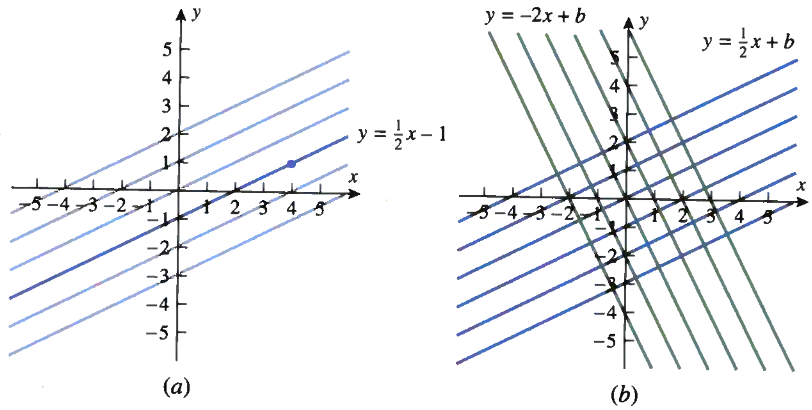


Figure 1.6.3

THE FAMILY $y = x^n$

A function of the form $f(x) = x^p$, where p is constant is called a ***power function***. If p is a positive integer, say $p = n$, then the power functions have the form $f(x) = x^n$. The graphs of the curves $y = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 1.6.4. The first graph is the line $y = x$ with slope 1 that passes through the origin, and the second is a parabola that opens up and has its vertex at the origin (see Appendix 2).

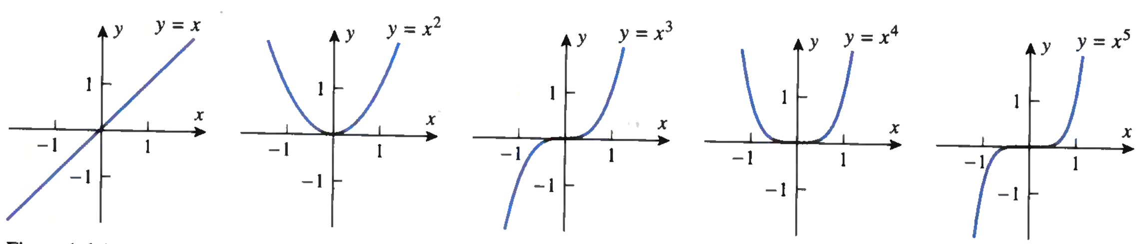


Figure 1.6.4

For $n > 2$ the shape of the graph of $y = x^n$ depends on whether n is even or odd (Figure 1.6.5). For even values of n the graphs have the same general shape as the parabola $y = x^2$ (though they are not actually parabolas if $n > 2$), and for odd values of n greater than 1 they have the same general shape as $y = x^3$. The graphs in the family $y = x^n$ share a number of important characteristics:

- For even values of n the functions $f(x) = x^n$ are even, and their graphs are symmetric about the y-axis; for odd values of n the functions $f(x) = x^n$ are odd, and their graphs are symmetric about the origin.
- For all values of n the graphs pass through the origin and the point $(1, 1)$. For even values of n the graphs pass through $(-1, 1)$, and for odd values of n they pass through $(-1, -1)$.
- Increasing n causes the graph to become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.

REMARK. The last characteristic can be explained numerically by considering the effect of raising a real number x to successively higher powers. If x is a fraction, that is, $-1 < x < 1$, then the absolute value of x^n *decreases* as n increases (try raising $\frac{1}{2}$ or $-\frac{1}{2}$ to higher and higher powers, for example). This explains why successive graphs in Figure 1.6.5 become flatter over the interval $-1 < x < 1$. On the other hand, if $x > 1$ or $x < -1$, then the

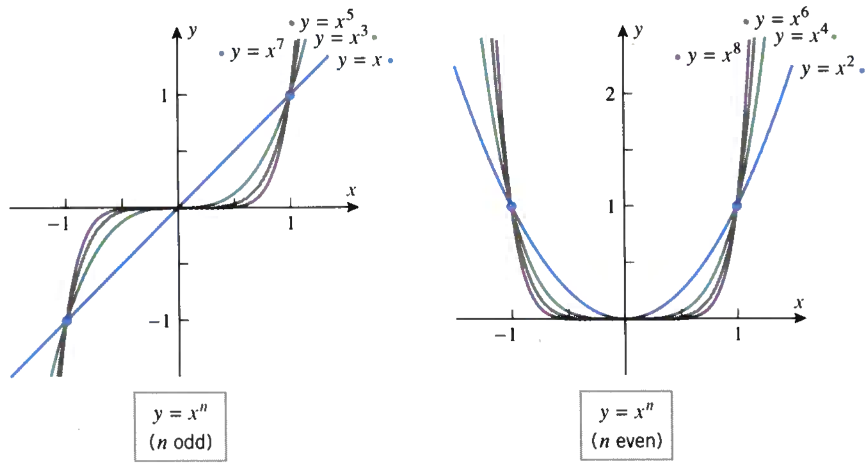


Figure 1.6.5

absolute value of x^n *increases* as n increases (try raising 2 or -2 to higher and higher powers). This explains why successive graphs become steeper if $x > 1$ or $x < -1$.

THE FAMILY $y = x^{-n}$

If p is a negative integer, say $p = -n$, then the power functions $f(x) = x^p$ have the form $f(x) = x^{-n} = 1/x^n$. Figure 1.6.6*a* shows the graphs of $y = 1/x$ and $y = 1/x^2$, and Figure 1.6.6*b* shows how these graphs relate to other members of the family. The graph of $y = 1/x$ is called an ***equilateral hyperbola*** (for reasons to be discussed later).

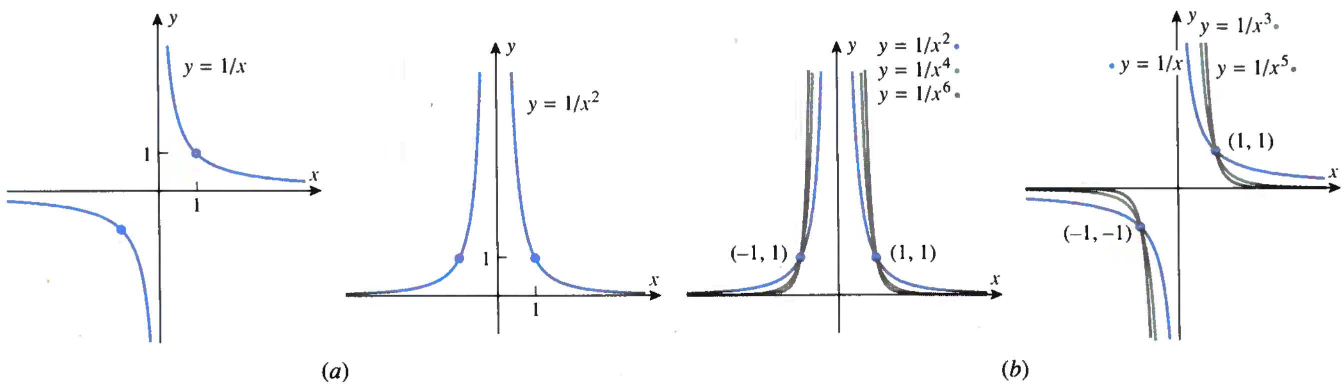


Figure 1.6.6

For odd values of n the graphs have the same general shape as $y = 1/x$, and for even values of n they have the same general shape as $y = 1/x^2$. The graphs in the family $y = 1/x^n$ share a number of important characteristics:

- For even values of n the functions $f(x) = 1/x^n$ are even, and their graphs are symmetric about the y-axis; for odd values of n the functions $f(x) = x^n$ are odd, and their graphs are symmetric about the origin.
- For all values of n the graphs pass through the point $(1, 1)$ and have a break (called a ***discontinuity***) at the origin. This is caused by the division by zero that occurs when $x = 0$. For even values of n the graphs pass through $(-1, 1)$, and for odd values of n they pass through $(-1, -1)$.
- Increasing n causes the graph to become steeper over the interval $-1 < x < 1$ and flatter over the intervals $x > 1$ and $x < -1$.

REMARK. The last characteristic can be explained numerically by considering the effect of raising the reciprocal of a number x to successively higher powers. If x is a nonzero fraction, then it lies in the interval $-1 < x < 1$, and its reciprocal satisfies $1/x > 1$ or $1/x < -1$. Thus, as n increases the absolute value of $1/x^n$ also increases. This explains why successive graphs in Figure 1.6.6 become successively steeper over the interval $-1 < x < 1$. On the other hand, if $x > 1$ or $x < -1$, then $-1 < 1/x < 1$. Thus, as n increases the absolute value of $1/x^n$ *decreases*. This explains why successive graphs in Figure 1.6.6 get successively flatter if $x > 1$ or $x < -1$.

THE FAMILY $y = x^{1/n}$

If $p = 1/n$, where n is a positive integer, then the power functions $f(x) = x^p$ have the form $f(x) = x^{1/n} = \sqrt[n]{x}$. In particular, if $n = 2$, then $f(x) = \sqrt{x}$, and if $n = 3$, then $f(x) = \sqrt[3]{x}$. The graphs of these functions are shown in parts (a) and (b) of Figure 1.6.7. Observe that the graph of $y = \sqrt[3]{x}$ extends over the entire x-axis because $f(x) = \sqrt[3]{x}$ is defined for all real values of x (every real number has a cube root); in contrast, the graph of $y = \sqrt{x}$ only extends over the nonnegative x-axis (negative numbers have imaginary square roots). Observe also that the graph of $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ (Figure 1.6.7c).

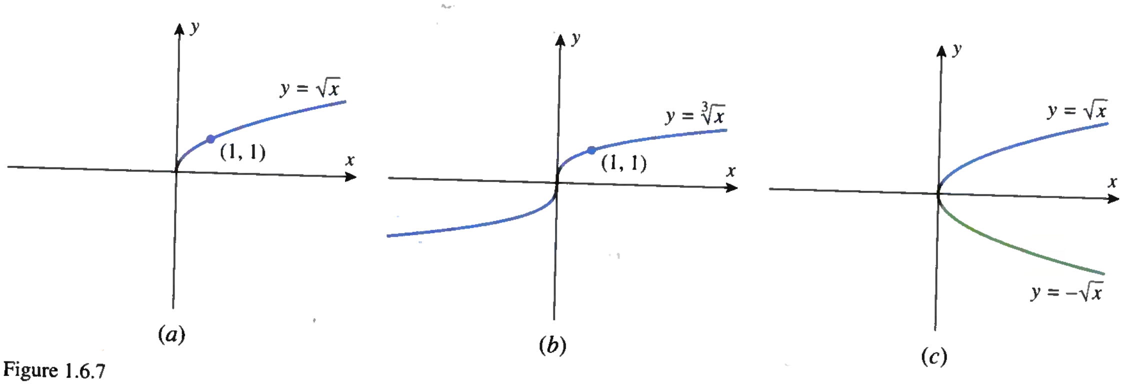


Figure 1.6.7

For even values of n the graphs of $y = \sqrt[n]{x}$ have the same general shape as $y = \sqrt{x}$, and for odd values of n they have the same general shape as $y = \sqrt[3]{x}$.

FOR THE READER. Sketch the graphs of $y = \sqrt[n]{x}$ for $n = 2, 4, 6$ on one set of axes and for $n = 3, 5, 7$ on another set. Use a graphing device to check your work.

POWER FUNCTIONS WITH FRACTIONAL AND IRRATIONAL EXPONENTS

Power functions can also have fractional or irrational exponents. For example,

$$f(x) = x^{2/3}, \quad f(x) = \sqrt[5]{x^3}, \quad f(x) = x^{-7/8}, \quad \text{and} \quad f(x) = x^{\sqrt{2}} \tag{3}$$

are all power functions of this type; we will discuss power functions of these forms in later sections.

FOR THE READER. Read the note preceding Exercise 29 of Section 1.3, and use a graphing utility to generate complete graphs of the functions in (3).

MODELS INVOLVING INVERSE PROPORTIONS

Recall that a variable y is said to be ***inversely proportional to a variable x*** if there is a positive constant k, called the ***constant of proportionality***, such that

$$y = \frac{k}{x} \tag{4}$$

Since k is assumed to be positive, the graph of this equation has the same basic shape as $y = 1/x$ but is compressed or stretched in the x-direction.

Observe that in Formula (4) doubling x decreases y by a factor of $1/2$, tripling x decrease y by a factor of $1/3$, and, more generally, increasing x by a factor of r decreases y by factor of $1/r$.

Models involving inverse proportion arise in various laws of physics. For exampl ***Boyle's law*** in physics states that at a constant temperature the pressure P exerted by fixed quantity of an ideal gas is inversely proportional to the volume V occupied by the gas that is,

$$P = \frac{k}{V}$$

(Figure 1.6.8).

If y is inversely proportional to x, then it follows from (4) that the product of y an x is constant, since $yx = k$. This provides a useful way of identifying inverse proportio models in experimental data.

Example 2

Table 1.6.1 shows some experimental data.

Table 1.6.1

EXPERIMENTAL DATA

x	0.8	1	2.5	4	6.25	10
y	6.25	5	2	1.25	0.8	0.5

(a) Explain why the data suggest that y is inversely proportional to x.

(b) Express y as a function of x.

(c) Graph your function and the data together for $x \geq 0$.

Solution. For every data point we have $xy = 5$, so y is inversely proportional to x and $y = 5/x$. The graph of this equation with the data points is shown in Figure 1.6.9. ◀

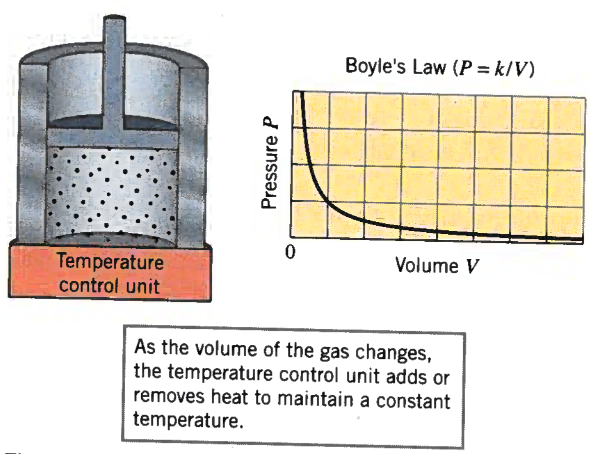


Figure 1.6.8

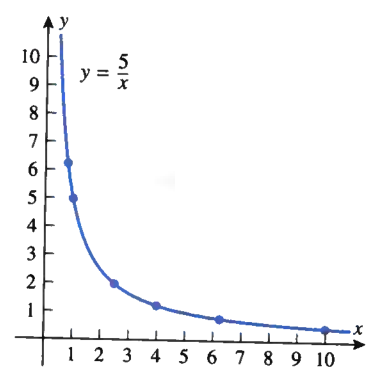


Figure 1.6.9

A QUICK REVIEW OF POLYNOMIALS

A detailed review of polynomials is given in Appendix F, but for convenience we will review some of the terminology here.

A ***polynomial in x*** is a function that is expressible as a sum of finitely many terms of the form cx^n, where c is a constant and n is a nonnegative integer. Some examples of polynomials are

$$2x+1, \quad 3x^2+5x-\sqrt{2}, \quad x^3, \quad 4\,(=4x^0), \quad 5x^7-x^4+3$$

The function $(x^2-4)^3$ is also a polynomial because it can be expanded by the binomial formula (see the inside front cover) and expressed as a sum of terms of the form cx^n:

$$(x^2-4)^3 = (x^2)^3 - 3\,(x^2)^2\,(4) + 3(x^2)(4^2) - (4^3) = x^6 - 12x^4 + 48x^2 - 64 \qquad (5)$$

A general polynomial can be written in either of the following forms, depending on whether one wants the powers of x in ascending or descending order:

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

The constants $c_0, c_1, \ldots, c_n$ are called the ***coefficients*** of the polynomial. When a polynomial is expressed in one of these forms, the highest power of x that occurs with a nonzero coefficient is called the ***degree*** of the polynomial. Constants are considered to have degree 0, since we can write $c = cx^0$. Polynomials of degree 1, 2, 3, 4, and 5 are described as ***linear***, ***quadratic***, ***cubic***, ***quartic***, and ***quintic***, respectively. For example,

$3+5x$	Has degree 1 (linear)
x^2-3x+1	Has degree 2 (quadratic)
$2x^3-7$	Has degree 3 (cubic)
$8x^4-9x^3+5x-3$	Has degree 4 (quartic)
$\sqrt{3}+x^3+x^5$	Has degree 5 (quintic)
$(x^2-4)^3$	Has degree 6 [see (5)]

The natural domain of a polynomial in x is $(-\infty, +\infty)$, since the only operations involved are multiplication and addition; the range depends on the particular polynomial. We already know that the graphs of polynomials of degree 0 and 1 are lines and that the graphs of polynomials of degree 2 are parabolas. Figure 1.6.10 shows the graphs of some typical polynomials of higher degree. Later, we will discuss polynomial graphs in detail, but for now it suffices to observe that graphs of polynomials are very well behaved in the sense that they have no discontinuities or sharp corners. As illustrated in Figure 1.6.10, the graphs of polynomials wander up and down for awhile in a roller-coaster fashion, but eventually that behavior stops and the graphs steadily rise or fall indefinitely as one travels along the curve in either the positive or negative direction. We will see later that the number of peaks and valleys is determined by the degree of the polynomial.

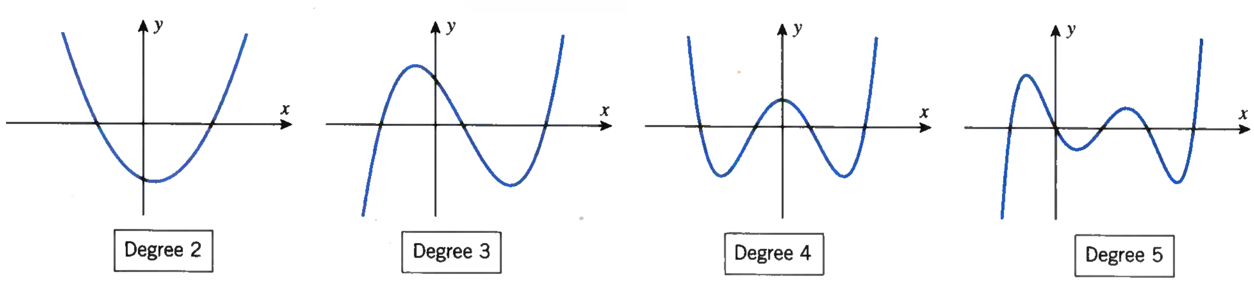


Figure 1.6.10

RATIONAL FUNCTIONS

A function that can be expressed as a ratio of two polynomials is called a ***rational function***. If $P(x)$ and $Q(x)$ are polynomials, then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

consists of all values of x such that $Q(x) \neq 0$. For example, the domain of the ration: function

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

consists of all values of x, except $x = 1$ and $x = -1$. Its graph is shown in Figure 1.6.1 along with the graphs of two other typical rational functions.

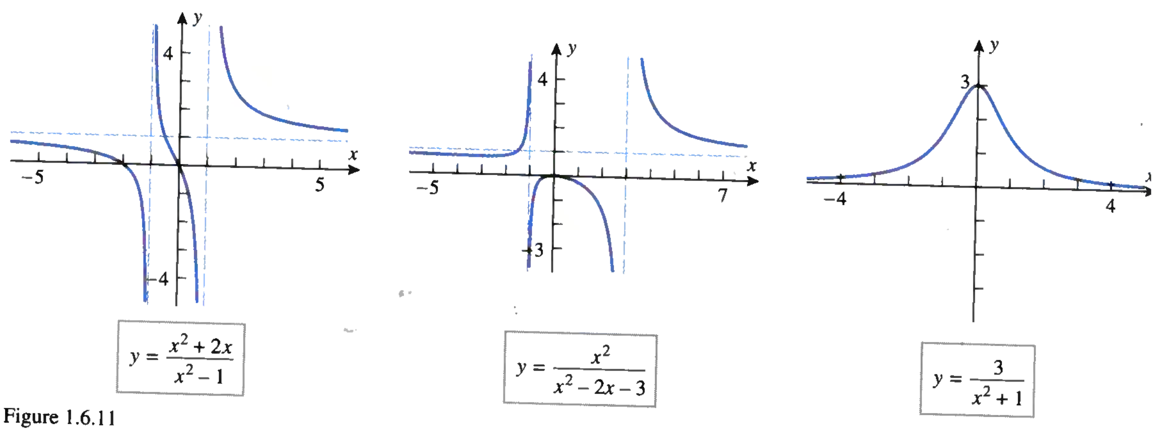


Figure 1.6.11

The graphs of rational functions with nonconstant denominators differ from the graphs of polynomials in some essential ways:

- Unlike polynomials whose graphs are continuous (unbroken) curves, the graphs of rational functions have discontinuities at the points where the denominator is zero.
- As x gets closer and closer to a point of discontinuity, the graph rises or falls indefinitely, getting closer and closer to a vertical line, called a ***vertical asymptote***; these are represented by the dashed vertical lines in Figure 1.6.11.
- Unlike the graphs of polynomials, which eventually rise or fall indefinitely, the graphs of many (but not all) rational functions eventually get closer and closer to some horizontal line, called a ***horizontal asymptote***, as one travels along the curve in either the positive or negative direction; these are represented by the dashed horizontal lines in the first two parts of Figure 1.6.11. In the third part of the figure the x-axis is a horizontal asymptote.

ALGEBRAIC FUNCTIONS

Functions that can be constructed from polynomials by applying finitely many algebraic operations (addition, subtraction, division, and root extraction) are called ***algebraic functions***. Some examples are

$$f(x) = \sqrt{x^2 - 4}, \quad f(x) = 3\sqrt[3]{x}(2 + x), \quad f(x) = x^{2/3}(x + 2)^2$$

As illustrated in Figure 1.6.12, the graphs of algebraic functions vary widely, so it is difficult to make general statements about them. Later in this text we will develop general calculus methods for analyzing such functions.

A QUICK REVIEW OF TRIGONOMETRIC FUNCTIONS

A detailed review of trigonometric functions is given in Appendix E, but for convenience we will summarize some of the main ideas here.

It is often convenient to think of the trigonometric functions in terms of circles rather than triangles. For this purpose, consider a point that moves either clockwise or counter-

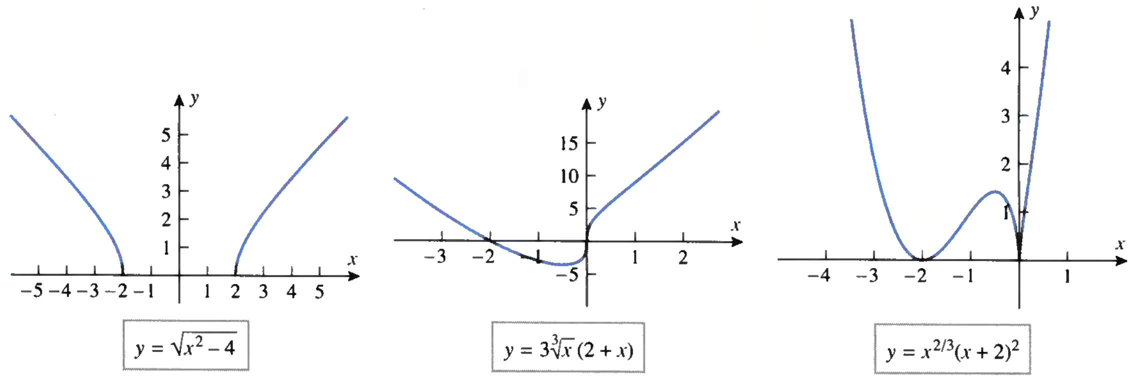


Figure 1.6.12

clockwise along the ***unit circle*** $u^2+v^2=1$ in the uv-plane, starting at $(1, 0)$ and stopping at a point P (Figure 1.6.13*a*). Let x denote the ***signed*** arc length traveled by the moving point, taking x to be positive for counterclockwise motion and negative for clockwise motion. (We allow for the possibility that the point may traverse the circle more than once.) When convenient, the variable x can also be interpreted as the angle in radians that is swept out by the radial line from the origin to P, with the usual convention that angles are positive if generated by counterclockwise rotations and negative if generated by clockwise rotations. We can *define* $\cos x$ to be the u-coordinate of P and $\sin x$ to be the v-coordinate of P (Figure 1.6.13*b*).

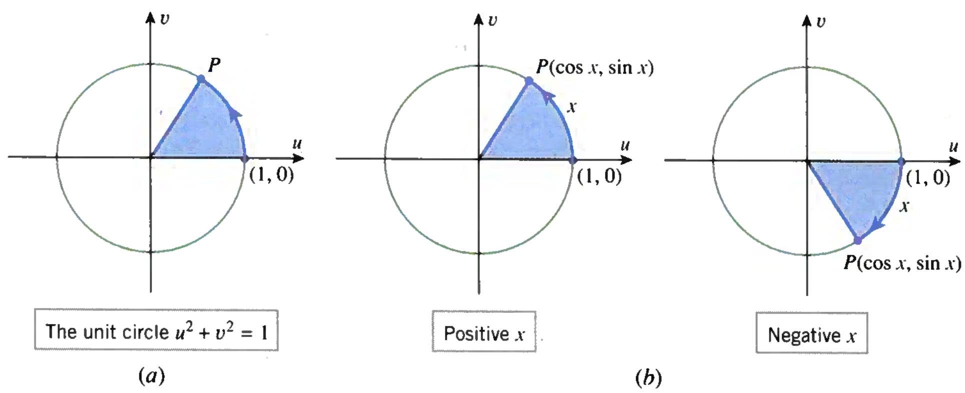


Figure 1.6.13

The remaining trigonometric functions can be defined in terms of the functions $\sin x$ and $\cos x$:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

The graphs of the six trigonometric functions in Figure 1.6.14 should already be familiar to you, but try generating them using a graphing utility, making sure to use radian measure for x.

REMARK. In this text we will always assume that the independent variable in a trigonometric function is in radians unless specifically stated otherwise.

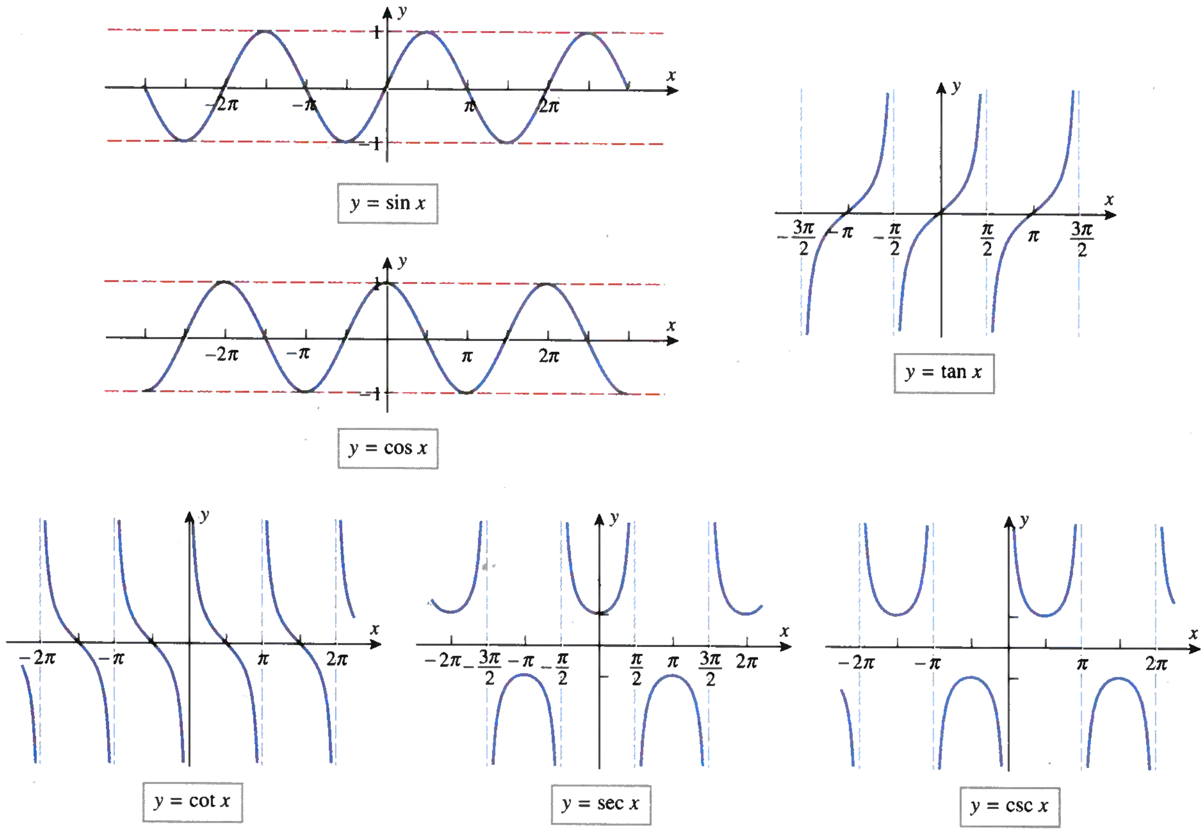


Figure 1.6.14

PROPERTIES OF sin *x*, cos *x*, AND tan *x*

Many of the basic properties of $\sin x$ and $\cos x$ can be deduced from the circle definitions of these functions. For example:

- As the point $P(\cos x, \sin x)$ moves around the unit circle, its coordinates vary between -1 and 1, and hence

  $$-1 \le \sin x \le 1 \quad \text{and} \quad -1 \le \cos x \le 1$$

- If x increases or decreases by 2π radians, then the point $P(\cos x, \sin x)$ makes one complete revolution around the unit circle, and the coordinates return to their starting values. Thus, $\sin x$ and $\cos x$ have period 2π; that is,

  $$\sin(x \pm 2\pi) = \sin x$$
  $$\cos(x \pm 2\pi) = \cos x$$

- As $P(\cos x, \sin x)$ moves around the unit circle, $\sin x$ is zero when P is on the horizontal axis (which occurs when x is an integer multiple of π), and $\cos x$ is zero when P is on the vertical axis (which occurs when x is an odd multiple of $\pi/2$). Thus,

  $$\sin x = 0 \quad \text{if and only if} \quad x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$
  $$\cos x = 0 \quad \text{if and only if} \quad x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$$

- As $P(\cos x, \sin x)$ moves around the unit circle $u^2 + v^2 = 1$, its coordinates satisfy this equation for all x, which produces the fundamental trigonometric identity

  $$\cos^2 x + \sin^2 x = 1$$

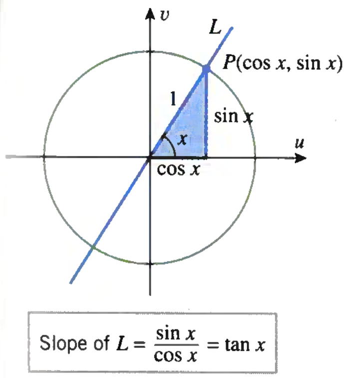


Figure 1.6.15

Observe that the graph of $y = \tan x$ has vertical asymptotes at the points $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$. This is to be expected since $\tan x = \sin x / \cos x$, and these are the values of x at which $\cos x$ is zero. What is less obvious, however, is the fact that $\tan x$ repeats every π radians (i.e., has period π), even though $\sin x$ and $\cos x$ have period 2π. This can be explained by interpreting

$$\tan x = \frac{\sin x}{\cos x}$$

as the slope of the line L that passes through the origin and the point $P(\cos x, \sin x)$ on the unit circle in the uv-plane (Figure 1.6.15). Each time x increases or decreases by π radians, the point P traverses half the circumference, and the line L rotates π radians, so its starting and ending slope are the same.

RADIANS AS A DIMENSIONLESS UNIT

The choice of radian measure as opposed to degree measure depends on the nature of the problem being considered; degree measure is usually chosen in engineering problems involving measurements of angles, and radian measure is usually chosen when the function properties of $\sin x$, $\cos x$, $\tan x$, ... are the primary focus. Radian measure is also usually chosen in problems involving arc lengths on circles because of the basic result in trigonometry which states that the arc length s of a sector with radius r and a central angle of θ (radians) is given by

$$s = r\theta \tag{6}$$

(Figure 1.6.16).

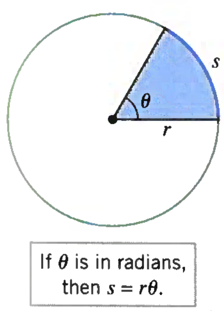


Figure 1.6.16

In applications involving angles, radians require special treatment to ensure that quantities are assigned proper units. To see why this is so, let us rewrite (6) as

$$\theta = \frac{s}{r}$$

The left side of this equation is in radians, and the right side is the ratio of two lengths, say meters/meters or feet/feet. However, because these units of length cancel, the right side of this equation is actually *dimensionless* (has no units). Thus, to ensure consistency between the two sides of the equation, we would have to omit the units of radians on the left side to make it dimensionless as well. In practical terms this means that units of radians can be used in intermediate computations, when convenient, but they need to be omitted in the end result to ensure consistency of units. This is confusing, to say the least, but the following example should clarify the idea.

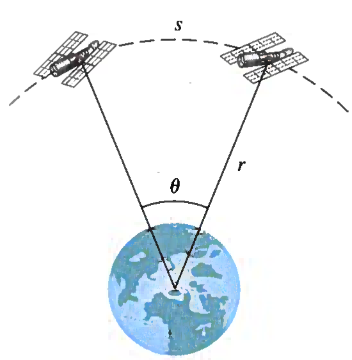


Figure 1.6.17

Example 3

Suppose that two satellites circle the equator in an orbit of radius $r = 4.23 \times 10^7$ m (Figure 1.6.17). Find the arc length s that separates the satellites if they have an angular separation of $\theta = 2.00°$.

Solution. To apply Formula (6), we must convert the angular separation to radians:

$$2.00° = \frac{\pi}{180}(2.00) \approx 0.0349 \text{ rad}$$

Thus, from (6)

$$s = r\theta = (4.23 \times 10^7 \text{ m})(0.0349 \text{ rad}) = 1.48 \times 10^6 \text{ m}$$

In this computation the product $r\theta$ produces units of meters × radians, but if we treat radians as dimensionless, we have meters × radians = meters, which correctly produces units of meters (m) for the arc length s. ◀

THE FAMILIES $y = A \sin Bx$ AND $y = A \cos Bx$

Many important applications lead to trigonometric functions of the form

$$f(x) = A\sin(Bx - C) \quad \text{and} \quad g(x) = A\cos(Bx - C) \tag{7}$$

where A, B, and C are nonzero constants. The graphs of such functions can be obtained by stretching, compressing, translating, and reflecting the graphs of $y = \sin x$ and $y = \cos x$ appropriately. To see why this is so, let us start with the case where $C = 0$ and consider how the graphs of the equations

$$y = A\sin Bx \quad \text{and} \quad y = A\cos Bx$$

relate to the graphs of $y = \sin x$ and $y = \cos x$. If A and B are positive, then the effect of the constant A is to stretch or compress the graphs of $y = \sin x$ and $y = \cos x$ vertically by a factor of A, and the effect of the constant B is to compress or stretch the graphs of $\sin x$ and $\cos x$ horizontally by a factor of B. For example, the graph of $y = 2\sin 4x$ can be obtained by stretching the graph of $y = \sin x$ vertically by a factor of 2 and compressing it horizontally by a factor of 4. (Recall from Section 1.4 that the multiplier of x *stretches* when it is less than 1 and *compresses* when it is greater than 1.) Thus, as shown in Figure 1.6.18, the graph of $y = 2\sin 4x$ varies between -2 and 2, and repeats every $2\pi/4 = \pi/2$ units.

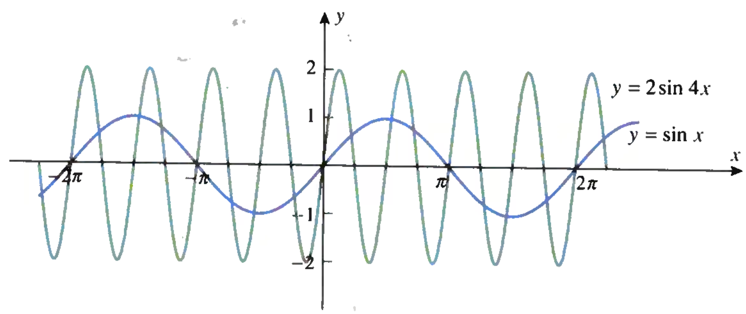


Figure 1.6.18

In general, if A and B are positive numbers, then the graphs of

$$y = A\sin Bx \quad \text{and} \quad y = A\cos Bx$$

oscillate between $-A$ and A and repeat every $2\pi/B$ units, so we say that these functions have ***amplitude*** A and ***period*** $2\pi/B$. In addition, we define the ***frequency*** of these functions to be the reciprocal of the period, that is, the frequency is $B/2\pi$. If A or B is negative, then these constants cause reflections of the graphs about the axes as well as compressing or stretching them; and in this case the amplitude, period, and frequency are given by $|A|$, $2\pi/|B|$, and $|B|/2\pi$, respectively.

Example 4

Make sketches of the following graphs that show the period and amplitude.

(a) $y = 3\sin 2\pi x$ (b) $y = -3\cos 0.5x$ (c) $y = 1 + \sin x$

Solution (a). The equation is of the form $y = A\sin Bx$ with $A = 3$ and $B = 2\pi$, so the graph has the shape of a sine function, but with amplitude $A = 3$ and period $2\pi/B = 2\pi/2\pi = 1$ (Figure 1.6.19*a*).

Solution (b). The equation is of the form $y = A\cos Bx$ with $A = -3$ and $B = 0.5$, so the graph has the shape of a cosine function that has been reflected about the x-axis (because $A = -3$ is negative), but with amplitude $|A| = 3$ and period $2\pi/B = 2\pi/0.5 = 4\pi$ (Figure 1.6.19*b*).

Solution (c). The graph has the shape of a sine function that has been translated up 1 unit (Figure 1.6.19*c*). ◀

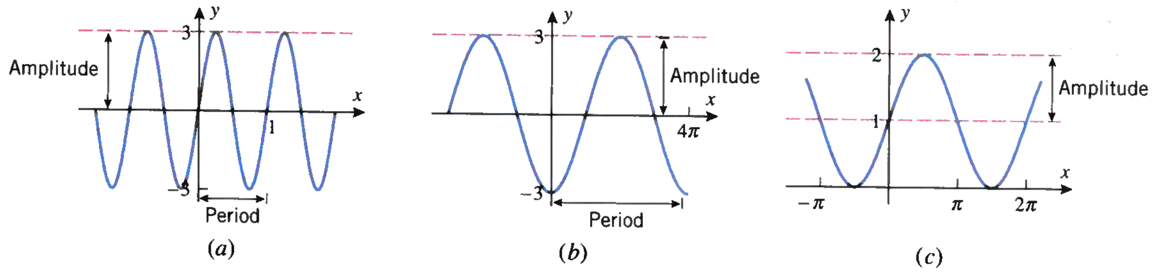


Figure 1.6.19

THE FAMILIES $y = A\sin(Bx - C)$ AND $y = A\cos(Bx - C)$

To investigate the graphs of the more general families

$$y = A\sin(Bx - C) \quad \text{and} \quad y = A\cos(Bx - C)$$

it will be helpful to rewrite these equations as

$$y = A\sin\left[B\left(x - \frac{C}{B}\right)\right] \quad \text{and} \quad y = A\cos\left[B\left(x - \frac{C}{B}\right)\right]$$

In this form we see that the graphs of these equations can be obtained by translating the graphs of $y = A\sin Bx$ and $y = A\cos Bx$ to the left or right, depending on the sign of C/B. For example, if $C/B > 0$, then the graph of

$$y = A\sin[B(x - C/B)] = A\sin(Bx - C)$$

can be obtained by translating the graph of $y = A\sin Bx$ to the right by C/B units (Figure 1.6.20). The quantity C/B is called the ***phase shift*** of the function; a positive phase shift corresponds to right translation, and a negative phase shift corresponds to a left translation.

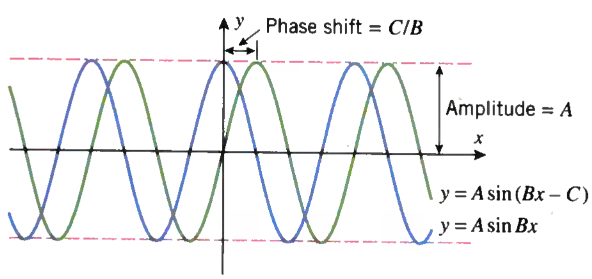


Figure 1.6.20

Example 5

Find the amplitude, period, and phase shift of

$$y = 3\cos\left(2x + \frac{\pi}{2}\right)$$

and confirm your results by graphing the equation on a calculator or computer.

Solution. The equation can be rewritten as

$$y = 3\cos\left[2x - \left(-\frac{\pi}{2}\right)\right] = 3\cos\left[2\left(x - \left(-\frac{\pi}{4}\right)\right)\right]$$

which is of the form

$$y = A\cos\left[B\left(x - \frac{C}{B}\right)\right]$$

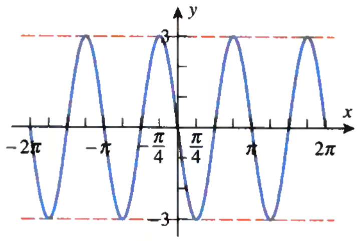


Figure 1.6.21

with $A = 3$, $B = 2$, and $C/B = -\pi/4$; thus, the graph has the shape of a cosine function, but with amplitude $A = 3$, period $2\pi/B = \pi$, and phase shift $C/B = -\pi/4$ (Figure 1.6.21). ◀

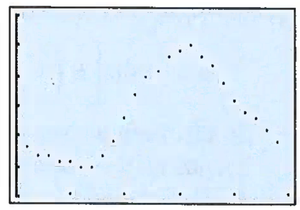
Example 6

Figure 1.6.22*a* shows a table and scatter plot of temperature data recorded over a 24-hour period in the city of Philadelphia.* Find a function that models the data, and graph your function and data together.

PHILADELPHIA TEMPERATURES
FROM 1:00 A.M. TO 12:00 MIDNIGHT ON 27 AUGUST 1993
(t = HOURS AFTER MIDNIGHT AND T = DEGREES FAHRENHEIT)

	A.M.		P.M.	
	t	T	t	T
1:00	1	78°	13	91°
2:00	2	77°	14	93°
3:00	3	77°	15	94°
4:00	4	76°	16	95°
5:00	5	76°	17	93°
6:00	6	75°	18	92°
7:00	7	75°	19	89°
8:00	8	77°	20	86°
9:00	9	79°	21	84°
10:00	10	83°	22	83°
11:00	11	87°	23	81°
12:00	12	90°	24	79°

Source: *Philadelphia Inquirer*, 28 August 1993.


Scatter plot of data
[0, 25] × [70, 100]
t T
(*a*)

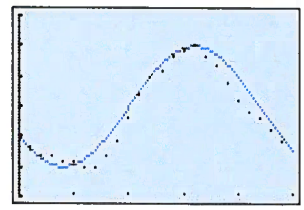

Model for data
$T = 85 + 10\sin[(\pi/12)(t - 10)]$
[0, 25] × [70, 100]
t T
(*b*)

Figure 1.6.22

Solution. The pattern of the data suggests that the relationship between the temperature T and the time t can be modeled by a sinusoidal function that has been translated both horizontally and vertically, so we will look for an equation of the form

$$T = D + A\sin[Bt - C] = D + A\sin\left[B\left(t - \frac{C}{B}\right)\right] \tag{8}$$

Since the highest temperature is 95°F and the lowest temperature is 75°F, we take $2A = 20$ or $A = 10$. The midpoint between the high and low is 85°F, so we have a vertical shift of $D = 85$. The period seems to be about 24, so $2\pi/B = 24$ or $B = \pi/12$. The phase shift appears to be about 10 (verify), so $C/B = 10$. Substituting these values in (8) yields the equation

$$T = 85 + 10\sin\left[\frac{\pi}{12}(t - 10)\right]$$

(Figure 1.6.22*b*). ◀

*This example is based on the article "Everybody Talks About It!—Weather Investigations," by Gloria S. Dion and Iris Brann Fetta, *The Mathematics Teacher*, Vol. 89, No. 2, February 1996, pp. 160–165.

OTHER FAMILIES

In addition to the functions mentioned in this section, there are exponential and logarithmic functions, which we will study later, and various special functions that arise in physics and engineering. There are also many kinds of functions that have no names; indeed, one of the important themes of calculus is to provide methods for analyzing new types of functions.

EXERCISE SET 1.6 Graphing Calculator

1. (a) Find an equation for the family of lines whose members have slope $m = 3$.
   (b) Find an equation for the member of the family that passes through $(-1, 3)$.
   (c) Sketch some members of the family, and label them with their equations. Include the line in part (b).
2. Find an equation for the family of lines whose members are perpendicular to those in Exercise 1.
3. (a) Find an equation for the family of lines with y-intercept $b = 2$.
   (b) Find an equation for the member of the family whose angle of inclination is $135°$.
   (c) Sketch some members of the family, and label them with their equations. Include the line in part (b).
4. Find an equation for
   (a) the family of lines that pass through the origin
   (b) the family of lines with x-intercept $a = 1$
   (c) the family of lines that pass through the point $(1, -2)$
   (d) the family of lines parallel to $2x + 4y = 1$.

In Exercises 5 and 6, state a geometric property common to all lines in the family, and sketch five of the lines.

5. (a) The family $y = -x + b$
   (b) The family $y = mx - 1$
   (c) The family $y = m(x + 4) + 2$
   (d) The family $x - ky = 1$
6. (a) The family $y = b$
   (b) The family $Ax + 2y + 1 = 0$
   (c) The family $2x + By + 1 = 0$
   (d) The family $y - 1 = m(x + 1)$
7. Find an equation for the family of lines tangent to the circle with center at the origin and radius 3.
8. Find an equation for the family of lines that pass through the intersection of $5x - 3y + 11 = 0$ and $2x - 9y + 7 = 0$.
9. The U.S. Internal Revenue Service uses a 10-year linear depreciation schedule to determine the value of various business items. This means that an item is assumed to have a value of zero at the end of the tenth year and that at intermediate times the value is a linear function of the elapsed time. Sketch some typical depreciation lines, and explain the practical significance of the y-intercepts.
10. Find all lines through $(6, -1)$ for which the product of the x- and y-intercepts is 3.
11. In each part, match the equation with one of the accompanying graphs.
    (a) $y = \sqrt[5]{x}$ (b) $y = 2x^5$
    (c) $y = -1/x^8$ (d) $y = 8^x$
    (e) $y = \sqrt[4]{x - 2}$ (f) $y = 1/8^x$

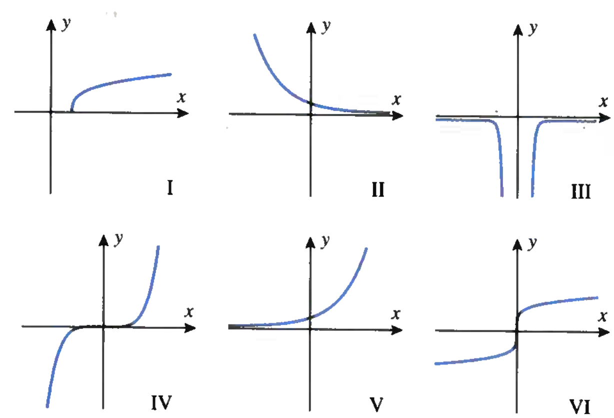


Figure Ex-11

12. The table in the accompanying figure gives approximate values of three functions: one of the form kx^2, one of the form kx^{-3}, and one of the form $kx^{3/2}$. Identify which is which, and estimate k in each case.

x	0.25	0.37	2.1	4.0	5.8	6.2	7.9	9.3
$f(x)$	640	197	1.08	0.156	0.0513	0.0420	0.0203	0.0124
$g(x)$	0.0312	0.0684	2.20	8.00	16.8	19.2	31.2	43.2
$h(x)$	0.250	0.450	6.09	16.0	27.9	30.9	44.4	56.7

Figure Ex-12

In Exercises 13 and 14, sketch the graph of the equation for $n = 1, 3,$ and 5 in one coordinate system and for $n = 2, 4,$ and 6 in another coordinate system. Check your work with a graphing utility.

13. (a) $y = -x^n$ (b) $y = 2x^{-n}$ (c) $y = (x-1)^{1/n}$

14. (a) $y = 2x^n$ (b) $y = -x^{-n}$
(c) $y = -3(x+2)^{1/n}$

15. (a) Sketch the graph of $y = ax^2$ for $a = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(b) Sketch the graph of $y = x^2 + b$ for $b = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(c) Sketch some typical members of the family of curves $y = ax^2 + b$.

16. (a) Sketch the graph of $y = a\sqrt{x}$ for $a = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(b) Sketch the graph of $y = \sqrt{x} + b$ for $b = \pm 1, \pm 2,$ and ± 3 in a single coordinate system.
(c) Sketch some typical members of the family of curves $y = a\sqrt{x} + b$.

In Exercises 17–20, sketch the graph of the equation by making appropriate transformations to the graph of a basic power function. Check your work with a graphing utility.

17. (a) $y = 2(x+1)^2$ (b) $y = -3(x-2)^3$
(c) $y = \dfrac{-3}{(x+1)^2}$ (d) $y = \dfrac{1}{(x-3)^5}$

18. (a) $y = 1 - \sqrt{x+2}$ (b) $y = 1 - \sqrt[3]{x+2}$
(c) $y = \dfrac{5}{(1-x)^3}$ (d) $y = \dfrac{2}{(4+x)^4}$

19. (a) $y = \sqrt[3]{x+1}$ (b) $y = 1 - \sqrt{x-2}$
(c) $y = (x-1)^5 + 2$ (d) $y = \dfrac{x+1}{x}$

20. (a) $y = 1 + \dfrac{1}{x-2}$ (b) $y = \dfrac{1}{1+2x-x^2}$
(c) $y = -\dfrac{2}{x^7}$ (d) $y = x^2 + 2x$

21. Sketch the graph of $y = x^2 + 2x$ by completing the square and making appropriate transformations to the graph of $y = x^2$.

22. (a) Use the graph of $y = \sqrt{x}$ to help sketch the graph of $y = \sqrt{|x|}$.
(b) Use the graph of $y = \sqrt[3]{x}$ to help sketch the graph of $y = \sqrt[3]{|x|}$.

23. The table in the accompanying figure provides data about the relationship between distance d traveled in meters and elapsed time t in seconds for an object dropped near the Earth's surface. Plot time versus distance and make a guess at a "square-root function" that provides a reasonable model for t in terms of d. Use a graphing utility to confirm the reasonableness of your guess.

d (meters)	0	2.5	5	10	15	20	25
t (seconds)	0	0.7	1.0	1.4	1.7	2	2.3

Figure Ex-23

24. (a) The table below provides data on five moons of the planet Saturn. In this table r is the *orbital radius* (the average distance between the moon and Saturn) and t is the time in days required for the moon to complete one orbit around Saturn. For each data pair calculate $tr^{-3/2}$, and use your results to find a reasonable model for r as a function of t.
(b) Use the model from part (a) to estimate the orbital radius of the moon Enceladus, given that its orbit time is $t \approx 1.370$ days.
(c) Use the model from part (a) to estimate the orbit time of the moon Tethys, given that its orbital radius is $r \approx 2.9467 \times 10^5$ km.

Moon	Radius (100,000 km)	Orbit Time (days)
1980S28	1.3767	0.602
1980S27	1.3935	0.613
1980S26	1.4170	0.629
1980S3	1.5142	0.694
1980S1	1.5147	0.695

25. As discussed in this section, Boyle's law states that at a constant temperature the pressure P exerted by a gas is related to the volume V by the equation $P = k/V$.
(a) Find the appropriate units for the constant k if pressure (which is force per unit area) is in newtons per square meter (N/m^2) and volume is in cubic meters (m^3).
(b) Find k if the gas exerts a pressure of 20,000 N/m^2 when the volume is 1 liter (0.001 m^3).
(c) Make a table that shows the pressures for volumes of 0.25, 0.5, 1.0, 1.5, and 2.0 liters.
(d) Make a graph of P versus V.

26. A manufacturer of cardboard drink containers wants to construct a closed rectangular container that has a square base and will hold $\frac{1}{10}$ liter (100 cm^3). Estimate the dimension of the container that will require the least amount of material for its manufacture.

A variable y is said to be ***inversely proportional to the square of a variable x*** if y is related to x by an equation of the form $y = k/x^2$, where k is a nonzero constant, called the ***constant of proportionality***. This terminology is used in Exercises 27 and 28.

27. According to Coulomb's law, the force F of attraction between positive and negative point charges is inversely proportional to the square of the distance x between them.

(a) Assuming that the force of attraction between two point charges is 0.0005 newton when the distance between them is 0.3 meter, find the constant of proportionality (with proper units).

(b) Find the force of attraction between the point charges when they are 3 meters apart.

(c) Make a graph of force versus distance for the two charges.

(d) What happens to the force as the particles get closer and closer together? What happens as they get farther and farther apart?

28. It follows from Newton's Universal Law of Gravitation that the weight W of an object (relative to the Earth) is inversely proportional to the square of the distance x between the object and the center of the Earth, that is, $W = C/x^2$.

(a) Assuming that a weather satellite weighs 2000 pounds on the surface of the Earth and that the Earth is a sphere of radius 4000 miles, find the constant C.

(b) Find the weight of the satellite when it is 1000 miles above the surface of the Earth.

(c) Make a graph of the satellite's weight versus its distance from the center of the Earth.

(d) Is there any distance from the center of the Earth at which the weight of the satellite is zero? Explain your reasoning.

29. In each part, match the equation with one of the accompanying graphs, and give the equations for the horizontal and vertical asymptotes.

(a) $y = \dfrac{x^2}{x^2 - x - 2}$ (b) $y = \dfrac{x - 1}{x^2 - x - 6}$

(c) $y = \dfrac{2x^4}{x^4 + 1}$ (d) $y = \dfrac{4}{(x + 2)^2}$

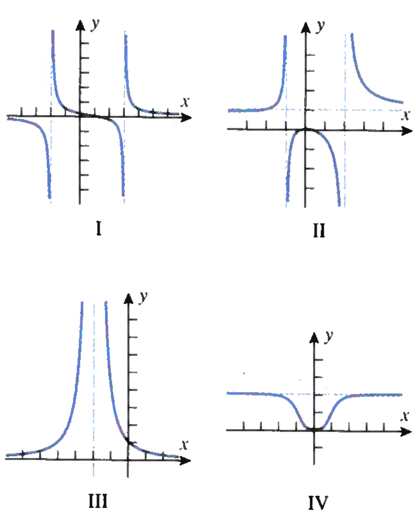


Figure Ex-29

30. Find an equation of the form $y = k/(x^2 + bx + c)$ whose graph is a reasonable match to that in the accompanying figure. Check your work with a graphing utility.

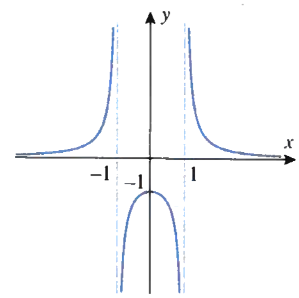


Figure Ex-30

In Exercises 31 and 32, draw a radial line from the origin with the given angle, and determine whether the six trigonometric functions are positive, negative, or undefined for that angle.

31. (a) $\dfrac{\pi}{3}$ (b) $-\dfrac{\pi}{2}$ (c) $\dfrac{2\pi}{3}$

(d) -1 (e) $\dfrac{5\pi}{4}$ (f) $\dfrac{11\pi}{6}$

32. (a) $\dfrac{3\pi}{2}$ (b) $-\dfrac{5\pi}{4}$ (c) π

(d) $\dfrac{5\pi}{2}$ (e) 4 (f) $-\dfrac{33\pi}{7}$

In Exercises 33 and 34, use a calculating utility set to the radian mode to confirm the approximations $\sin(\pi/5) \approx 0.588$ and $\cos(\pi/8) \approx 0.924$, and then use these values to approximate the given expressions by hand calculation. Check your answers using the trigonometric function operations of your calculating utility.

33. (a) $\sin \dfrac{4\pi}{5}$ (b) $\cos\left(-\dfrac{\pi}{8}\right)$ (c) $\sin \dfrac{11\pi}{5}$

(d) $\cos \dfrac{7\pi}{8}$ (e) $\sin \dfrac{2\pi}{5}$ (f) $\cos^2 \dfrac{\pi}{5}$

34. (a) $\sin \dfrac{16\pi}{5}$ (b) $\cos\left(-\dfrac{17\pi}{8}\right)$ (c) $\sin \dfrac{41\pi}{5}$

(d) $\sin\left(-\dfrac{\pi}{16}\right)$ (e) $\cos \dfrac{27\pi}{8}$ (f) $\tan^2 \dfrac{\pi}{8}$

35. Assuming that $\sin\alpha = a$, $\cos\beta = b$, and $\tan\gamma = c$, express the stated quantities in terms of a, b, and c.

(a) $\sin(-\alpha)$ (b) $\cos(-\beta)$ (c) $\tan(-\gamma)$

(d) $\sin\left(\dfrac{\pi}{2} - \alpha\right)$ (e) $\cos(\pi - \beta)$ (f) $\sin(\alpha + \pi)$

(g) $\sin(2\beta)$ (h) $\cos(2\beta)$ (i) $\sec(\beta + 2\pi)$

(j) $\csc(\alpha + \pi)$ (k) $\cot(\gamma + 5\pi)$ (l) $\sin^2\left(\dfrac{\beta}{2}\right)$

36. A ship travels from a point near Hawaii at 20° N latitude directly north to a point near Alaska at 56° N latitude.
(a) Assuming the Earth to be a sphere of radius 4000 mi, find the actual distance traveled by the ship.
(b) What fraction of the Earth's circumference did the ship travel?

37. The Moon completes one revolution around the Earth in approximately 29.5 days. Assuming that the Moon's orbit is a circle with a radius of 0.38×10^9 m from the center of the Earth, find the arc length traveled by the Moon in 1 day.

38. A spoked wheel with a diameter of 3 ft rolls along a flat road without slipping. How far along the road does the wheel roll if the spokes turn through 225°?

39. As illustrated in the accompanying figure, suppose that you hold one quarter flat against a table while you rotate a second quarter around it without slippage. Through what angle will the second quarter have turned about its own center when it returns to its original location?

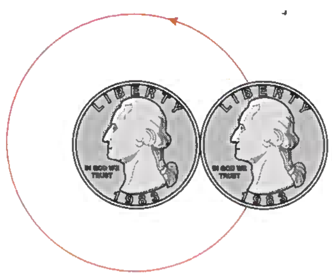


Figure Ex-39

40. Suppose that you begin cutting wedge-shaped pieces from a pie so that the arc length along the outer crust of each piece is equal to the radius. What fraction of the pie will remain after all pieces that can be cut in this way are eaten?

> In Exercises 41 and 42, find an equation for the graph assuming that there is no phase shift.

41.

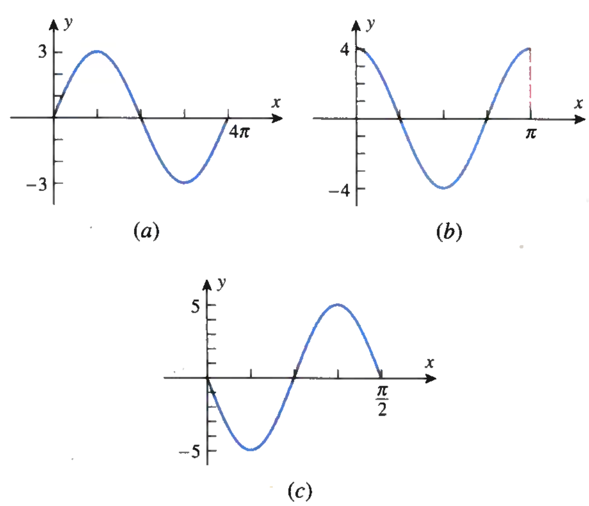


Figure Ex-41

42.

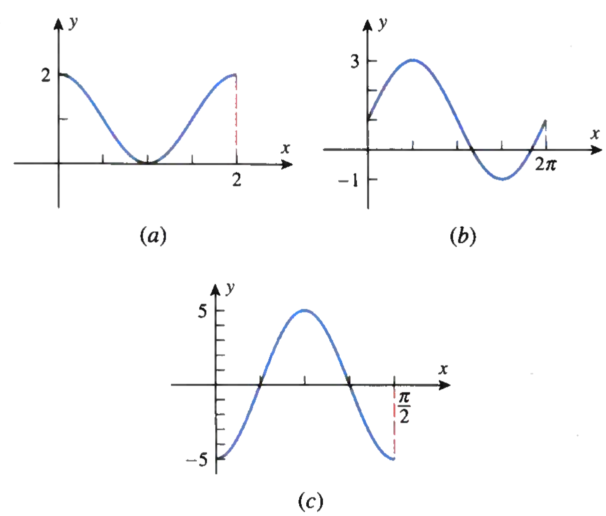


Figure Ex-42

43. In each part, find an equation for the graph that has the form $y = y_0 + A \sin(Bx - C)$.

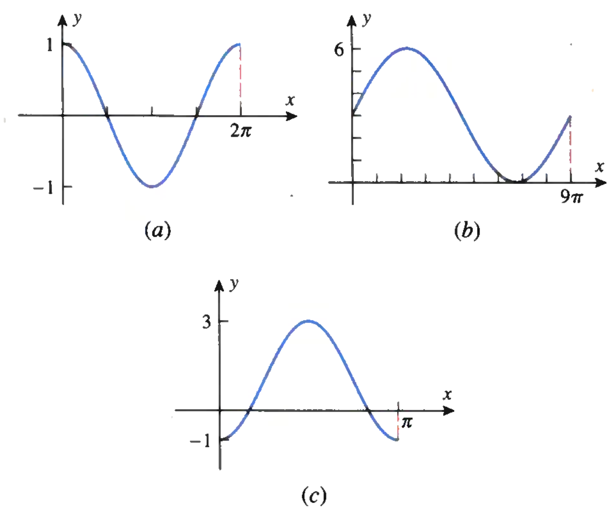


Figure Ex-43

44. In the United States, a standard electrical outlet supplies sinusoidal electrical current with a maximum voltage of $V = 120\sqrt{2}$ volts (V) at a frequency of 60 cycles per second. Write an equation that expresses V as a function of the time t, assuming that $V = 0$ if $t = 0$.

> In Exercises 45 and 46, find the amplitude, period, and phase shift, and sketch at least two periods of the graph by hand. Check your work with a graphing utility.

45. (a) $y = 3 \sin 4x$ (b) $y = -2 \cos \pi x$
(c) $y = 2 + \cos\left(\dfrac{x}{2}\right)$